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Optimal Control of a Class of Discrete Multivariable Nonlinear Systems. Application to a Fermentation Process.

The optimal control of a class of discrete multivariable nonlinear systems given by: $x_{k+1} = a(x_k) + B(x_k)u_k$, $y_k = Cx_k$, is analyzed. A closed-loop structure is obtained with the proposed performance index. The addition of numerical integrators to the output error and the design of an optimal control law for the resultant augmented system lead to a very robust control structure. The performance of this control law is evaluated by applying it to a simulated continuous culture fermentation process.

Introduction

The synthesis of control laws for nonlinear systems is a problem which has recently begun to be studied from different points of view [1-6]. However, most of the published papers are concerned only with the continuous case and the implementation of different analog control devices is not very performant. This is mainly due to the presence of multipliers and dividers needed to synthesize the control law [6].

The improvement in the use of microcomputers for process control in the last few years has made the research on control of discrete nonlinear systems be very attractive, because of the facility for programing any control algorithm.

In this paper a technique for synthesizing a control law for a class of discrete nonlinear systems is proposed. This control law is optimal with respect to a given performance index. Some properties of the resultant closed-loop system are presented and the elimination of the steady state error is discussed. A scheme based on a set of integrators and amplifiers is proposed in order to obtain a very robust control structure.

The control law is applied to a simulated continuous culture fermentation process. There has been some attempts to accomplish a digital optimal control of this kind of processes - [7, 8]; however, it still remains some problems unsolved. They are originated by the high nonlinearity of the plant model and the typical parameter variations presented when changes in the environment occur [10]. This work probably represents a first solution to these problems.

Discrete Nonlinear Systems

The nonlinear systems considered are those described by the following equations:

$$\mathbf{x}_{k+1} = \mathbf{a}(\mathbf{x}_k) + B(\mathbf{x}_k)\mathbf{u}_k \tag{1}$$

$$\mathbf{y}_k = C\mathbf{x}_k \tag{2}$$

where $\mathbf{x}_k \in R^n$, $\mathbf{u}_k \in R^r$ and $\mathbf{y}_k \in R^m$ are the state, control and output vectors, respectively; $\mathbf{a}(\mathbf{x}_k)$ and $B(\mathbf{x}_k)$ are vector and matrix functions of class C^∞. C is a constant matrix of proper dimensions. Moreover, in this paper the following hypothesis are proposed:

H1) The state $\mathbf{x}$ is measurable. This condition is necessary in order to obtain an optimal controller, as it will be shown later.

H2) $m=r$, that is, the system has an equal number of inputs and outputs. This is a frequently used hypothesis in the design of control algorithms for multivariable systems - [1, 9].

Measure of Performance

The general problem can be stated as to find a control sequence which transfers the system output to a specified value and maintains it there even under the presence of internal or external perturbations.

The design of the control law is performed by means of optimal control theory. It is then necessary to define the performance measure to be minimized when the control law is applied. This measure is selected as follows: if it is desired to drive the output vector to a given one (reference), it must contain an output error function. This may be accomplished by means of a non-negative definite and differentiable penalty function of the error, $h(\mathbf{e}_k)$. Furthermore, if it is desired to prevent large excursions of the control vector from one sampling time to the next, a penalty function of the control or, more precisely, of the difference $\mathbf{u}_k - \mathbf{u}_{k-1}$ can be introduced. However, the introduction of this function makes very difficult to obtain an explicit solution of the control, thus precluding a closed-loop control scheme. It is then proposed not to consider the control difference, $\mathbf{u}_k - \mathbf{u}_{k-1}$, but the error difference $\mathbf{e}_{k+1} - \mathbf{e}_k$. As it will be seen later, this consideration permits to obtain an explicit solution of the control.

The following performance measure is then proposed:

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$$J = \sum_{k=0}^{N-1} (h(\mathbf{e}_k) + (\mathbf{e}_{k+1} - \mathbf{e}_k)^T (\mathbf{e}_{k+1} - \mathbf{e}_k)) \tag{3}$$

where $[O, N]$ is the observation time and:

$$\mathbf{e}_k \triangleq \mathbf{z}_k - \mathbf{y}_k \tag{4}$$

$\mathbf{e}_k$ is the output error and $\mathbf{z}_k$ the reference vector. The problem can then be reformulated as to find an optimal control sequence, $\mathbf{u}_k^*$, $k = 0, 1, \ldots, N-1$, such that the performance measure given by (3) be minimized over the trajectories defined by (1) and (2).

Optimal Control Law

For the system given by (1) and (2), under the hypothesis H1 and H2, we will show in this section that the control law minimizing the performance measure (3) is given by:

$$\mathbf{u}_k^* = (CB(\mathbf{x}_k))^{-1}(\mathbf{z}_{k+1} - C\mathbf{a}(\mathbf{x}_k) + (\tfrac{1}{2}M_k - Q - \mathrm{I})\mathbf{e}_k) \tag{5}$$

where I is the identity matrix, M_k and Q are (mxm)-matrix that will be defined later.

The Hamiltonian of the system is:

$$\mathrm{H}(\mathbf{e}_k, \mathbf{u}_k, \mathbf{p}_{k+1}, k) \triangleq \mathrm{H}_k = h(\mathbf{e}_k) + (\mathbf{e}_{k+1} - \mathbf{e}_k)^T (\mathbf{e}_{k+1} - \mathbf{e}_k) + \\ + \mathbf{p}_{k+1}^T \mathbf{e}_{k+1} \tag{6}$$

where $\mathbf{p}_k$ is the system costate vector.

The optimal control is the control that minimizes the Hamiltonian. Since $\mathbf{u}_k \in R^r$ is not constrained, the derivative of equation (6) can be taken with respect to $\mathbf{u}_k$ in order to obtain an expression for this control. On the other hand, from (1), (2), and (4) it follows that:

$$\mathbf{e}_{k+1} = \mathbf{z}_{k+1} - C\mathbf{a}(\mathbf{x}_k) - CB(\mathbf{x}_k)\mathbf{u}_k \tag{7}$$

Substituting (7) in (6), taking the derivative of the resultant equation with respect to $\mathbf{u}_k$ and making it equal to zero, we obtain:

$$-2(CB(\mathbf{x}_k))^T (\mathbf{z}_{k+1} - C\mathbf{a}(\mathbf{x}_k) - CB(\mathbf{x}_k)\mathbf{u}_k - \mathbf{e}_k + \tfrac{1}{2}\mathbf{p}_{k+1}) = \mathbf{0} \tag{8}$$

By hypothesis H2, $CB(\mathbf{x}_k)$ is a (mxm)-matrix. Let's assume that $CB(\mathbf{x}_k)$ is nonsingular. The control law is then given by:

$$\mathbf{u}_k^* = (CB(\mathbf{x}_k))^{-1}(\mathbf{z}_{k+1} - C\mathbf{a}(\mathbf{x}_k) - \mathbf{e}_k + \tfrac{1}{2}\mathbf{p}_{k+1}) \tag{9}$$

It can be observed that $\partial^2 \mathrm{H}_k / \partial \mathbf{u}_k^2 = 2(CB(\mathbf{x}_k))^T (CB(\mathbf{x}_k))$ is a positive definite matrix; then the calculated control vector minimizes the performance index.

Note: It has been assumed that matrix $CB(\mathbf{x}_k)$ is nonsingular. If a region of the state space exists where this matrix is singular, the optimal input signal may become unbounded. This doesn't invalidate the optimality of the solution, but it implies that the optimal control law cannot be implemented in practice.

We can now obtain the equation for $\mathbf{p}_k$. This equation is given by $\mathbf{p}_k = \partial \mathrm{H}_k / \partial \mathbf{e}_k$. From (6), (7), and (9) it follows that:

$$\mathbf{p}_{k+1} = -\frac{\partial h(\mathbf{e}_k)}{\partial \mathbf{e}_k} + \mathbf{p}_k \tag{10}$$

and from (7), (9), and (10) we obtain:

$$\mathbf{e}_{k+1} = \mathbf{e}_k + \tfrac{1}{2}\frac{\partial h(\mathbf{e}_k)}{\partial \mathbf{e}_k} - \tfrac{1}{2}\mathbf{p}_k \tag{11}$$

Expressions (9) to (11) form a system of equations which must be satisfied for (3) to be a minimum.

Let us propose:

$$h(\mathbf{e}_k) = \mathbf{e}_k^T Q \mathbf{e}_k \tag{12}$$

where Q is a (mxm) positive definite matrix which, by simplicity, is taken as a diagonal matrix. Then equations (10) and (11) become:

$$\mathbf{e}_{k+1} = (I+Q)\mathbf{e}_k - \tfrac{1}{2}\mathbf{p}_k \tag{13}$$

$$\mathbf{p}_{k+1} = -2Q\mathbf{e}_k + \mathbf{p}_k \tag{14}$$

It can be shown (see Appendix A) that the solution of system (13)–(14), for the costate vector, gives a linear error function:

$$\mathbf{p}_k = M_k \mathbf{e}_k \tag{15}$$

M_k being a time varying matrix. Then, from (9), (14), and (15) it follows equation (5).

An expression for M_k can easily be obtained from (13) to (15) as:

$$(M_{k+1}(I+Q) - \tfrac{1}{2}M_{k+1}M_k - M_k + 2Q)\mathbf{e}_k = \mathbf{0}$$

This equation must be satisfied for any $\mathbf{e}_k$. Therefore:

$$M_{k+1}(I+Q) - \tfrac{1}{2}M_{k+1}M_k - M_k + 2Q = 0 \tag{16}$$

Equation (16) is a first order quadratic equation. It can be solved starting from the boundary conditions given by $\mathbf{p}_N = \mathbf{0}$ [11]. Then, from (15), $M_N = 0$.

In summary, to obtain the control law it is necessary to solve equation (16) starting from the boundary condition $M_N = 0$. Notice that Q is assumed to be diagonal, then M_k is also diagonal for every k. Equation (16) is then a decoupled first order system of quadratic equations. After M_k is obtained for $k = N-1, N-2, \ldots, 0$, these values are used to calculate $\mathbf{u}_k$, $k = 0, 1, \ldots, N-1$, by means of equation (5).

Optimal Control Law When $N \to \infty$. For practical purposes it is useful to consider an infinite observation horizon ($N \to \infty$). In this case it is possible to show that equation (16) converges to a unique and constant solution (see Appendix B). This solution is given by:

$$M = \mathrm{diag}(q_i + \sqrt{q_i(q_i+4)})_{mxm} \tag{17}$$

where q_i is the weighting factor corresponding to the ith error component ($Q = \mathrm{diag}\,(q_i)$). The control law is then

$$\mathbf{u}_k^* = (CB(\mathbf{x}_k))^{-1}(\mathbf{z}_{k+1} - C\mathbf{a}(\mathbf{x}_k) + \hat{M}\mathbf{e}_k) \tag{18}$$

where $\hat{M} = \mathrm{diag}\,(\tfrac{1}{2}\sqrt{q_i(q_i+4)} - \tfrac{1}{2}q_i - 1)_{mxm}$

We can obtain an expression for the error from (13), (15), and (17):

$$\mathbf{e}_{k+1} = -\hat{M}\mathbf{e}_k \tag{19}$$

$\hat{M}$ is a diagonal matrix such that its elements have magnitude less than one if $q_i > 0$. Then the proposed control law makes the error converge to zero. It can be observed, from (19), that all the outputs are decoupled since $\hat{M}$ is a diagonal matrix. Moreover, q_i is directly related to the dynamics of the closed-loop system and it acts as an acceleration factor. This fact may be considered in order to propose a satisfactory value for q_i.

Proportional-Integral Control Law

An efficient controller must be robust; that is, it must keep the output at the desired steady state value even when external or internal (parametric) disturbances are present. The results obtained in the above section about the error convergence are valid only when the controller is designed using the real process parameter values. However, when these values are not known precisely or a parameter variation occurs and this is not taken into account by the controller, a non-zero steady state error may exist.

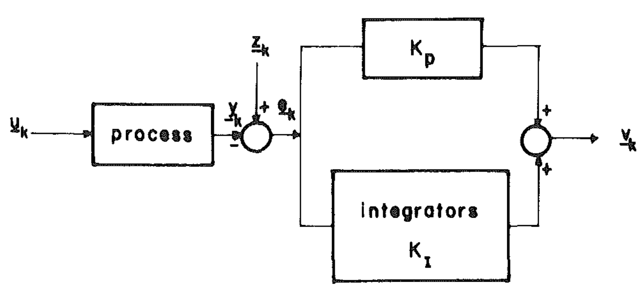

Fig. 1 Addition of error amplifiers and integrators

If unknown nonzero mean external disturbances are present, their effect may be eliminated by addition of a set of integrators on the output error. On the other hand, the transient response can be set by addition of error amplifiers. This results in the scheme shown in Fig. 1.

Let us now define an augmented system whose state vector is formed by $\mathbf{x}$ and $\mathbf{v}$, the input vector is $\mathbf{u}$ and the output vector is $\mathbf{v}$. Notice that, if $\mathbf{v}$ is taken as the regulated (constant) output of the augmented system, the error signal must be zero at steady state.

Let us consider a process described by the equations (1) and (2). The output error is given by equation (4). The relation between $\mathbf{e}_k$ and $\mathbf{v}_k$ is as follows:

$$\mathbf{v}_{k+1} = \mathbf{v}_k + (K_p + K_I)\,\mathbf{e}_{k+1} - K_p \mathbf{e}_k \tag{20}$$

where K_p and K_I are weighting factors of the proportional and integral actions, respectively.

From (1), (2), (4), and (20) we can obtain:

$$\mathbf{v}_{k+1} = \mathbf{v}_k + K_p C\mathbf{x}_k - (K_p+K_I)\,C\mathbf{a}(\mathbf{x}_k) + (K_p+K_I)\,\mathbf{z}_{k+1} - K_p\mathbf{z}_k - (K_p+K_I)\,CB(\mathbf{x}_k)\,\mathbf{u}_k \tag{21}$$

The equations representing the augmented system are the following:

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{a}}(\hat{\mathbf{x}}_k) + \hat{B}(\hat{\mathbf{x}}_k)\,\mathbf{u}_k \tag{22}$$

$$\hat{\mathbf{y}}_k = \hat{C}\hat{\mathbf{x}}_k \tag{23}$$

$$\hat{\mathbf{e}}_k = \hat{\mathbf{z}}_k - \hat{\mathbf{y}}_k \tag{24}$$

where $\hat{\mathbf{x}}_k = \begin{pmatrix} \mathbf{x}_k \\ \mathbf{v}_k \end{pmatrix}$; $\hat{\mathbf{a}}(\hat{\mathbf{x}}_k) =$

$$\begin{pmatrix} \mathbf{a}(\mathbf{x}_k) \\ \mathbf{v}_k + K_p C\mathbf{x}_k - (K_p+K_I)\,C\mathbf{a}(\mathbf{x}_k) + (K_p+K_I)\,\mathbf{z}_{k+1} - K_p\mathbf{z}_k \end{pmatrix}$$

$$\hat{B}(\hat{\mathbf{x}}_k) = \begin{pmatrix} B(\mathbf{x}_k) \\ -(K_p+K_I)\,CB(\mathbf{x}_k) \end{pmatrix} ;\ \hat{\mathbf{y}}_k = \mathbf{v}_k ;\ \hat{C} = (0\ I)$$

and $\hat{\mathbf{z}}_k$ is the reference vector for $\mathbf{v}_k$. As it has been mentioned above, $\hat{\mathbf{z}}_k$ must be constant ($\hat{\mathbf{z}}_k = \hat{\mathbf{z}}$) and it can be chosen arbitrarily.

Considering a performance index given by (3), where $\mathbf{e}_k$ is replaced by $\hat{\mathbf{e}}_k$, and from the above equations, the optimal control can be expressed as:

$$\mathbf{u}_k^* = \frac{(CB(\mathbf{x}_k))^{-1}}{K_p+K_I}\,\big(K_p\mathbf{y}_k - (K_p+K_I)\,C\mathbf{a}(\mathbf{x}_k) - (I+\hat{M})\,(\hat{\mathbf{z}}-\mathbf{v}_k) + (K_p+K_I)\,\mathbf{z}_{k+1} - K_p\mathbf{z}_k\big) \tag{25}$$

where $\hat{M} = \frac{1}{2} M - Q - I$. In this case, the observation time is considered to be infinite.

Application to a Fermentation Process

The control law obtained (25) has been applied to a simulated continuous culture fermentation process. This

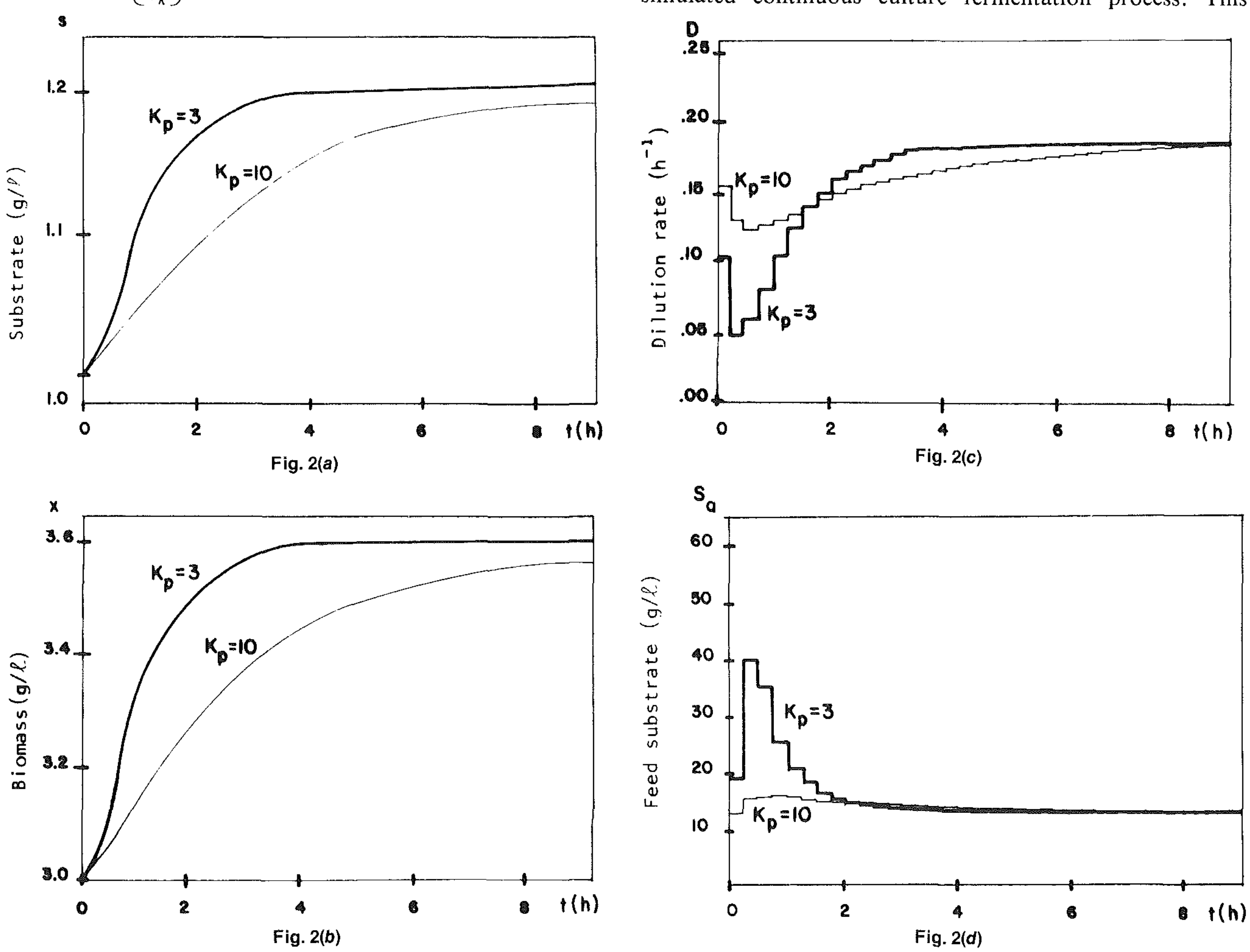

Fig. 2 Output and control time responses for a change in the reference vector

process produces single cell protein from yeast grown on methanol. More details about the process are given in [10].

Process Model. The process is described by the following equations:

$$\dot{x} = (\mu - D)x \tag{26}$$

$$\dot{s} = D(S_a - s) - \frac{\mu}{R}x \tag{27}$$

$$\mu = \mu_m \frac{s}{K+s} \tag{28}$$

where x and s are the biomass and substrate concentrations (g/l), respectively, D is the dilution rate (h^{-1}), S_a is the feed substrate concentration and μ_m, R, and K are characteristic parameters of the process.

In an industrial continuous culture fermentation process to produce biomass, one of the main requirements is to operate it at optimal productivity conditions. Normally, the control variables are kept constant at the maximum productivity point. For a continuous culture fermentation process, productivity is defined as:

$$P = Dx \tag{29}$$

where P is given in ($g/l.h$)

The optimal substrate concentration, that is, the value of s which maximize P, is given by: [12]

$$s^* = (K)^{1/2}\{(K+S_a)^{1/2} - (K)^{1/2}\} \tag{30}$$

s^* is function only of K and S_a and is independent of μ_m and R. Most of the problems in a fermentation process are reflected in μ_m and R variations; then we can make a static optimization by controlling s at its optimal value s^*. S_a can be measured from the process and K is a constant process parameter.

Another important requirement in an industrial fermentation process is to maintain a certain biomass rate production. Then it is necessary to regulate the biomass concentration x at a specified value x^*.

The two requirements mentioned above make necessary to design a multivariable control law. From the physical point of view the selection of D and S_a as the control variables is adequate because of the facility to handle them.

Discrete Model. We have applied the forward Euler algorithm to discretize the analog process model (26)–(28) and obtain the discrete model needed to calculate the control law. In spite of the existence of better choices, we have found this algorithm well adapted to this problem, as it will be shown later.

Applying this algorithm to the analog process model we obtain the following:

$$\mathbf{x}_{k+1} = \mathbf{a}(\mathbf{x}_k) + B(\mathbf{x}_k)\mathbf{u}_k \tag{31}$$

$$\mathbf{y}_k = C\mathbf{x}_k \tag{32}$$

where $\mathbf{x}_k = (x_k\, s_k)^T$; $\mathbf{u}_k = (D_k\, D_k \cdot S_{ak})^T$

$$\mathbf{a}(\mathbf{x}_k) = \begin{pmatrix} x_k + H\mu_k x_k \\ -\dfrac{H}{R}\mu_k x_k + s_k \end{pmatrix}; B(\mathbf{x}_k) = H\begin{pmatrix} -x_k & 0 \\ -s_k & 1 \end{pmatrix}$$

$$C = I; \mu_k = \mu_m \frac{s_k}{K+s_k}$$

H is the sampled period. As it may be seen, $\det(CB(\mathbf{x}_k)) = -Hx_k = 0$ if $x_k = 0$. An analysis of the physical process

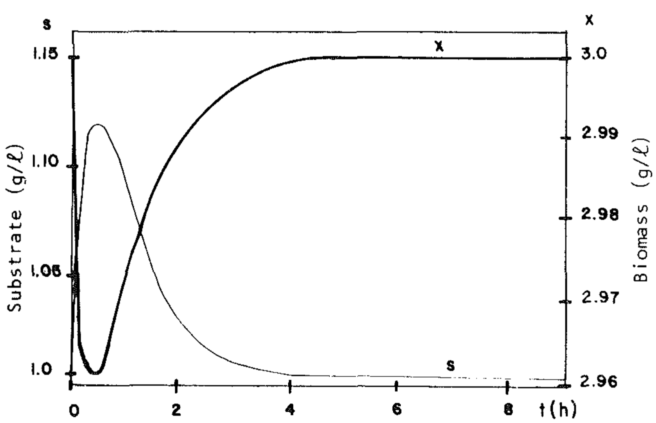

Fig. 3(a)

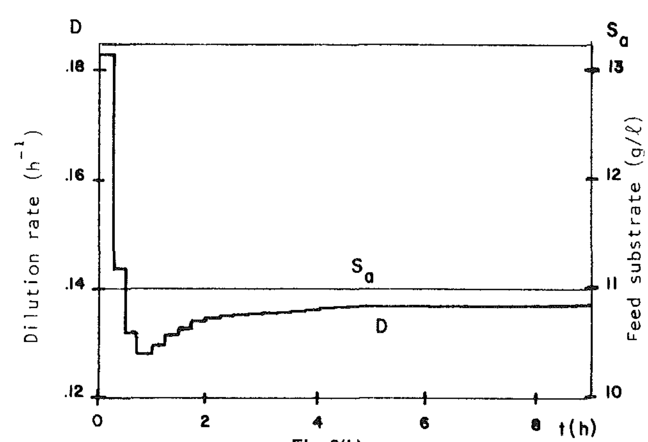

Fig. 3(b)

Fig. 3 Output and control time responses for a disturbance of 25 percent on the dilution rate

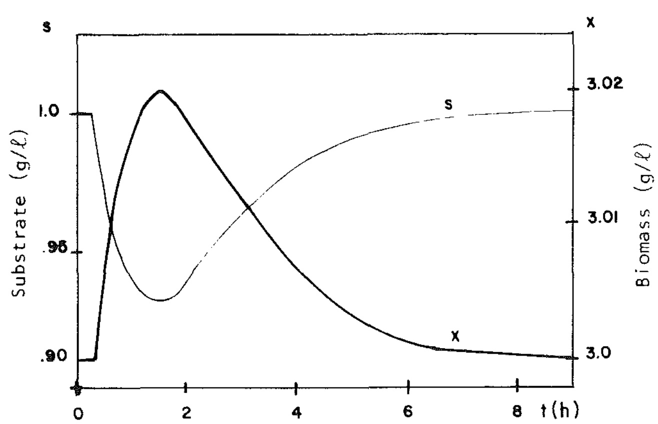

Fig. 4(a)

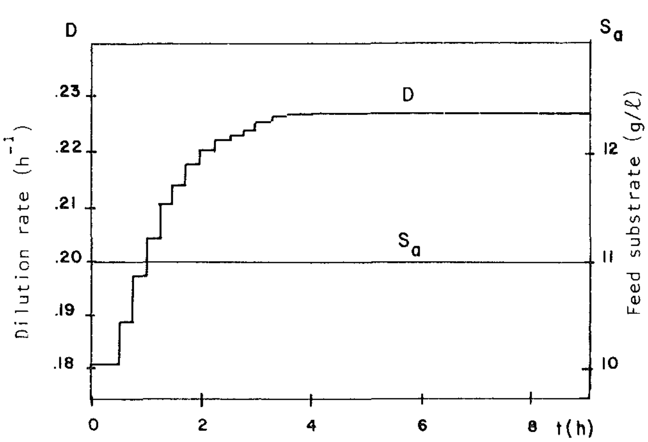

Fig. 4(b)

Fig. 4 System response for a change of 25 percent in μ_m

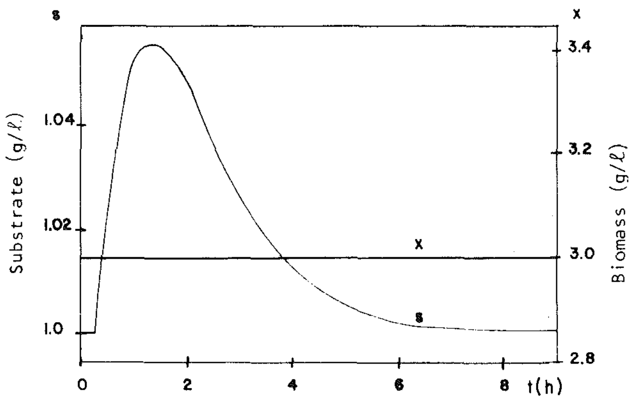

Fig. 5(*a*)

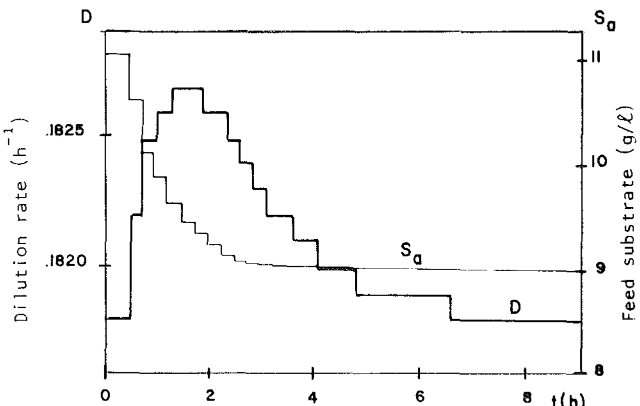

Fig. 5(*b*)

Fig. 5 System response for a change of 25 percent in R

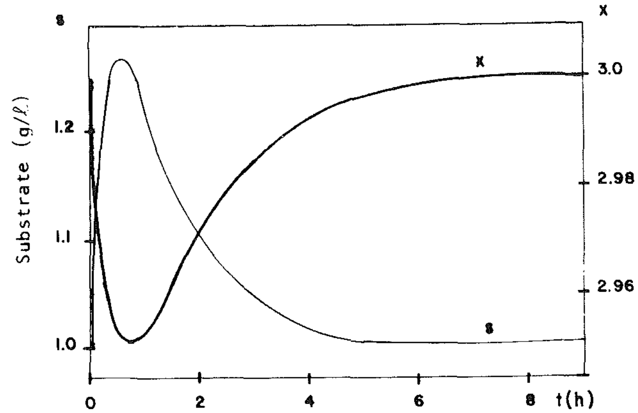

Fig. 6(*a*)

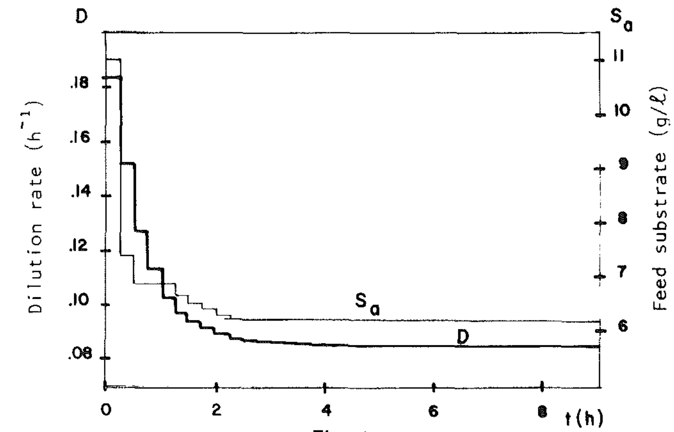

Fig. 6(*b*)

Fig. 6 System response for changes in D (25 percent), S_a (25 percent), R (25 percent), μ_m ($-$25 percent) and K(50 percent)

shows that this situation seldom occurs in practice, so we can say that matrix $CB(\mathbf{x}_k)$ is nonsingular and the control law will be bounded.

Control Law. We will show the case when regulation is desired, that is, the output reference is a constant vector: $\mathbf{z}_k = (x^*, s^*)^T$. Also, we assume that $\hat{\mathbf{z}} = \mathbf{z}_k$.

Application of equation (25) gives the following expressions for D_k^* and S_{ak}^* ($S_{ak}^* = (D_k \cdot S_{ak})^*/D_k^*$):

$$D_k^* = \frac{1}{K_p + K_I}\left(\frac{1-K_I}{Hx_k}x^* + \frac{K_I}{H} + (K_p + K_I)\,\mu_{ck} - \frac{1}{Hx_k}v_{xk} + \right.$$
$$\left. + \frac{\tilde{K}_x}{Hx_k}(x^* - v_{xk})\right) \qquad (33)$$

$$S_{ak}^* = s_k - \frac{1}{(K_p + K_I)HD_k^*}\left((1-K_I)s^* + K_I s_k\right.$$
$$- \frac{(K_p + K_I)H}{R_c}\mu_{ck}x_k -$$
$$\left. - v_{sk} + \tilde{K}_s(s^* - v_{sk})\right) \qquad (34)$$

where $\mu_{ck} = \mu_{mc}\, s_k/(K_c + s_k)$. The subindex c is associated to the controller parameters. v_{xk} and v_{sk} are the added state components when the PI block is included. $\tilde{K}_x$ and $\tilde{K}_s$ are given by expressions of the form $\tilde{K}_i = \frac{1}{2}\sqrt{q_i(q_i+4)} - \frac{1}{2}\ q_i - 1$ and q_i is the ith component of the diagonal of the Q matrix.

Simulation Results. The control law (33)-(34) was applied to the process whose characteristics are given by $\mu_m = 0.2$ (h^{-1}), $R = 0.3$, $K = 0.1$ (g/l).

The reference was fixed to $\mathbf{z} = (x^*, s^*)^T = (3, 1)^T$. Also, we have chosen $Q = I$ and $K_I = 1$.

Figure 2 shows what happens when the reference vector changes by a prespecified amount (x^*: 3 → 3.6; s^*: 1 → 1.2). The output and control responses are shown for two values of K_p: 3 and 10. Notice that if the proportional action increases, the system becomes slower and the control vector has smaller excursions from one sampling time to the next.

For the remaining experiments, K_p has been chosen equal to 3. Figure 3 shows the output and control evolution in time when a step disturbance of 0.045 (25 percent) on the dilution rate has been introduced. This is a typical event occurring in the real process. Both output variables return to their nominal values in about 4 hours after the disturbance has been produced.

In the aforementioned experiments it has been assumed that the control parameters (μ_{mc}, R_c, and K_c) are equal to the process parameters (μ_m, R, and K). If changes in the process parameters occur and are undetected by the controller, responses shown in Figs. 4, 5, and 6 are obtained.

Figure 4 shows what happens when a change in μ_m is produced after the system has attained a steady state. μ_m has changed from 0.2 to 0.25 (25 percent) following a first order system dynamics with a constant time of approximately one hour. This is a more realistic parametric change than a step disturbance. Moreover, the dynamics of the simulated parametric change is rather fast, so it could be expected a better performance if the perturbation has a slower dynamics. A good regulation of the process output can be observed.

Figure 5 shows the results obtained when R is changed from 0.3 to 0.375 (25 percent) on the same conditions mentioned for μ_m. Only the substrate concentration is disturbed in this case, and it returns to its nominal value in about five hours.

Finally, in Fig. 6 it is shown what happens when disturbances are presented on D(25 percent), S_a(25 percent), R(25 percent), μ_m($-$25 percent) and K(50 percent). As it can be observed, in spite of the strong disturbances introduced into the process, the controller is able to effect a very good regulation.

Conclusions

In this paper a technique for the synthesis of control laws of a certain class of discrete nonlinear systems has been proposed. The use of optimal control theory and an adequate selection of the performance index allowed us to obtain a control law which is an explicit function of the state vector, so a closed loop scheme is obtained. By appropriate selection of the penalty function of the error vector, it is possible to change easily the system dynamics or to bound the maximum magnitude of the control.

On the other hand, the addition of a block of error amplifiers-integrators has reduced the steady state error to zero when internal or external disturbances are present.

The obtained control law has been applied to a simulated continuous culture fermentation process with good results. However, since this approach assumes that the state vector is accessible and in most fermentation processes there is no access to the whole state vector, it is necessary to build an observer whose design is still an unsolved problem for the general case of discrete nonlinear systems.

In the experiments presented in this paper we only try to show the robustness of the scheme to regulate s and x. It must be noted that S_a has been taken as a control input, so when S_a varies, s^* must follow these variations by means of equation (30) in order to always be operating at the optimal productivity conditions.

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APPENDIX A

Equations (13) and (14) can be expressed as:

$$\mathbf{w}_{k+1} = A_d \mathbf{w}_k \tag{35}$$

where

$$\mathbf{w} = \begin{pmatrix} \mathbf{e} \\ \mathbf{p} \end{pmatrix} \text{ and } A_d = \begin{pmatrix} I+Q & -\tfrac{1}{2}I \\ -2Q & I \end{pmatrix} \tag{36}$$

A_d is a matrix whose determinant is equal to one, so A_d^{-l} exists for every positive integer l.

The solution of (35) is:

$$\mathbf{w}_k = A^{k-k_0}\mathbf{w}_0 \tag{37}$$

which is valid for every k and k_0. Let $k = N$ and $k_0 = k$, then

$$\mathbf{w}_N = A_d^{N-k}\mathbf{w}_k$$

thus

$$\mathbf{w}_k = A_d^{k-N}\mathbf{w}_N = A_d^{-l}\mathbf{w}_N \tag{38}$$

where $l = N-k$.
From (36) and (38):

$$\mathbf{e}_k = A_{11k}\mathbf{e}_N + A_{12k}\mathbf{p}_N \tag{39}$$

$$\mathbf{p}_k = A_{21k}\mathbf{e}_N + A_{22k}\mathbf{p}_N \tag{40}$$

where A_{ijk} are submatrix of A_d^{-l} of dimension mxm.

$$A_d^{k-N} = \begin{pmatrix} A_{11k} & A_{12k} \\ A_{21k} & A_{22k} \end{pmatrix}$$

From the boundary conditions we have $\mathbf{p}_N = \mathbf{0}$. Then, from (39) and (40):

$$\mathbf{p}_k = A_{21k}A_{11k}^{-1}\mathbf{e}_k \triangleq M_k\mathbf{e}_k \tag{41}$$

APPENDIX B

Since Q was assumed to be diagonal and it was shown that M is also diagonal, namely:

$$Q = \mathrm{diag}(q^i) \text{ and } M = \mathrm{diag}(m^i)$$

In this case, analysis of equation (16) can be reduced to the analysis of the following scalar equation:

$$(1+q)m_{k+1} - \tfrac{1}{2}m_{k+1}m_k - m_k + 2q = 0 \tag{42}$$

where the superindex (i) has been omitted.
Then

$$m_k = \frac{(1+q)m_{k+1} + 2q}{1 + \tfrac{1}{2}m_{k+1}} \tag{43}$$

Since m_k is evaluated from $k=N-1$ to $k=0$, (43) is calculated as follows:

$$m_{N-1} = \frac{(1+q)m_N + 2q}{1+\tfrac{1}{2}m_N} \quad \ldots \quad m_o = \frac{(1+q)m_1 + 2q}{1+\tfrac{1}{2}m_1}$$

This is equivalent to have a system like the following:

$$m_{j+1} = \frac{(1+q)m_j + 2q}{1+\tfrac{1}{2}m_j}\; j = 0, \ldots, N-1, m_0 = 0 \tag{44}$$

When $N \to \infty$ ($j \to \infty$), (44) can be seen as the problem of evaluating the roots of the function:

$$f(m) = m - \frac{(1+q)m + 2q}{1+\tfrac{1}{2}m} = 0$$

by means of the successive approximation method. It is known that this method converges when

$$\left| \frac{dm_{j+1}}{dm_j} \right| < 1$$

Then, from (44):

$$\left| \frac{dm_{j+1}}{dm_j} \right| = \frac{1}{(1+\tfrac{1}{2}m_j)^2} < 1 \text{ for } m_j > 0 \text{ or } m_j < -4$$

If we start from $m_o = 0$ then $m_1 = 2q > 0$. On the other hand, if $j \to \infty$ the equilibrium points of (44) are:

$$m_\infty^+ = q + \sqrt{q(q+4)} > 0$$

and

$$m_\infty^- = q - \sqrt{q(q+4)} < 0$$

System (44) then converges, starting from $m_1 = 2q$, to m_∞^+.