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Slow Viscous Flow in a Syringe[1]

The slow viscous flow in a syringe is modeled by the quasi-steady axisymmetric Stokes equation with a point sink for the needle hole. The governing equations are approximated using nonstandard finite difference formulas optimized for the boundary conditions, and solved numerically using a SOR technique. Streamlines and pressure profiles are computed for a variety of syringe configurations.

1 Introduction

The syringe is probably the most important and most indispensable instrument in medicine. The syringe consists of three parts, the needle, the glass cylinder and the plunger. As the plunger is squeezed, the fluid inside the glass cylinder is ejected through the hollow needle.

The present paper is concerned with the low Reynolds number flow inside a syringe. This situation occurs in the slow controlled intravenous infusion of fluids and drugs in the hospital or laboratory. For example, typical values for the Reynolds number in indicator-dilution experiments are Re = 0.07 (50cc syringe at 1cc/min) or Re = 0.015 (10cc syringe at 0.1 cc/min). The infusion rates are kept constant by a calibrated infusion pump.

In Fig. 1 the plunger is moving to the left in a circular cylinder. Fluid is forced through a hollow needle of diameter usually less than 3 percent of the diameter of the cylinder. Due to the geometry the fluid dynamics can be separated into two parts: the flow in the needle and the flow in the cylinder. The flow in the needle (orifice) has been solved by Dagan, Weinbaum and Pfeffer [1], who concluded the velocity profile is dominantly Poiseuille, with end effects limited to a distance of only 1/4 the inner needle diameter. In what follows we shall study the cylinder region, with the needle hole approximated by a sink.

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2 Formulation

Let the origin of cylindrical coordinates (r, z) be situated at the sink. The cylinder is at $z' = 0$, $r' \leq a$ and $z' \geq 0$, $r' = a$. At the time of investigation, the plunger is at $z' = l$, $r' \leq a$ traveling with velocity U. Let v' and w' be the velocity components in the r' and z' directions, respectively.

Since the Reynolds number is much less than one, the flow is governed by the quasi-steady axisymmetric Stokes equation

$$\left(\frac{\partial^2}{\partial r'^2}-\frac{1}{r'}\frac{\partial}{\partial r'}+\frac{\partial^2}{\partial z'^2}\right)^2\psi'=0 \tag{1}$$

where ψ' is a stream function defined by

$$v'=-\frac{1}{r'}\frac{\partial\psi'}{\partial z'},\quad w'=\frac{1}{r'}\frac{\partial\psi'}{\partial r'} \tag{2}$$

The boundary conditions for the domain $0 \leq z' \leq l$, $0 \leq r' \leq a$ are

$$\begin{aligned} \text{on } z'=0,&\quad v'=w'=0, \text{ except at the sink at the origin,}\\ z'=l,&\quad v'=0,\ w'=-U\\ r'=a,&\quad v'=w'=0\\ r'=0,&\quad v'=\frac{\partial w'}{\partial r'}=0 \end{aligned} \tag{3}$$

We normalize all lengths by a, the velocities by U, the stream function by $-\frac{1}{2}a^2U$ and drop primes. The governing equations become

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)^2\psi=0$$

$$\text{on } z=0,\qquad \psi=1,\ \frac{\partial\psi}{\partial z}=0, \text{ except at the origin,}$$

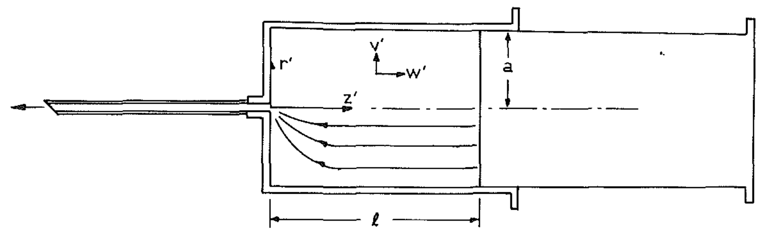

Fig. 1 Syringe geometry and coordinate system

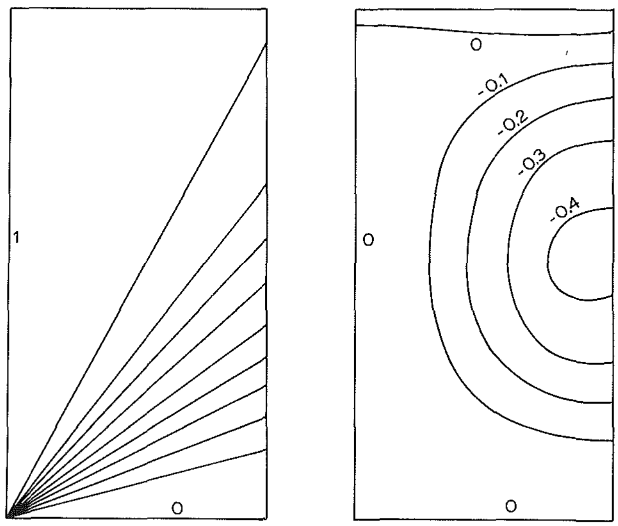

Fig. 2 ψ_0 ($\Delta\psi_0 = 0.1$) and χ lines for $\beta = 0.50$

$$
\begin{aligned}
z=\beta\equiv\frac{l}{a}, \quad & \frac{\partial\psi}{\partial z}=0,\ \frac{1}{r}\frac{\partial\psi}{\partial r}=2, \\
r=0, \quad & \psi=0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=0, \\
r=1, \quad & \psi=1,\ \frac{\partial\psi}{\partial r}=0
\end{aligned}
\tag{4}
$$

β is the instantaneous ratio of the axial length of the cylinder to its radius. There exists a weak singularity on the circle $z=\beta$, $r=1$ where the velocity w is discontinuous and a strong singularity at the origin where ψ is discontinuous. These singularities pose immediate problems in solving for the stream function directly. To remove the strong singularity analytically, we set

$$\psi(r,z)=\psi_0(r,z)+\chi(r,z) \tag{5}$$

where ψ_0 represents the Stokes flow due to a sink on an infinite solid plane [2]

$$\psi_0=1-z^3(r^2+z^2)^{-3/2} \tag{6}$$

Thus

$$\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)^2\chi=0 \tag{7}$$

The boundary conditions are

$$\text{on } z=0, \quad \chi=0,\ \frac{\partial\chi}{\partial z}=0 \tag{8}$$

$$
\begin{aligned}
\text{on } z=\beta, \quad & \chi=r^2-1+\beta^3(r^2+\beta^2)^{-3/2}, \\
& \frac{\partial\chi}{\partial z}=3\beta^2r^2(r^2+\beta^2)^{-5/2}
\end{aligned}
\tag{9}
$$

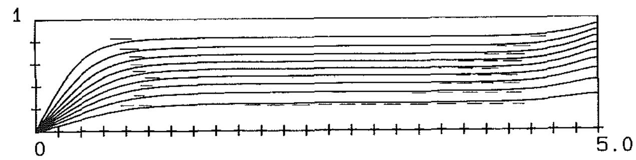

Fig. 3 ψ streamlines for $\beta = 5.0$ (solid) and Poiseuille streamlines (dashed)

$$\text{on } r=0, \quad \chi=0,\ \lim_{r\to 0}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial r}\right)=0 \tag{10}$$

$$
\begin{aligned}
\text{on } r=1, \quad & \chi=z^3(1+z^2)^{-3/2}, \\
& \frac{\partial\chi}{\partial r}=-3z^3(1+z^2)^{-5/2}
\end{aligned}
\tag{11}
$$

Although equations (7)–(11) are regular at the origin, a discontinuity in the derivative of χ still exists on $z = \beta$, $r = 1$. The boundary conditions are complicated enough such that an analytical solution is not worthwhile. In what follows we shall use numerical means to solve the problem.

3 Description of the Numerical Method

The partial differential equation for χ is solved numerically using finite difference methods [3], which produce an approximation for χ at each grid point. In the middle away from all boundaries, standard difference equations are used to approximate the partial derivatives. Near the boundaries, different nonstandard formulas are needed. These new difference formulas, using both values and partials of χ on the boundaries, were derived using the symbolic manipulation package SMP [4]. The derivation technique is standard in numerical analysis, but because of the high order partials and boundary derivative information involved, the mathematical derivation is extremely tedious and prone to error. That is why SMP was used after an initial manual attempt. To save the reader the effort involved in reproducing these formulas, all the finite difference approximations used are given in an appendix.

The finite difference approximation results in a linear system of equations $Ax = b$, with dimension 2500 for a modest 50×50 grid. Since the boundary conditions are not symmetric and difference formulas using known derivatives on the boundaries are employed, the coefficient matrix A is not symmetric. The matrix A is nearly diagonally dominant, however, suggesting that successive over-relaxation (SOR) may work. SOR does not, in fact, converge for all choices of the relaxation parameter ω anywhere between 0 and 2 (A is not symmetric and positive definite). Nevertheless, excellent convergence for SOR was achieved with $\omega = 1.6$.

4 The Streamlines

After χ (r, z) is computed, it is added to ψ_0 to obtain the stream function ψ. Figure 2 shows the solution χ and ψ_0 for $\beta = 0.5$. Figure 3 shows the sum, the stream function ψ, for $\beta = 5$ or when the plunger is 5 radius' distance from the sink. For such large β values, the fluid flow shows 3 distinct

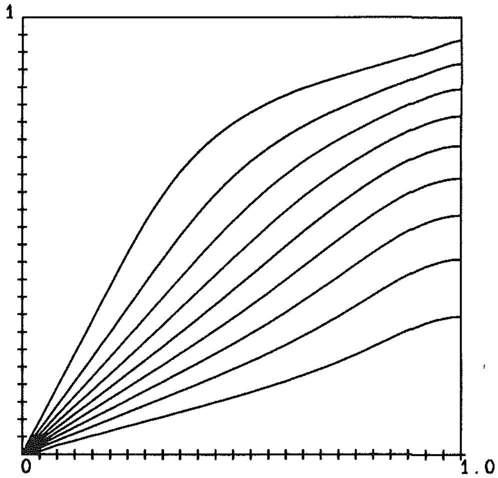

Fig. 4 ψ streamlines for $\beta = 1.0$, ($\Delta\psi = 0.1$)

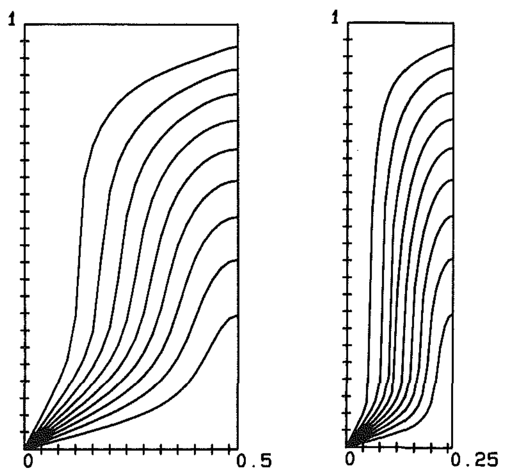

Fig. 5 ψ streamlines for $\beta = 0.5$ and $\beta = 0.25$ ($\Delta\psi = 0.1$)

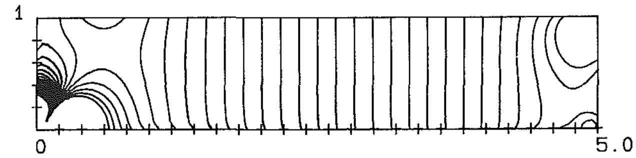

Fig. 6 Constant pressure lines for $\beta = 5.0$ ($\Delta p = 3$)

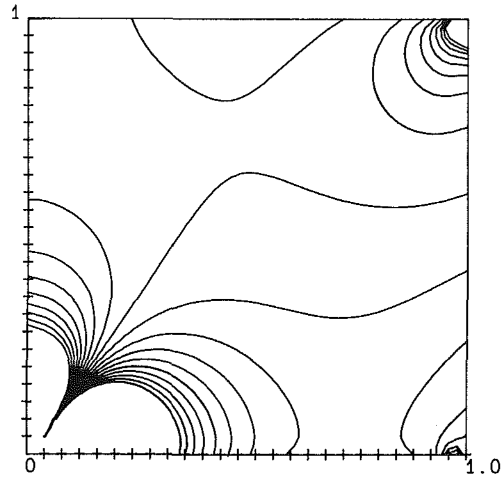

Fig. 7 Constant pressure lines for $\beta = 1.0$ ($\Delta p = 5$)

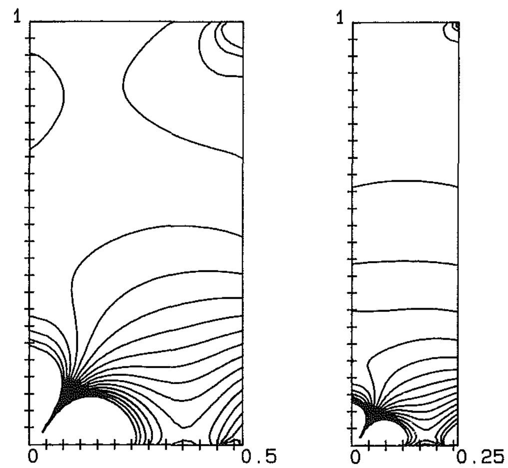

Fig. 8 Constant pressure lines for $\beta = 0.50$ ($\Delta p = 30$) and $\beta = 0.25$ ($\Delta p = 200$)

regions: the plunger region at right, the middle Poiseuille region and the sink region at left. The flow in the plunger region is similar to the entry flow into a circular tube [5], where the uniform velocity profile is changed into the parabolic profile. Our results agree with Lew and Fung's [5] analytical result and Wagner's [6] numerical result for the infinite tube at zero Reynolds number. The entrance length is about one diameter. The middle regions show almost Poiseuille parallel flow. For comparison, the dashed lines are the Poiseuille streamlines

$$\psi = 2r^2 - r^4 \tag{12}$$

In the sink region the fluid turns and flows into the origin. The streamlines are dominated by the Stokes sink represented by ψ_0. There are no regions of recirculation although the velocity is quite low in the corner at $r = 1, z = 0$. If a Moffatt-type eddy [7] does exist, its strength is estimated to be two thousandths of the main flow, and thus undetectable. Figures 4, 5 show the changes in streamline patterns as the plunger moves in closer. The three distinct regions disappear due to mutual interaction. At $\beta = 0.25$ the velocity becomes almost radially inwards in contrast to the mainly axial Poiseuille flow at $\beta = 5$.

5 The Pressure Distribution

The pressure normalized by $U\mu/a$ is numerically computed by finite differences from the normalized Stokes formula:

$$\begin{aligned} \frac{\partial p}{\partial r} &= -\frac{1}{r}\left(\frac{\partial^3\psi}{\partial z\partial r^2} + \frac{\partial^3\psi}{\partial z^3}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial z\partial r}, \\ \frac{\partial p}{\partial z} &= \frac{1}{r}\left(\frac{\partial^3\psi}{\partial r^3} + \frac{\partial^3\psi}{\partial r\partial z^2}\right) - \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^3}\frac{\partial\psi}{\partial r} \end{aligned} \tag{13}$$

The constant pressure lines are shown in Figs. 6–8. Regions in the lower left and upper right corners are not contoured because the isobars are too dense. We see a high pressure region exists near the corner at $z = \beta$, $r = 1$ while near the center of the plunger ($z = \beta$, $r = 0$) the pressure shows a local high for large β and a local low for $\beta < 1$. In the vicinity of the sink, the pressure is high on the surface $z = 0$ and is low on the axis $r = 0$. The pressure pattern near the sink persists for all β and agrees with the Stokes sink pressure distribution [2]. Notice that for $\beta = 5$ there exists a middle Poiseuille region where pressure is independent of radial distance and the pressure gradient is uniform axially.

6 Discussion

The eigenfunction expansion method of reference [1] is unsuitable for the present problem. This is due to the fact that the needle hole is so small compared to the cylinder diameter that the convergence of the infinite series becomes too slow. If the singularity is removed by subtraction as in this paper, then eigenfunctions cease to exist.

The Stokes flow in a syringe is now solved. Although the streamlines do not show recirculation, the flow in a syringe is not uniform. The pressure distribution is more complicated. Regions of local high and local low pressure may cause particulate matter in the fluid to migrate, resulting in inhomogeneity of the injectate.

Stokes flow is not valid in the immediate neighborhood of the sink where velocities increase indefinitely. To eliminate the singularity, one must consider a hole of finite radius at the origin. Dagan, Weinbaum and Pfeffer [1] considered a finite cylindrical hole in an infinite plate and computed the streamlines and pressure distribution. They found the effect of the hole is limited to one hole radius in the vicinity of the junction. We conclude that the shape and size of the hole has little effect on the flow in the syringe presented in this paper.

What is the force on the plunger? Due to the difference in size, almost all hydrodynamic resistance comes from the flow through the needle. Reference [1] showed the pressure difference through a finite sized hole of radius a_h and length L_h is a linear sum of Poiseuille resistance and Sampson resistance [8]. Thus the force on the plunger is a constant

$$F=\pi a^2\left(\frac{8L_h}{\pi a_h}+3\right) \tag{14}$$

Of course, this formula would not be valid towards the end of the injection when the plunger is several needle radii from the hole.

References

1 Dagan, Z., Weinbaum, S., and Pfeffer, R., "An Infinite-Series Solution for the Creeping Motion Through an Orifice of Finite Length," *Journal of Fluid Mechanics*, Vol. 115, 1982, pp. 505–523.

2 Happel, J., and Brenner, H., *Low Reynolds Number Hydrodynamics*, Chap. 4, Prentice Hall, N.J., 1965.

3 Smith, G. D., *Numerical Solution of Partial Differential Equations*, Oxford University Press, N.Y., 1965.

4 Wolfram, S., "Symbolic Mathematical Computation," *Comm. ACM*, Vol. 28, 1985, pp. 390–395.

5 Lew, H. S., and Fung, Y. C., "On the Low-Reynolds Number Entry Flow Into a Circular Cylindrical Tube," *Journal of Biomechanics*, Vol. 2, 1969, pp. 105–119.

6 Wagner, M. H., "Developing Flow in Circular Conduits: Transition From Plug Flow to Tube Flow," *Journal of Fluid Mechanics*, Vol. 72, 1975, pp. 257–268.

7 Moffatt, H. K., "Viscous and Resistive Eddies Near a Sharp Corner," *Journal of Fluid Mechanics*, Vol. 18, 1964, pp. 1–18.

8 Sampson, R. A., "On Stoke's Current Function," *Philosophical Transactions of the Royal Society of London*, Vol. A182, 1891, pp. 449–518.

APPENDIX

Finite Difference Formulas

For completeness, all the finite difference approximations used are listed here. These approximations are optimal in the sense of using all derivative information available near the boundaries. Except for the first few listed, these are all nonstandard and had to be derived using SMP [4].

For the first partial with respect to r at all points

$$\frac{\partial\chi}{\partial r}(z,r)=\frac{\chi(z,r+h)-\chi(z,r-h)}{2h}$$

For the second partial with respect to r at all points

$$\frac{\partial^2\chi}{\partial r^2}(z,r)=\frac{\chi(z,r+h)-2\chi(z,r)+\chi(z,r-h)}{h^2}$$

For the third partial with respect to r in the middle away from the boundaries $r = 0$ and $r = 1$

$$\frac{\partial^3\chi}{\partial r^3}(z,r)=\frac{\chi(z,r+2h)-2\chi(z,r+h)+2\chi(z,r-h)-\chi(z,r-2h)}{2h^3}$$

Near the boundary $r = 0$ (i.e., at points (z, h))

$$\frac{\partial^3\chi}{\partial r^3}(z,r)=\frac{-3\chi(z,r)+2h\,\frac{\partial\chi}{\partial r}(z,r-h)+\frac{8}{3}\chi(z,r-h)+\frac{1}{3}\chi(z,r+2h)}{h^3}$$

Near the boundary $r = 1$, (i.e., at points $(z, 1-h)$)

$$\frac{\partial^3\chi}{\partial r^3}(z,r)=\frac{3\chi(z,r)+2h\,\frac{\partial\chi}{\partial r}(z,r+h)-\frac{8}{3}\chi(z,r+h)-\frac{1}{3}\chi(z,r-2h)}{h^3}$$

For the fourth partial with respect to r away from the boundaries $r = 0$ and $r = 1$

$$\frac{\partial^4\chi}{\partial r^4}(z,r)=\frac{\chi(z,r-2h)-4\chi(z,r-h)+6\chi(z,r)-4\chi(z,r+h)+\chi(z,r+2h)}{h^4}$$

Near $r = 0$,

$$\frac{\partial^4 \chi}{\partial r^4}(z,r) = \frac{16\chi(z,r) - \frac{1}{4}\chi(z,r+3h) + \frac{8}{3}\chi(z,r+2h) - 9\chi(z,r+h) - \frac{113}{12}\chi(z,r-h) - 5h\frac{\partial \chi}{\partial r}(z,r-h)}{h^4}$$

Near $r = 1$,

$$\frac{\partial^4 \chi}{\partial r^4}(z,r) = \frac{16\chi(z,r) - \frac{1}{4}\chi(z,r-3h) + \frac{8}{3}\chi(z,r-2h) - 9\chi(z,r-h) - \frac{113}{12}\chi(z,r+h) + 5h\frac{\partial \chi}{\partial r}(z,r+h)}{h^4}$$

For the fourth partial with respect to z away from the boundaries $z = 0$ and $z = \beta$

$$\frac{\partial^4 \chi}{\partial z^4}(z,r) = \frac{\chi(z-2k,r) - 4\chi(z-k,r) + 6\chi(z,r) - 4\chi(z+k,r) + \chi(z+2k,r)}{k^4}.$$

Near $z = 0$,

$$\frac{\partial^4 \chi}{\partial z^4}(z,r) = \frac{16\chi(z,r) - \frac{1}{4}\chi(z-3k,r) + \frac{8}{3}\chi(z-2k,r) - 9\chi(z-k,r) - \frac{113}{12}\chi(z+k,r) + 5k\frac{\partial \chi}{\partial z}(z+k,r)}{k^4}.$$

Near $z = \beta$

$$\frac{\partial^4 \chi}{\partial z^4}(z,r) = \frac{16\chi(z,r) - \frac{1}{4}\chi(z-3k,r) + \frac{8}{3}\chi(z-2k,r) - 9\chi(z-k,r) - \frac{113}{12}\chi(z+k,r) + 5k\frac{\partial \chi}{\partial z}(z+k,r)}{k^4}.$$

For the mixed fourth partial in the middle away from all boundaries

$$\begin{aligned}12h^2k^2 \frac{\partial^4 \chi}{\partial r^2 \partial z^2}(z,r) = \\ &2(30\chi(z,r) + \chi(z,r-2h) - 16\chi(z,r-h) - 16\chi(z,r+h) + \chi(z,r+2h)) \\ &-(30\chi(z-k,r) + \chi(z-k,r-2h) - 16\chi(z-k,r-h) - 16\chi(z-k,r+h) + \chi(z-k,r+2h)) \\ &-(30\chi(z+k,r) + \chi(z+k,r-2h) - 16\chi(z+k,r-h) - 16\chi(z+k,r+h) + \chi(z+k,r+2h))\end{aligned}$$

Near $z = 0$,

$$\begin{aligned}h^2k^2 \frac{\partial^4 \chi}{\partial r^2 \partial z^2}(z,r) = \\ &-2\Big(\frac{-10}{3}\chi(z,r) + \frac{257}{144}\chi(z-k,r) + \frac{7}{4}\chi(z+k,r) - \frac{2}{9}\chi(z+2k,r) \\ &\quad + \frac{1}{48}\chi(z+3k,r) + \frac{5k}{12}\frac{\partial \chi}{\partial z}(z-k,r)\Big) \\ &+\Big(\frac{-10}{3}\chi(z,r-h) + \frac{257}{144}\chi(z-k,r-h) + \frac{7}{4}\chi(z+k,r-h) - \frac{2}{9}\chi(z+2k,r-h) \\ &\quad + \frac{1}{48}\chi(z+3k,r-h) + \frac{5k}{12}\frac{\partial \chi}{\partial z}(z-k,r-h)\Big)\end{aligned}$$

$$+\left(\frac{-10}{3}\chi(z,r+h)+\frac{257}{144}\chi(z-k,r+h)+\frac{7}{4}\chi(z+k,r+h)-\frac{2}{9}\chi(z+2k,r+h)\right.$$

$$\left.+\frac{1}{48}\chi(z+3k,r+h)+\frac{5k}{12}\frac{\partial\chi}{\partial z}(z-k,r+h)\right)$$

Near $z=\beta$

$$h^2k^2\frac{\partial^4\chi}{\partial r^2\partial z^2}(z,r)=$$

$$-2\left(\frac{-10}{3}\chi(z,r)+\frac{257}{144}\chi(z+k,r)+\frac{7}{4}\chi(z-k,r)-\frac{2}{9}\chi(z-2k,r)\right.$$

$$\left.+\frac{1}{48}\chi(z-3k,r)-\frac{5k}{12}\frac{\partial\chi}{\partial z}(z+k,r)\right)$$

$$+\left(\frac{-10}{3}\chi(z,r-h)+\frac{257}{144}\chi(z+k,r-h)+\frac{7}{4}\chi(z-k,r-h)-\frac{2}{9}\chi(z-2k,r-h)\right.$$

$$\left.+\frac{1}{48}\chi(z-3k,r-h)-\frac{5k}{12}\frac{\partial\chi}{\partial z}(z+k,r-h)\right)$$

$$+\left(\frac{-10}{3}\chi(z,r+h)+\frac{257}{144}\chi(z+k,r+h)+\frac{7}{4}\chi(z-k,r+h)-\frac{2}{9}\chi(z-2k,r+h)\right.$$

$$\left.+\frac{1}{48}\chi(z-3k,r+h)-\frac{5k}{12}\frac{\partial\chi}{\partial z}(z+k,r+h)\right)$$

Near $r=0$,

$$h^2k^2\frac{\partial^4\chi}{\partial r^2\partial z^2}(z,r)=$$

$$-2\left(\frac{-10}{3}\chi(z,r)+\frac{257}{144}\chi(z,r-h)+\frac{7}{4}\chi(z,r+h)-\frac{2}{9}\chi(z,r+2h)\right.$$

$$\left.+\frac{1}{48}\chi(z,r+3h)+\frac{5h}{12}\frac{\partial\chi}{\partial r}(z,r-h)\right)$$

$$+\left(\frac{-10}{3}\chi(z-k,r)+\frac{257}{144}\chi(z-k,r-h)+\frac{7}{4}\chi(z-k,r+h)-\frac{2}{9}\chi(z-k,r+2h)\right.$$

$$\left.+\frac{1}{48}\chi(z-k,r+3h)+\frac{5h}{12}\frac{\partial\chi}{\partial r}(z-k,r-h)\right)$$

$$+\left(\frac{-10}{3}\chi(z+k,r)+\frac{257}{144}\chi(z+k,r-h)+\frac{7}{4}\chi(z+k,r+h)-\frac{2}{9}\chi(z+k,r+2h)\right.$$

$$\left.+\frac{1}{48}\chi(z+k,r+3h)+\frac{5h}{12}\frac{\partial\chi}{\partial r}(z+k,r-h)\right)$$

Near $r=1$,

$$h^2k^2\frac{\partial^4\chi}{\partial r^2\partial z^2}(z,r)=$$

$$-2\left(\frac{-10}{3}\chi(z,r)+\frac{257}{144}\chi(z,r+h)+\frac{7}{4}\chi(z,r-h)-\frac{2}{9}\chi(z,r-2h)\right.$$

$$\left.+\frac{1}{48}\chi(z,r-3h)-\frac{5h}{12}\frac{\partial\chi}{\partial r}(z,r+h)\right)$$

$$+\left(\frac{-10}{3}\chi(z-k,r)+\frac{257}{144}\chi(z-k,r+h)+\frac{7}{4}\chi(z-k,r-h)-\frac{2}{9}\chi(z-k,r-2h)\right.$$

$$\left.+\frac{1}{48}\chi(z-k,r-3h)-\frac{5h}{12}\frac{\partial\chi}{\partial r}(z-k,r+h)\right)$$

$$+\left(\frac{-10}{3}\chi(z+k,r)+\frac{257}{144}\chi(z+k,r+h)+\frac{7}{4}\chi(z+k,r-h)-\frac{2}{9}\chi(z+k,r-2h)\right.$$

$$+\frac{1}{48}\chi(z+k,r-3h)-\frac{5h}{12}\frac{\partial\chi}{\partial r}(z+k,r+h)\Big)$$

For the mixed third partial in the middle away from the boundaries $z = 0$ and $z = \beta$

$$24hk^2\frac{\partial^3\chi}{\partial r\partial z^2}(z,r) =$$

$$30\chi(z,r-h)+\chi(z-2k,r-h)-16\chi(z-k,r-h)-16\chi(z+k,r-h)+\chi(z+2k,r-h)$$
$$+30\chi(z,r+h)+\chi(z-2k,r+h)-16\chi(z-k,r+h)-16\chi(z+k,r+h)+\chi(z+2k,r+h)$$

Near $z = 0$,

$$2hk^2\frac{\partial^3\chi}{\partial r\partial z^2}(z,r) =$$

$$-\Big(\frac{-10}{3}\chi(z,r-h)+\frac{257}{144}\chi(z-k,r-h)+\frac{7}{4}\chi(z+k,r-h)-\frac{2}{9}\chi(z+2k,r-h)$$
$$+\frac{1}{48}\chi(z+3k,r-h)+\frac{5k}{12}\frac{\partial\chi}{\partial z}(z-k,r-h)\Big)$$
$$+\Big(\frac{-10}{3}\chi(z,r+h)+\frac{257}{144}\chi(z-k,r+h)+\frac{7}{4}\chi(z+k,r+h)-\frac{2}{9}\chi(z+2k,r+h)$$
$$+\frac{1}{48}\chi(z+3k,r+h)+\frac{5k}{12}\frac{\partial\chi}{\partial z}(z-k,r+h)\Big)$$

Near $z = \beta$

$$2hk^2\frac{\partial^3\chi}{\partial r\partial z^2}(z,r) =$$

$$-\Big(\frac{-10}{3}\chi(z,r-h)+\frac{257}{144}\chi(z+k,r-h)+\frac{7}{4}\chi(z-k,r-h)-\frac{2}{9}\chi(z-2k,r-h)$$
$$+\frac{1}{48}\chi(z-3k,r-h)-\frac{5k}{12}\frac{\partial\chi}{\partial z}(z+k,r-h)\Big)$$
$$+\Big(\frac{-10}{3}\chi(z,r+h)+\frac{257}{144}\chi(z+k,r+h)+\frac{7}{4}\chi(z-k,r+h)-\frac{2}{9}\chi(z-2k,r+h)$$
$$+\frac{1}{48}\chi(z-3k,r+h)-\frac{5k}{12}\frac{\partial\chi}{\partial z}(z+k,r+h)\Big)$$