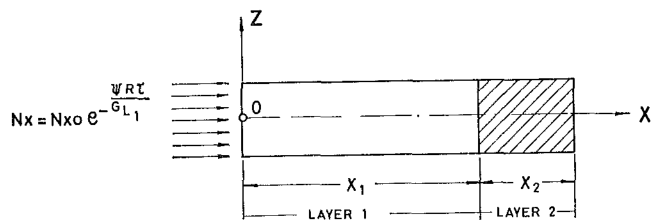

Fig. 1 Two-layered shell subjected to impulsive load

$$[W] = 0 \tag{9}$$

$$[N_x] = -\alpha[U], \tag{10}$$

$$[N_{\theta 1}] = \nu_1{}^2[N_x], \tag{11}$$

and

$$[N_{\theta 2}] = \nu_1\nu_2[N_x]. \tag{12}$$

The variables along the impulsive leading wave are now determined and the remainder solution domain is integrated by the characteristics finite-difference technique.

An Initial Boundary-Value Problem

An initial boundary-value problem is considered where the developed scheme is demonstrated. The problem consists of a two-layered cylindrical membrane shell subjected to an impulsive axisymmetric load applied at one end of the finite shell, Fig. 1. The applied load is an exponential overpressure and is presented in the following dimensionless form:

$$N_x = N_{x_0}e^{-\psi R\tau/G_{L_1}}$$

where N_{x_0} is the peak magnitude taken to be unity and ψ is a damping factor taken to be 10^5 1/sec for this example. The dimensionless length of the first layer is taken to be 0.5 and the second layer is 0.125 long. Material properties are as follows: $\rho_1 = \rho_2 = 2.4 \times 10^{-4}$ lb-sec²/in.⁴, $G_{L_1} = 2.45 \times 10^5$ ips, $\nu_1 = {}^1/_3$, $\nu_2 = {}^1/_6$, $E_1 = 12.8 \times 10^6$ psi, $E_2 = 3.5 \times 10^6$ psi, and $R = 24.5$ in. Time integration increment is taken to be $d\tau = 0.01$.

Conclusions and Results

The study presented a method for the solution of impulsive stress wave-propagation problems in layered shells. A two-layered shell impacted by a longitudinal axisymmetric load was solved in detail. By examining the results, Fig. 2, it can be seen that the influence of the second layer in the shell on its deformation is as follows: as the leading compression wave approaches the interface at τ just < 0.5, $N_x = 1$. Upon its arrival at the interface ($\tau = 0.5$, $x = 0.5$) $N_x = {}^2/_3$. Therefore, the refracted leading wave at $\tau > 0.5$ propagates into the second layer of the smaller impedance as a compression wave, but with a smaller amplitude in N_x. The reflected wave at $\tau > 0.5$ propagates into the first layer as a tension wave having a smaller amplitude in N_x than in the second layer. These events can be observed in Fig. 1, at $\tau = 0.75$. It was also found that the radial particle velocity W is amplified largely at the interface only, due to the Love wave phenomenon which readily revealed itself through the presented method.

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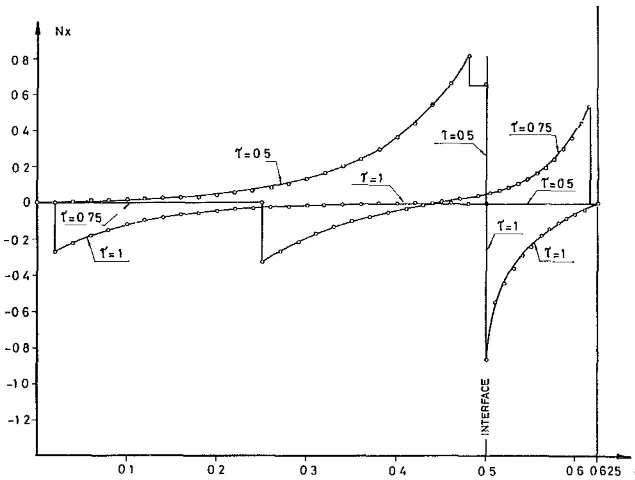

Fig. 2 Longitudinal stress resultant N_x at times $\tau = 0.5$, $\tau = 0.75$, and $\tau = 1$

Shear Correction Factors for Orthotropic Laminates Under Static Load[1]

J. M. WHITNEY[2]

Introduction

BECAUSE of the increasing use of fiber-reinforced composites, their is considerable interest in laminated anisotropic plate analysis. Theories which include the effects of transverse shear deformation [1, 2],[3] as well as bending-extensional coupling effects, follow the basic approach used by Mindlin [3] for homogeneous plates, including the use of a shear correction factor k. In the work of Yang, Norris, and Stavsky [1], the k factor is determined from dynamic considerations. There are indications, however, that k factors calculated in this manner may not yield the closest approximation to the exact solution for laminates subjected to static loading [4]. This is due to the fact that the transverse shear stresses for static bending of laminated plates

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Manuscript received by ASME Applied Mechanics Division, July 1972.

are nearly parabolic [5], even at span-to-thickness ratios as small as four, while in the dynamic case the presence of inertia forces can induce a more complicated shear distribution, especially at intermediate or short wavelengths [4, 6]. In addition, practical laminates often consist of many plies in which the fiber direction varies from layer to layer. Thus it is desirable to treat an approach which does not require an exact solution, as they become very cumbersome for laminates with many layers.

Chow [7] applied two k factors, k_1 associated with γ_{xz} and k_2 associated with γ_{yz}, to symmetric laminates having an orthotropic axis of symmetry in each ply parallel to the x-axis of the plate. A procedure which follows the approach of Reissner [8] for homogeneous isotropic plates is used to determine numerical values for the k factors. Chow's work does not, however, include an assessment of the validity of the proposed approach.

In this Note the procedure of Chow is extended to orthotropic laminates of nonsymmetric construction, and the accuracy of the approach is demonstrated by comparing the static bending solution for various laminated plates to solutions obtained by satisfying exact theory of elasticity in each ply as well as the interface continuity conditions. Numerical results show the values of k_1^2, and k_2^2 depend on detailed laminate construction. Significant difference between k_1^2 and k_2^2 is also noted for symmetric laminates.

Determination of k_1^2 and k_2^2

Denoting partial differentiation by a comma, the shear constitutive relations of reference [2] for orthotropic laminates take the form

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} k_1^2 A_{55} & 0 \\ 0 & k_2^2 A_{44} \end{bmatrix} \begin{bmatrix} \psi_x + w_{,x} \\ \psi_y + w_{,y} \end{bmatrix} \qquad (1)$$

where Q_x and Q_y are shear resultants defined in the usual manner, and

$$A_{ii} = \int_{-h/2}^{h/2} C_{ii} dz$$

The kinematic variables w and ψ_i are plate deflections and rotations, respectively, while C_{ii} are ply shear stiffnesses. Since the procedure to determining k_1^2 and k_2^2 is the same, a detailed discussion of k_1^2 is sufficient.

Consider an orthotropic laminate subjected to the static cylindrical bending conditions

$$v^0 = \psi_y = 0, \qquad u^0 = u(x), \qquad \psi_x = \psi_x(x), \qquad w = w(x) \qquad (2)$$

where u^0 and v^0 are the midplane displacements in the x, y-directions, respectively. According to the theory developed in reference [2]

$$\sigma_{x,x}{}^m = \frac{-Q_{11}{}^m}{D} (B_{11} - A_{11} z) Q_x \qquad (3)$$

where

$$D = (D_{11} A_{11} - B_{11}{}^2)$$

$$(A_{11}, B_{11}, D_{11}) = \int_{-h/2}^{h/2} Q_{11}{}^m (1, z, z^2) dz$$

and $Q_{11}{}^m$ is a plane stress reduced stiffness of the mth layer. Equation (3) in conjunction with the equilibrium equation of elasticity

$$\sigma_{x,x}{}^m + \tau_{xz,z}{}^m = 0 \qquad (4)$$

yields the result

$$\tau_{xz}{}^m = \frac{1}{2D} [a^m + Q_{11}{}^m z (2B_{11} - A_{11} z)] Q_x \qquad (5)$$

where a^m's are constants determined from interface continuity conditions and the requirement that τ_{xz}'s vanish on the bottom surface of the plate. Vanishing of τ_{xz} on the upper surface is then assured by the governing equations of the laminated plate theory.

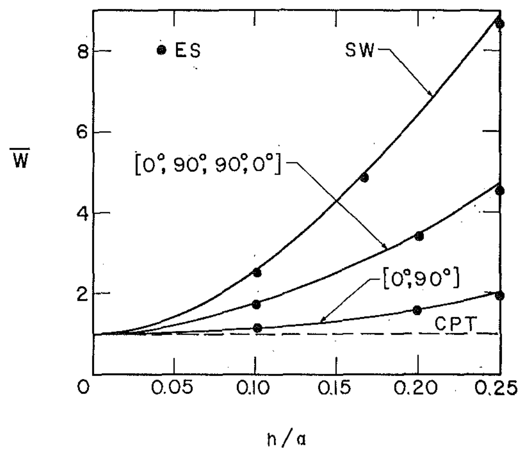

Fig. 1

The strain-energy density for each ply becomes

$$2V^m = g^m(z) Q_x \qquad (6)$$

where

$$g^m(z) = S_{55}{}^m \left[a^m + \frac{Q_{11}{}^{m_z}}{2D} (2B_{11} - A_{11} z) \right]^2$$

and $S_{55}{}^m$ is the shear compliance. Integrating equation (6) across the plate thickness, and equating the result to an analogous expression obtained from the constitutive relations, equation (1) yields

$$k_1^2 = \left[A_{55} \int_{-h/2}^{h/2} g^m(z) dz \right]^{-1} \qquad (7)$$

For symmetric laminates, $B_{11} = 0$, and equation (7) reduces to results obtained by Chow [7]. For homogeneous plates $k_1^2 = 5/6$, the classic value determined by Reissner [8].

Numerical Examples

To illustrate the accuracy of the proposed procedure, consider a rectangular orthotropic laminate of dimensions a, b having simply supported edge conditions, and subjected to the surface load $p = p_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$. The laminated plate solution is found in reference [2] and the exact elasticity solution in reference [5]. Numerical results for the maximum midplane deflection of a graphite/epoxy laminate are shown in Fig. 1, where each ply has the following unidirectional properties:

$$E_L = 25 \times 10^6 \text{ psi}, \qquad E_T = 10^6 \text{ psi}, \qquad \nu_{LT} = 0.25,$$
$$G_{LT} = 0.5 \times 10^6 \text{ psi}, \qquad G_{TT} = 0.2 \times 10^6 \text{ psi}$$

where L signifies the direction parallel to the fibers, T the transverse direction, and ν_{LT} is Poisson's ratio measuring strain in the transverse direction in the presence of uniaxial normal stress parallel to the fibers. Both a 4-layer symmetric and a 2-layer unsymmetric square plate are considered. Orientation of the fibers in each layer relative to the x-axis of the plate is indicated. The deflections are normalized by solutions obtained from laminated plate theory with transverse shear deformation neglected (CPT). Exact elasticity solutions (ES) are indicated by the dots. For the two-ply laminate $k_1^2 = k_2^2 = 0.8212$, and for the four-ply laminate $k_1^2 = 0.5952$ and $k_2^2 = 0.7205$.

To provide a more severe test for the proposed method of determining the shear correction factors, a sandwhich plate (SW) is also considered in Fig. 1. The face sheets are unidirectional graphite/epoxy oriented at 0 deg with respect to the x-axis and have a thickness of $h/10$. The core properties are as follows:

$$E_{xx} = E_{yy} = 0.04 \times 10^6 \text{ psi}, \qquad E_{zz} = 0.5 \times 10^6 \text{ psi},$$

$$G_{xz} = G_{yz} = 0.06 \times 10^6 \text{ psi}, \qquad G_{xy} = 0.016 \times 10^6 \text{ psi},$$

$$\nu_{zx} = \nu_{zy} = \nu_{xy} = 0.25$$

In addition, $k_1^2 = 0.4098$ and $k_2^2 = 0.6915$. Excellent agreement is obtained between the laminated plate theory and the exact elasticity solution for all three examples shown in Fig. 1.

Acknowledgment

The author wishes to acknowledge Prof. C. T. Sun of Purdue University for his helpful comments concerning this work.

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Liapunov Functional for a Rectangular, Nonuniform Density Membrane

M. W. DIXON[1]

The question of stability of equilibrium of a general rectangular, nonuniform density membrane adjacent to a supersonic airstream arbitrarily directed with respect to the edges of the membrane is considered using the direct method of Liapunov. The governing equation of the system, derived using piston theory, is used with the developed Liapunov functional to show that the equilibrium of the system is stable in the absence of structural damping, or for structural damping of a certain form.

Introduction

ASHLEY and Zartarian [1][2] showed by solving the equation of motion developed using piston theory that the equilibrium configuration of the uniform density membrane of finite length and infinite width adjacent to a supersonic airstream flowing parallel to the length is stable when structural damping is not present.

Here the stability of equilibrium of the general rectangular, nonuniform density membrane adjacent to a supersonic airstream flowing in a arbitrary direction with respect to the edges of the membrane will be studied using Liapunov's direct method. The membrane, considered as a distributed parameter system, will be studied with and without structural damping.

There are two main problem areas in the application of Liapunov's direct method to distributed parameter systems. First, there is the problem of creating a theoretical basis which is comparable to that now existing for discrete systems. The second problem area is the actual development of specific Liapunov functionals for specific problems. This is in many ways the more difficult problem and has never been fully solved even for discrete systems. The area of concern here is the development of a Liapunov functional to be used with the equation of motion of the membrane and the existing theoretical basis to reach conclusions as to the stability of equilibrium of the membrane.

Statement of Problem

If a nonuniform density membrane, adjacent to a supersonic airstream which is arbitrarily directed with respect to the edges of the membrane, is tightly stretched such that the variation in tension with lateral displacement is negligible, its equation of motion becomes

$$\ddot{y} + \hat{C}\dot{y} + Ky = 0, \quad y \in Y \subset H, \tag{1}$$

where $\dot{y} \equiv \partial y/\partial t$, $\ddot{y} \equiv \partial^2 y/\partial t^2$, $\hat{C} = (C + |\mathbf{U}|\ \rho_\infty/\mathbf{M})/\rho(x_1, x_2)$, and $Ky = -\{(T/\rho(x_1, x_2))\nabla^2 y - \rho_\infty|\mathbf{U}|\ (u_1\partial_1 y + u_2\partial_2 y)/[\mathbf{M}\rho(x_1, x_2)]\}$. C is the differential operator in the spatial coordinates representing the structural damping, u_1 and u_2 are the components of the free-stream velocity $\mathbf{U}$ in the x_1 and x_2-directions respectively, $\partial_1 \equiv \partial/\partial x_1$ and $\partial_2 \equiv \partial/\partial x_2$, ρ_∞ is free-stream density of the air, and $\mathbf{M}$ is the Mach number. $T > \epsilon_1 > 0$ and $\rho(x_1, x_2) > \epsilon_2 > 0$ are, respectively, the constant tension and variable density of the membrane. For simplicity, $-\rho_\infty|\mathbf{U}|(u_1\partial_1 y + u_2\partial_2 y)/\mathbf{M}\rho(x_1, x_2)$ will be written as $\boldsymbol{\sigma}\cdot\nabla y$, i.e.,

$$\boldsymbol{\sigma}\cdot\nabla y \equiv -\rho_\infty|\mathbf{U}|(u_1\partial_1 y + u_2\partial_2 y)/[\mathbf{M}\rho(x_1, x_2)].$$

Defining the orthogonal axes x_1 and x_2 to be directed along any pair of orthogonal edges of the rectangular membrane yield the following boundary conditions:

$$y(0, x_2) = y(a, x_2) = y(x_1, 0) = y(x_1 b) = 0$$

where a and b are the dimensions of the membrane. Y is defined by the boundary conditions and appropriate smoothness conditions, and H is defined to be the real Hilbert space.

Candidates for Energylike Liapunov Functionals

It should be noted that $\boldsymbol{\sigma}\cdot\nabla y$ is circulatory [2] therefore, an energy functional does not exist for this system. Energylike candidates for Liapunov functionals, called V-candidates, will now be generated using methods recently developed [3, 4]. For ease of writing

$$A \equiv -Te^{\mathbf{R}\cdot\mathbf{x}}[\partial_1 e^{-\mathbf{R}\cdot\mathbf{x}}\,\partial_1 + \partial_2 e^{-\mathbf{R}\cdot\mathbf{x}}\,\partial_2]e^{\mathbf{R}\cdot\mathbf{x}} - \lambda\rho(x_1, x_2)e^{\mathbf{R}\cdot\mathbf{x}}$$

and

$$B \equiv e^{-\mathbf{R}\cdot\mathbf{x}}/\rho(x_1, x_2)$$

where B is defined on Y and maps Y into the range of B, R_B, and A is defined on R_B and maps R_B into H. $Y \subset R_B$. $\mathbf{R}\cdot\mathbf{x} \equiv \rho_\infty|\mathbf{U}|(u_1x_1 + u_2x_2)/(TM)$ and λ is an arbitrary scalar constant.

V-candidates, V_1 and V_2, are found to be

$$V_1 = \int_0^a \int_0^b (\dot{y}B^{-1}\dot{y} + yB^{-1}Ky)dx_1dx_2 \qquad y, \dot{y} \in Y \tag{2}$$

and

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Manuscript received by ASME Applied Mechanics Division, June, 1972; final revision, October, 1972.