Algebraic dynamics of certain Gamma function values

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Abstract

We present significant numerical evidence, based on the entropy analysis by lumping of the binary expansion of certain values of the Gamma function, that some of these values correspond to incompressible algorithmic information. In particular, the value $\Gamma(1/5)$ corresponds to a peak of non-compressibility as anticipated on a priori grounds from number-theoretic considerations. Other fundamental constants are similarly considered.

This work may be viewed as an invitation for other researchers to apply information theoretic and decision theory techniques in number theory and analysis.

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1 Introduction

Nature provides us with a wide variety of symbolic strings ranging from the sequences generated by the symbolic dynamics of nonlinear systems to RNA and DNA sequences or DLA patterns (*diffusion limited aggregation* patterns are a classical subject in Nonlinear Chemistry) [4, 38, 39].

Entropy-like quantities are a very useful tool for the analysis of such sequences. Of special interest are the *block entropies*, extending Shannon's classical definition of the entropy of a single state to the entropy of a succession of states [38]. In particular, it has been shown in the literature that *scaling* the block entropies by length sometimes yields interesting information on the structure of the sequence [18, 19].

In particular, one of the present authors has derived an entropy criterion for the specialized, yet important algorithmic property of *automaticity* of a sequence. We recall that, a sequence is called *automatic* if it is generated by a finite automaton (the lowest level Turing machine). For more details about automatic sequences the reader is referred to [14], and for their role in Physics to [1].

This criterion is based on entropy analysis by *lumping*. *Lumping* is the reading of the symbolic sequence by 'taking portions' (see expression (1)), as opposed to *gliding* where one has essentially a 'moving frame'. Notice that gliding is the standard approach in the literature. Reading a symbolic sequence in a specific way is also called *decimation* of the sequence.

The paper is articulated as follows. In Section two we recall some useful facts. In Section three we present the mathematical formulation of the entropy analysis by lumping. In Section four we present our intuitive motivation based on algorithmic arguments while in Section five we present a central example of an automatic sequence, taken from the world of nonlinear Science, namely the Feigenbaum sequence. In Section six we present our main results. In Section seven we speak about automaticity and algorithmic compressibility measures. In section eight we analyse $\exp(\pi/\sqrt(2))$. Finally, in Section nine we draw our main conclusions and discuss future work.

2 Some definitions

We first recall some useful facts from elementary number theory. As is well known, *rational* numbers can be written in the form of a fraction p/q, where p and q are integers and *irrational* ones cannot take this form. The k-ary expansion of a rational number (for instance the decimal or binary expansion) is periodic or eventually periodic and conversely. Irrational numbers form two categories: *algebraic irrational* and *transcendental*, according to whether they can be obtained as roots of a polynomial with rational coefficients or not. The k-ary expansion of an irrational number is necessarily *aperiodic*. Note that transcendental numbers are *well approximated* by fractions. In 1874 G. Cantor showed that 'almost all' real numbers are transcendental.

A *normal* number in base $k \geq 2$ is a real number x such that, for each integer $d \geq 1$, each block of length d occurs in the k-ary expansion of x with (equal) asymptotic frequency $1/k^d$. A rational number is never normal, while there exist numbers which are normal and transcendental, like Champernowne's number. This number is obtained by concatenating the decimal expansions of consecutive integers [13]

$$0.1234567891011121314...$$

and it is simultaneously transcendental and normal in base 10.

There is an important and widely believed conjecture, according to which all algebraic irrational numbers are believed to be normal. But present techniques fall woefully short on this matter, see [2]. It seems that E. Borel was the first who explicitly formulated such a conjecture in the early fifties [6]. Actually, normality is not the best criterion to distinguish between algebraic irrational and transcendental numbers. In fact, there exist transcendental numbers which are normal, like Champernowne's number [13], [12], [1] and probably π [39], [40] [1]. One of the first systematic studies towards this direction dates back to ENIAC also some fifty years ago [34, 7]. No truly 'natural' transcendental number has been shown to be normal in any base, hence the interest in computation.

3 Entropy analysis by lumping

For reasons both of completeness and for later use, we compile here the basic ideas of the method of entropy analysis by lumping. We consider a subsequence of length N selected out of a very long (theoretically infinite) symbolic sequence. We stipulate that this subsequence is to be read in terms of distinct 'blocks' of length n,

$$\ldots \underbrace{A_1 \ldots A_n}_{B_1} \underbrace{A_{n+1} \ldots A_{2n}}_{B_2} \ldots \underbrace{A_{jn+1} \ldots A_{(j+1)n}}_{B_{j+1}} \ldots$$

We call this reading procedure *lumping*. We shall employ lumping throughout the sequel. The following quantities characterize the information content of the sequence [33, 18]

i) The *dynamical (Shannon-like) block-entropy* for blocks of length n is given by

$$H(n) := -\sum_{(A_1,\ldots A_n)} p^{(n)}(A_1,\ldots,A_n)\cdot\ln p^{(n)}(A_1,\ldots,A_n) \tag{1}$$

where the probability of occurrence of a block $A_1\ldots A_n$, denoted $p^{(n)}(A_1,...,A_n)$, is defined (when it exists) in the statistical limit as

$$p^{(n)}(A_1,\ldots,A_n) = \frac{\#\text{of blocks of the form} A_1\ldots A_n \text{ found when lumping}}{\text{Total}\,\#\text{ of blocks found}} \tag{2}$$

starting from the beginning of the sequence, and the associate entropy per letter

$$h^{(n)} = \frac{H(n)}{n}. \tag{3}$$

ii) The *conditional entropy* or entropy excess associated with the addition of a symbol to the right of an n-block

$$h_{(n)} = H(n+1) - H(n). \tag{4}$$

iii) The *entropy of the source* (a topological invariant), defined as the limit (if it exists)

$$h = \lim_{n\to\infty} h_{(n)} = \lim_{n\to\infty} h^{(n)} \tag{5}$$

which is the discrete analogue of metric or Kolmogorov entropy.

We now turn to the selection problem, that is to the possibility of emergence of some preferred configurations (blocks) out of the complete set of different possibilities. The number of all possible symbolic sequences of length n (*complexions* in the sense of Boltzmann) in a K-letter alphabet is

$$N_K = K^n. \tag{6}$$

Yet not all of these configurations are necessarily realized by the dynamics, nor are they equiprobable. A remarkable theorem due to McMillan [33], gives a partial answer to the selection problem asserting that for stationary and ergodic sources the probability of occurrence of a block $(A_1,\ldots,A_n)$ is

$$p_n(A_1,\ldots,A_n) \sim e^{-H(n)} \tag{7}$$

for almost all blocks $(A_1, \ldots, A_n)$. In order to determine the abundance of long blocks one is thus led to examine the scaling properties of $H(n)$ as a function of n.

It is well known that numerically, block entropy is underestimated. This underestimation of $H(n)$ for large values of n is due to the simple fact that not all words will be represented adequately if one looks at long enough samples. The situation becomes more and more prominent for calculating $H(n)$ by 'lumping' instead of 'gliding'. Indeed in the case of 'lumping' an exponentially fast decaying tail towards value zero follows after an initial plateau.

Since the probabilities of the words of length m are calculated by their frequencies, i.e. $p_n = N_1/N_{[sample]}$ where $N_{[sample]}$ is the size of the available data-sample i.e. the length of the 'text' under consideration, then as $N_1 \to 0$ for long words, the block entropy calculated will reach a maximum value, its *plateau*, at

$$H_{MAX} = \log_{[K]}(N_{[sample]})$$

where K the length of the alphabet. Indeed, this corresponds to the maximum value of the entropy for this sample, given when

$$p_n = 1/N_{[sample]}.$$

This value corresponds also to an effective maximum word length

$$n_{max} = \ln N_{[sample]}$$

in view of eqs. (1), (6) and (7).

For instance, if we have a binary sequence with 10,000 terms, of course $b = 2$ and $N_{[sample]} = 10^4$. This way, the value of H_{MAX} can determine a safe border for finite size effects. In our case

$$H_{MAX} = n_{max} = \ln\left(10^4\right) = 9.2\ldots, \tag{8}$$

so that $n_{max} = 9$ and we can safely consider the entropies until $n = 8$.

After this small digression, we recall here the main result of the entropy analysis by lumping, see also [28, 29]. Let m^k be the length of a block encountered when lumping, $H(m^k)$ the associated block entropy. We recall that, in view of a result by Cobham (Theorem 3 of [14]), a sequence is called *m-automatic* if it is the image by a letter to letter projection of the fixed point of a set of substitutions of constant length m. A substitution

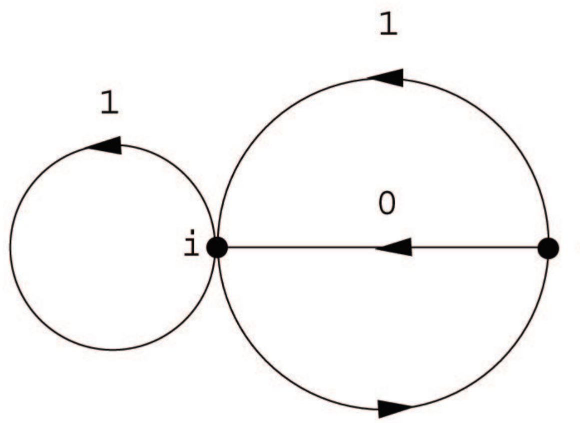

Figure 1: Deterministic finite automaton described by Cobham's algorithmic procedure. This automaton contains two states: i and a and to each state corresponds by the function of exit F a symbol; either $F(i) = R = 1$ or $F(a) = L = 0$. To calculate the n^{th} term of the 2^{∞} sequence we first express the number n in its binary form and then we start running the automaton from its initial state, according to the binary digits of n. In this trip we read the symbols contained in the binary expansion of n from the left to the right following the targets indicated by the letters. For instance $n = 3 =$ (11 base 2) gives the run $i \rightarrow i \rightarrow i$ so that $u(3) = R = 1$, while $n = 9 =$ (1001 base 2) gives the run $i \rightarrow i \rightarrow a \rightarrow i \rightarrow i$ so that $u(9) = R = 1$.

is called *uniform* or *of constant length* if all the images of the letters have the same length. For instance, the Feigenbaum symbolic sequence can in an equivalent manner be generated by the Metropolis, Stein and Stein algorithm [35, 25], or as the fixed point $(\sigma^F)^{\infty}(R)$ of the set of substitutions of length 2: $\sigma^F(R) = RL, \sigma^F(L) = RR$ starting with R, or by the finite automaton of Figure 1 (see also Section five).

The term 'automatic' comes from the fact that an automatic sequence is generated by a finite automaton.

The following properties then holds:

- If the symbolic sequence $(u_n)_{n \in \mathcal{N}}$ is m-automatic, then

$$\exists \quad k_o \in \{0, 1\}, \quad m \in \mathcal{N}^*, \quad \forall \quad k \geq k_o : \quad H(m^{k_o}) = H(m^k) \qquad (9)$$

  when lumping, starting from the beginning of the sequence.

The meaning of the previous proposition is that for m-automatic sequences there is always an envelope in the diagram $H(n)/n$ versus n, falling off exponentially as $\sim m^{-k}$ for blocks of a length m^k, $k = 1, 2, ...$ For infinite ergodic strings, the conclusion does not depend on the starting point. Similar conclusions hold if instead of a one-to-one letter projection we have a one-to-many letters projection of constant length. In particular, we have the following result.

- If the symbolic sequence $(u_n)_{n\in\mathcal{N}}$ is the image of the fixed point of a set of substitutions of length m by a projection of constant length μ, then

$$\exists \quad k_o \in \{0,1\}, \quad m \in \mathcal{N}^*, \quad \forall \quad k \geq k_o : \quad H(\mu \cdot m^{k_o}) = H(\mu \cdot m^k) \tag{10}$$

  when lumping, starting from the beginning of the sequence.

Our propositions give an interesting *diagnostic* for automaticity. When one is given an unknown symbolic sequence and numerically applies entropy analysis by lumping, then if the sequence does not obey such an invariance property predicted by the propositions, it is certainly non-automatic. In the opposite case, if one observes evidence of an invariance property, then the sequence is a good candidate to be automatic.

For stochastic automata, the following proposition also holds (see [31]).

- If the symbolic sequence $(u_n)_{n\in\mathcal{N}} \in \{0,1\}$ is generated by a *Cantorian stochastic automaton,* then [31]

$$\forall \quad k \geq 1 : \quad H(m^k) = k \cdot H(m) \tag{11}$$

  when lumping, starting from the beginning of the sequence.

4 The example of the Feigenbaum sequence

Before proceeding to the analysis of binary expansions of the values of the gamma function $\Gamma(1/n)$ (which as we shall see presently seems not to be automatic) we first give an example of entropy analysis by lumping of a 2-automatic sequence: the period-doubling or *Feigenbaum sequence*, much studied in the literature [23, 19, 25].

The Feigenbaum symbolic sequence can in an equivalent manner be generated by the Metropolis, Stein and Stein algorithm [35, 25], or as the fixed point $(\sigma^F)^\infty(R)$ of the set of substitutions of length 2: $\sigma^F(R) = RL, \sigma^F(L) = RR$ starting with R, or by the finite automaton of Fig.1. According to our first proposition, this sequence satisfies

$$H(1) = H(2) = \cdots = H(2^k) \tag{12}$$

when lumping, while for any integer r,

$$H(2 \cdot r) = H(r) \tag{13}$$

as is shown in [25].

Thus, the Feigenbaum sequence appears to be extremely compressible from the viewpoint of algorithmic information theory—memorizing the finite automaton (instead of memorizing the full sequence) lets one reproduce every term and so, the complete sequence. We say that the information carried by the Feigenbaum sequence is 'algorithmically compressible'.

The period-doubling sequence, is the only one for which an exact functional relation between the block-entropies when *lumping* and when *gliding* exists in the literature, so that it is an especially instructive example.

5 Motivation for the Gamma function

The basis of reduced complexity computation of Gamma function values is illustrated by the cases of $\pi = \Gamma^2(1/2)$, and of $\Gamma(1/4)$, and $\Gamma(3/4)$. These algorithms are discussed at length in [8] and related material is to be found in [7]. Their origin is very classical relying on the early elliptic function discoveries of Gauss and Legendre but they do not appear to have been found earlier.

Algorithm. Let $x_0 := \sqrt{2}, \pi_0 = 2 + \sqrt{2}$ and $y_1 := 2^{1/4}$, compute

$$x_{n+1} := \frac{\left(\sqrt{x_n} + 1/\sqrt{x_n}\right)}{2}$$

for $n \geq 0$ and

$$y_{n+1} := \frac{\left(y_n\sqrt{x_n} + 1/\sqrt{x_n}\right)}{y_n + 1},$$

$$\pi_n := \frac{x_n+1}{y_n+1}\,\pi_{n-1}$$

for $n>0$. Then

$$0<\pi_n-\pi<10^{-2^{n+1}}.$$

Hence

$$(2+\sqrt{2})\prod_{n\geq 1}\frac{1+x_n}{1+y_n}=\Gamma^2\left(\frac{1}{2}\right)$$

while

$$(1+1/\sqrt{2})\prod_{n\geq 1}x_n\frac{1+x_n}{1+y_n}=\Gamma^4\left(\frac{3}{4}\right),$$

and

$$(1+1/\sqrt{2})^3\prod_{n\geq 1}x_n^{-1}\left(\frac{1+x_n}{1+y_n}\right)^3=\frac{1}{16}\,\Gamma^4\left(\frac{1}{4}\right),$$

provide corresponding quadratic algorithms for $\Gamma(1/2), \Gamma(3/4)$ and $\Gamma(1/4)$, [8, pp. 46–51].

There are similar algorithms for $\Gamma(1/6), \Gamma(1/3), \Gamma(2/3)$ and $\Gamma(5/6)$, and related elliptic integral methods for $\Gamma(k/24)$ for all positive integer k are given by [**?**]. For example,

$$\mathrm{K}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)=\frac{3^{-1/4}}{4}\,\beta\left(\frac{1}{3},\frac{1}{6}\right)=\frac{3^{1/4}}{4^{1/3}}\,\frac{\Gamma^3\left(\frac{1}{3}\right)}{\pi}.$$

In consequence, since elliptic function values are fast computable, we obtain algorithms for $\Gamma\left(k/6\right)$.

No such method is known for other rational Gamma values, largely because the needed elliptic integral and Gamma function identities are too few and do not allow one to separate $\Gamma(1/5)$ and $\Gamma(2/5)$, for example, while they do allow for their product to be computed.

This does not rule out the existence of other approaches but it suggests that the algorithmic complexity of $\Gamma(1/5)$ should be greater than that of $\Gamma(1/5)\,\Gamma(2/5)$, and that the algorithmic complexity of $\Gamma(1/5)$ or $\Gamma(1/9)$ should be greater than that of $\Gamma(1/12)$. This in part motivates our analysis.

Similarly, we note that

$$\Gamma\left(\frac{1}{20}\right)\Gamma\left(\frac{3}{20}\right)\Gamma\left(\frac{7}{20}\right)\Gamma\left(\frac{9}{20}\right)=160\,\sqrt{+\sqrt{5}-2}\,\pi\,\mathrm{K}\left(k_5\right)$$

where

$$k_5 = \sqrt{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{10}-\sqrt{2}}{4}.$$

Thus this Gamma product is fast computable, as are many others.

6 Results

In this work, we have considered the first 10,000 digits of the binary expansions of numbers of the form $\Gamma(1/n)$, where $n = 2, 3, 4, 5, 8, 9, 12, 24$. We have good statistics until a block length $n = 9$.

We can report the following results:

1. The binary expansion of $\Gamma(1/5)$ presents the maximum value of the entropy throughout almost the whole range.

2. The binary expansions of $\Gamma(1/5)\Gamma(2/5)$ and $\Gamma(1/2)$ present the minimum value of the entropy through almost the whole range. This corresponds to significant algorithmic compressibility.

3. The binary expansion of $\Gamma(1/3)$ presents (within the limits of the numerical precision) non-monotonic behaviour of the block entropy per letter (not recorded below), indicating a deep and unanticipated algorithmic structure for this number.

4. The binary expansions of the other numbers present intermediate behavior.

There is now the question of the error bars. In any case, due to finite-sample effects the values of the entropy are underestimated, as we have already explain in Section three. To estimate the error of these computations, suppose that, for $k = 1$ there is an error in one digit over 10,000 digits. Then the corresponding error in the entropy H_1 by lumping will be

$$\delta H_1 = -\frac{1}{10^4} \ln \frac{1}{10^4} \sim 0.00092 \simeq 0.001, \tag{14}$$

while due to lumping there is an error for the entropy H_8 (at the limit of our numerical precision) of 1 block per $\frac{10^4}{8} = 1250$ blocks of length 8, leading to a corresponding error in the entropy H_8 by lumping

$$\delta H_8 = -\frac{1}{1250} \ln \frac{1}{1250} \sim 0.0057 \simeq 0.006, \tag{15}$$

so that we can keep three significant digits of the entropy in the whole range.

In particular, we have the following results for $H(k)/k$ for k from 1 to 9, 12 and 24.

$n = 2$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.692
5	0.689
6	0.688
7	0.680
8	0.670

$n = 3$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.692
5	0.692
6	0.690
7	0.688
8	0.679

$n = 4$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.692
5	0.692
6	0.690
7	0.685
8	0.681

$n = 5$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.693
5	0.692
6	0.692
7	0.690
8	0.687

$n=8$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.692
5	0.692
6	0.689
7	0.688
8	0.679

$n=9$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.693
5	0.691
6	0.690
7	0.687
8	0.680

$n=12$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.693
5	0.692
6	0.690
7	0.687
8	0.681

$n=24$	
k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.693
5	0.692
6	0.689
7	0.687
8	0.681

The basic conclusion from these tables is that these Gamma function values correspond to little compressible information, as the entropy per letter $H(n)/n$ approaches in all cases its maximum value

$$h_{max}^{(n)} = \ln 2 = 0.693147.... \tag{16}$$

Furthermore, on inspecting the blocks that appear, one can check that (within the limits of our numerical precision), all possible blocks of letter occur in the binary expansions of these Gamma function values (as we would say in the language of the ergodic theory and dynamical systems, the system is "mixing"), a fact that validates both the statistics and the conclusions about the algorithmic incompressibility of the next Section.

We have also considered the first 5,000 digits of the binary expansion of $\Gamma(1/5)\Gamma(2/5)$. We have good statistics up to a block length $n = 9$. In particular, we obtain the following results for $H(k)/k$ for k from 1 to 8. This as conjectured shows significantly more compressibility.

k	$H(k)/k$
1	0.693
2	0.693
3	0.693
4	0.692
5	0.690
6	0.689
7	0.680
8	0.667

7 Automaticity measures

As we have already mentioned, when a symbolic sequence is generated by a deterministic finite automaton with m-states, then the block entropies measured by lumping respect an invariance property:

$$H(m) = H(m^2) = \cdots = H(m^j) = const.$$

for k integer, $k > 1$.

When this invariance property breaks, the sequence is not generated by a deterministic finite automaton with m-states. Still, one can still obtain a measure of algorithmic complexity (in particular of “”algorithmic compressibility”) taking values from 0 % to 100 % the index: (in our notation)

$$A(j) = \left| \frac{h(m^{j+1}) - h(m^j)}{h_{max}(m^{j+1})} \right|, \tag{17}$$

properly normalized, on dividing by $h_{max}(m^{j+1})$.

To fix the ideas, let us consider the 2-states automaticity measure (so $m = 2$) of order $j = 1$, which can be expressed as

$$A(2) = \left| \frac{h(8) - h(4)}{h_{max}(8)} \right| = 0.032 = 3.2\% \tag{18}$$

In terms of 2-states automata, the variation of these indices is as follows:

n	$A(2)$
2	3.2
3	1.9
4	1.6
5	0.9
8	1.9
9	1.9
12	1.7
24	1.7

from which our conclusion about the algorithmic non-compressibility of $\Gamma(1/5)$ follows. Indeed, the more incompressible the sequence, the smaller the index $A(k)$. In confirmation of our earlier analysis, the corresponding value of $A(2)$ for $\Gamma(1/5)\Gamma(2/5)$ is 3.6%, indicating the highest algorithmic compressibility.

We arrive at exactly the same conclusions if we treat the values of $h(m)$ individually (instead of taking the absolute differences), searching directly for an alternative index of algorithmic compressibility

$$A' = \frac{h(i)}{h_{max}(i)}\%. \tag{19}$$

8 Entropy analysis of the constant $e^{\pi/\sqrt{2}}$

It has been shown [15, 16, 24] that, for a wide class of Hamiltonian dynamical systems, the constant

$$\varrho := e^{\pi/\sqrt{2}}$$

plays the role that is played by the Feigenbaum constant δ for the logistic map and for dissipative systems in general [36, 20, 21, 10, 11, 22]. Thus, this constant (bifurcation ratio of period doubling bifurcations) is not universal, rather it depends on the particular dynamical system considered.

Recently, after the calculation of the Feigenbaum fundamental constants α and δ for the logistic map (quadratic non-linearity), to more than 1,000 digits by D. Broadhurst [10], a careful statistical analysis of these constants has been presented [32], indicating the real possibility that these constants are non-normal (so probably transcendental) numbers.

Now, it is easy to show that the constant ϱ is transcendental [41, 42, 43]. Indeed, according to the theorem of Gel'fond and Schneider—which resolved Hilbert's seventh problem—for a nonzero complex number λ and an irrational algebraic number b, one at least of the three numbers e^{λ}, $e^{b\lambda}$, b is transcendental. In our case, taking $\lambda = i\pi$ and $b = \sqrt{2}$, we easily obtain the transcendence of $e^{\frac{\pi}{\sqrt{2}}}$. As this constant is a combination of three fundamental constants π, e and $\sqrt{2}$, presumably all normal, it is reasonable to ask if ϱ also appears normal.

We first present an entropy analysis of the first 100,000 terms of the binary expansion of the constant $\varrho = e^{\pi/\sqrt{2}}$. We have reliable statistics for block lengths not exceeding $n = 10$.

Regarding the error bars now, we estimate the error of these computations as follows. Suppose that,for $k = 1$ there is an error in one digit over 100,000 digits. Then the corresponding error in the entropy H_1 by lumping will be

$$\delta H_1 = -\frac{1}{10^5} \ln \frac{1}{10^5} \sim 0.000115... \simeq 0.0001, \tag{20}$$

while due to lumping there is an error for the entropy H_{10} (at the limit of our numerical precision) of 1 block per $\frac{10^5}{10} = 10^4$ blocks of length 10, leading to a corresponding error in the entropy H_{10} by lumping

$$\delta H_8 = -\frac{1}{10^4} \ln \frac{1}{10^4} \sim 0.00092 \simeq 0.001. \tag{21}$$

For reasons of uniformity of our treatment, however, we keep three significant digits for the entropy per letter.

In particular, we record the following results for $H(k)/k$ as a function of k.

Length	Lumping of ϱ
1	0.693
2	0.693
3	0.693
4	0.693
5	0.693
6	0.693
7	0.693
8	0.692
9	0.691
10	0.688

This indicates serious evidence that ϱ is a normal number in base 2, since the entropy per letter approaches in all cases its maximum value $h_{max}^{(n)} = \ln 2 = 0.693....$

One should also notice that, all possible blocks of letters (*within the range computed*) appear in the binary expansions of ϱ (as we would say in the language of the ergodic theory and dynamical systems, the system is "mixing"), a fact that validates both the statistics and the conclusion about algorithmic incompressibility.

In order to observe the results of the change of the basis expansion, we also present here an entropy analysis of the first 100,000 terms of the decimal expansion of the constant $\varrho = e^{\pi/\sqrt{2}}$. We have reliable statistics for block lengths not exceeding $n = 5$.

For the error bars now, we estimate the error of these computations, suppose that,for $k = 1$ there is an error in one digit over 100,000 digits. Then the corresponding error in the entropy H_1 by lumping will be

$$\delta H_1 = -\frac{1}{10^5} \ln \frac{1}{10^5} \ln 10 \sim 0.000265... \simeq 0.0003, \tag{22}$$

while due to lumping there is an error for the entropy H_4 (at the limit of our numerical precision) of 1 block per $\frac{10^5}{4} = 25,000$ blocks of length 4, leading to a corresponding error in the entropy H_4 by lumping

$$\delta H_4 = -\frac{1}{25,000} \ln \frac{1}{25,000} \ln 10 \sim 0.000933 \simeq 0.001. \tag{23}$$

For reasons of uniformity, we also decided to keep three significant digits for the entropy per letter. In particular, we record the following results for $H(k)/k$.

Length	Lumping of ϱ
1	2.303
2	2.302
3	2.298
4	1.952

This again indicates serious evidence that ϱ would be a normal number in base 10, since the entropy per letter approaches in all cases its maximum value $h_{max}^{(n)} = \ln 10 = 2.303....$ Again, we notice that, one can check that all possible blocks of letters appear, a fact that validates both the statistics and the conclusion about the algorithmic incompressibility.

Finally, we note that in terms of algorithmic complexity ϱ is one of the most accessible constants. The following algorithm, a precursor to those given above for $\Gamma(1/n)$ [8, 7] provides $O(D)$ good digits with $\log D$ operations.

```
epi:=proc(r,N)  local k,n;
 k:=r; for n to N
         do k:=sqrt(1-k^2); k:=(1-k)/(1+k) od;
(k/4)^(-2^(-N));end;
```

Then $\mathrm{epi}(\sqrt{2}-1, N)$ returns roughly 2^N good digits of ϱ while $\mathrm{epi}(1/\sqrt{2}, N)$ does the same for $\exp(\pi)$.

9 Conclusions and outlook

We have performed an analysis of some binary expansions of the values of the Gamma function $\Gamma(1/n)$ by lumping. The basic novelty of this method is that, unlike use of the Fourier transform or conventional entropy analysis by gliding, it gives results that can be related to algorithmic characteristics of the sequences and, in particular, to the property of automaticity.

In light of the paucity of analytic techniques for establishing normality or other distributional facts about specific numbers, such experimental-computational tools are well worth exploring further and refining more.

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