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On the Strength Anisotropy of Bone and Wood

Formulas for the tensile, compressive, and shear strengths along an axis off the grain axis of a material with orthotropic symmetry are derived from the quadratic form of the Tsai-Wu strength theory. The results are compared with some shear strength data on wood and they are applied to data on human femoral bone and bovine Haversian bone. A strength criterion consistent with the Tsai-Wu theory and with the existing experimental strength data on bone is identified.

Introduction

Formulas are developed in this paper for the tensile, compressive, and shear strengths along an axis off the grain axis of a material with orthotropic symmetry such as bone and wood. These formulas are derived from a phenomenological strength theory.

The first formula for the strength along an axis off the grain axis was reported by Hankinson [1] in 1921. This formula, which has come to be known as Hankinson's formula, was developed empirically from compressive strength tests on spruce and other woods. If σ_1^- denotes the compressive strength along the grain, σ_2^- one of the compressive strengths in a cardinal direction transverse to the grain, then Hankinson predicted the compressive strength in a direction inclined at an angle θ to the grain by the formula

$$\sigma_\theta^- = \frac{\sigma_1^-\sigma_2^-}{\sigma_1^- \sin^2\theta + \sigma_2^- \cos^2\theta}. \tag{1}$$

Hankinson noted that this empirical formula applied to the compressive elastic limit as well as to the compressive ultimate strength.

In 1923 Rowse [2] studied the behavior of Douglas-fir in compression. Noting that the compressive strength perpendicular to the grain is difficult to determine because the wood crushes down indefinitely, Rowse determined the proportional limit stresses rather than strengths. He found that the formula of Hankinson fitted his experimental data reasonably well.

The book of Kollmann and Côté [3] presents Hankinson's formula with two differences from the original form given by Hankinson. First, it is indicated that Hankinson's formula applies only to tensile strengths and, second, Kollmann and Côté replace the power 2 to which the trigonometric terms in (1) are raised by an arbitrary power n, thus in the present notation (7.40) of [3] is

$$\sigma_\theta^+ = \frac{\sigma_1^+\sigma_2^+}{\sigma_1^+ \sin^n\theta + \sigma_2^+ \cos^n\theta}. \tag{2}$$

Kollman and Côté [3] report that Kollman [4] has shown that values of n between 1.5 and 2 are satisfactory. The analysis presented in the present paper will show clearly that n must equal two as Hankinson originally suggested. This fact is a direct consequence of the tensor transformation law for stress.

In 1971 Goodman and Bodig [5] reported an experimental study which concluded that, among the equations most often used to predict the strength of wood, Hankinson's formula was found to provide the best two-dimensional fit for the maximum compressive strength of the four species tested: Englemann spruce, Douglas-fir, oak, and aspen.

In 1974 Reilly [6] and Reilly and Burstein [7] reported an experimental study that involved the determination of the ultimate strengths of bone. They found that all bone types they tested followed Hankinson's criterion for both ultimate tensile strength and ultimate compressive strength. The type of bone tested included human femoral and bovine Haversian femoral.

Hankinson's formula predicts the strength along an axis off the grain axis from the strengths along the grain axis and transverse to the grain axis and it should be a derivable prediction from a complete theory of strength for orthotropic materials. Norris [8], in a work written in 1950 but not published until 1962, was the first to pursue the derivation of a formula of the Hankinson type from a complete strength theory. Norris [8] developed a strength theory of orthotropic materials, but the subsequent comprehensive analysis of Wu [9] has shown Norris' theory to be unacceptable because of inappropriate assumptions. The shortcomings of the Norris theory are also discussed by Kobetz and Krueger [10]. Norris derived a formula for the off the grain axis strength of orthotropic materials, but it was not Hankinson's formula. Norris' formula gave good results for plywood and a glass fabric laminate, but it did not give as good a representation of Rowse's Douglas-fir data as Hankinson's formula.

The present paper considers the same problem as Norris consid-

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ered, namely the derivation of formulas of the Hankinson type from complete theories of strength. In the years since Norris did his work there has been a great deal of work on theories of strength in connection with composite materials, and the present paper has an advantage over Norris' work in being able to draw upon these composite materials studies. Formulas of the Hankinson type will be derived here from special cases of the general theory of strength for anisotropic materials presented by Tsai and Wu [11] in 1972. This strength theory is phenomenological and contains a systematic scheme of approximation which permits any degree of accuracy desired. In the Appendix it will be shown that the first or linear approximation of the Tsai-Wu strength theory for orthotropic materials predicts the formula (1) of Hankinson. However, the linear approximation contains only terms linear in the normal stresses and it does not, as indeed Hankinson's criterion does not, account for the shear strength of the material. For this reason, and because of other deficiencies of the linear approximation, the quadratic approximation of the Tsai-Wu strength theory for orthotropic materials will be employed here. This quadratic approximation is developed from first principles in the following section.

Using the quadratic strength theory for orthotropic materials, formulas for the shear strength, the tensile strength, and the compressive strength as functions of the angle to the grain are obtained. The shear strength formula is compared with data on pine wood and the tensile and compressive strength formulas are compared with experimental data on bone. The shear strength formula is of the same type as Hankinson's formula except that normal stresses are replaced by the squares of shear stresses. It follows the general trend of the data on pine wood. The formulas for tensile and compressive strength are not of the same form as Hankinson's formula, but for the data on bone considered here, these formulas yield curves extremely close to the curves representing Hankinson's formula, and represent the data equally well or, in most cases, better. A form of a quadratic strength theory for transversely isotropic materials is suggested for use with bone in the conclusion.

The Quadratic Strength Theory for Orthotropic Materials

In this section a phenomenological strength criterion or theory for materials with orthotropic symmetry is developed. The criterion developed contains terms linear and quadratic in the stress and it is a special case of the anisotropic strength theory developed by Tsai and Wu [11] and Wu [9]. The philosophy followed here in developing the strength criterion for orthotropic materials is exactly the same as that described by Tsai and Wu [11] and Wu [9], but a simpler and more direct development is possible because it is assumed that the material has orthotropic symmetry.

The basic hypothesis underlying a phenomenological strength criterion is that there exists a function f of the stress state T_{xx}, T_{yy}, T_{zz}, T_{xy}, T_{xz}, T_{yz} such that, if the stress state is an ultimate stress state, the value of the function is a constant and, without loss of generality, the constant may be set equal to one, thus

$$f(T_{xx}, T_{yy}, T_{zz}, T_{xy}, T_{xz}, T_{yz}) = 1. \tag{3}$$

The strength function can also depend upon other material variables such as moisture content, age, etc., but the inclusion of these additional variables will not influence the general nature of the results obtained. It is required that the strength function f reflect the symmetry of the material to which it applies. If the material is assumed to be isotropic with respect to strength as metals are often assumed to be, then the function f depends upon the stress only through the three isotropic invariants. Bone and wood are generally considered to be orthotropic, that is to say they have three mutually perpendicular axes of symmetry, or equivalently, three mutually perpendicular planes of symmetry. In the present development a coordinate system denoted by x_1, x_2, x_3 is selected to coincide with the three mutually perpendicular axes of material symmetry at each point in the material body. In order that the function f reflect orthotropic symmetry, Green and Atkins [12, p. 14] show that f must depend upon the normal stresses T_{11}, T_{22}, and T_{33}, the squares of the values of the shear stresses T_{12}, T_{13}, and T_{23} and the determinant of the stress tensor $\mathbf{T}$; thus for an orthotropic material (3) is replaced by

$$f(T_{11}, T_{22}, T_{33}, T_{12}^2, T_{13}^2, T_{23}^2, \det \mathbf{T}) = 1. \tag{4}$$

If the material has transversely isotropic symmetry about the x_1 material axis, then (3) is replaced by

$$f(T_{11}, T_{22} + T_{33}, T_{12}^2 + T_{13}^2, T_{23}^2 - T_{23}^{22}, \det \mathbf{T}) = 1 \tag{5}$$

and if the material has isotropic strength symmetry (3) is replaced by

$$f(T_{11} + T_{22} + T_{33}, T_{12}^2 + T_{13}^2 + T_{23}^2 - T_{11}T_{22} - T_{22}T_{33} - T_{11}T_{33}, \det \mathbf{T}) = 1. \tag{6}$$

The systematic approach of Tsai and Wu [11] and Wu [9] is to expand the function f in a polynomial in the components of stress. The first approximation for f is the polynomial that retains only the linear terms, the nth approximation for f is the polynomial that retains terms of order n and all lower order. It is assumed that the $n + 1$st approximation to f will represent the data better than the nth approximation. It is found that the quadratic approximation is a reasonable compromise between accuracy and the burden of an extensive test program. Expanding (4) in a polynomial of its components, but retaining only linear terms and terms of order two we find

$$a_1T_{11} + a_2T_{22} + a_3T_{33} + a_4T_{11}^2 + a_5T_{22}^2 + a_6T_{33}^2 + a_7T_{12}^2 + a_8T_{13}^2 + a_9T_{23}^2 + a_{10}T_{11}T_{22} + a_{11}T_{11}T_{33} + a_{12}T_{22}T_{33} = 1, \tag{7}$$

where a_1 through a_{12} are constants. The constants a_1 through a_{12} can be determined by mechanical strength tests described by Wu [9]; thus (7) can be written

$$\left(\frac{1}{\sigma_1^+} - \frac{1}{\sigma_1^-}\right) T_{11} + \left(\frac{1}{\sigma_2^+} - \frac{1}{\sigma_2^-}\right) T_{22} + \left(\frac{1}{\sigma_3^+} - \frac{1}{\sigma_3^-}\right) T_{33} + \frac{T_{11}^2}{\sigma_1^+\sigma_1^-} + \frac{T_{22}^2}{\sigma_2^+\sigma_2^-} + \frac{T_{33}^2}{\sigma_3^+\sigma_3^-} + \frac{T_{12}^2}{\sigma_{12}^2} + \frac{T_{13}^2}{\sigma_{13}^2} + \frac{T_{23}^2}{\sigma_{23}^2} + 2F_{12}T_{11}T_{22} + 2F_{13}T_{11}T_{33} + 2F_{23}T_{22}T_{33} = 1, \tag{8}$$

where σ_1^+, σ_2^+, σ_3^+ and σ_1^-, σ_2^-, σ_3^- are the tensile and compressive strengths in the x_1, x_2, x_3-directions, respectively; σ_{12}, σ_{13}, and σ_{23} are the shear strengths and F_{12}, F_{13}, and F_{23} are quantities that should be determined in a biaxial normal stress situation and are called stress interaction coefficients. The optimal experimental determination of F_{12}, F_{13}, and F_{23} is discussed by Wu [13]. The shear strength on the i plane in the j-direction is denoted by σ_{ij}. It should be noted that the shear strengths are not necessarily symmetric in these indices like shear stresses

$$\sigma_{ij} \neq \sigma_{ji}, \quad i \neq j. \tag{9}$$

It follows then that there are actually six shear strengths for an orthotropic material. However, since the actual shear stress is symmetric it really does not matter if $\sigma_{12} > \sigma_{21}$, $\sigma_{13} > \sigma_{31}$ and $\sigma_{23} > \sigma_{32}$ or $\sigma_{21} > \sigma_{12}$, $\sigma_{31} > \sigma_{13}$ and $\sigma_{32} > \sigma_{23}$ since failure will occur on the weakest of the two mutually perpendicular planes. The material could be constrained to fail on the shear plane where the shear strength is higher, but it will be assumed that this is not the case. Henceforth the notation σ_{ij} will be used to denote the lesser of σ_{ij} and σ_{ji}.

In anticipation of a result to be obtained later the quantities σ_1 and δ_1 defined by

$$\sigma_1 = \sqrt{\sigma_1^+\sigma_1^-}, \quad \frac{1}{\delta_1} = \frac{1}{\sigma_1^+} - \frac{1}{\sigma_1^-}, \tag{10}$$

are introduced. σ_1^+ and σ_1^- are given in terms of σ_1 and δ_1 by

$$\frac{1}{\sigma_1^+} = \frac{1}{2\delta_1} + \sqrt{\frac{1}{4\delta_1^2} + \frac{1}{\sigma_1^2}}, \qquad \frac{1}{\sigma_1^-} = -\frac{1}{2\delta_1} + \sqrt{\frac{1}{4\delta_1^2} + \frac{1}{\sigma_1^2}}. \tag{11}$$

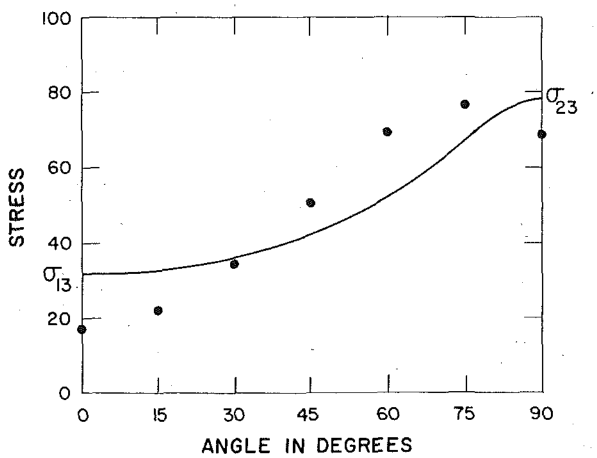

Fig. 1 A plot of equation (24) for the shear strength as a function of angle to the grain. The circles represent experimental data on pine from [14]. The units of stress are percentages of the ultimate compressive strength

The quantities σ_2, σ_3 and δ_2, δ_3 are similarly defined and analogous formulas can be obtained. Using these quantities the formula (8) representing the quadratic approximation of the orthotropic strength theory can be rewritten as

$$\frac{T_{11}}{\delta_1}+\frac{T_{22}}{\delta_2}+\frac{T_{33}}{\delta_3}+\frac{T_{11}^2}{\sigma_2^1}+\frac{T_{22}^2}{\sigma_2^2}+\frac{T_{33}^2}{\sigma_3^2}+\frac{T_{12}^2}{\sigma_{12}^2}+\frac{T_{13}^2}{\sigma_{13}^2}+\frac{T_{23}^2}{\sigma_{23}^2} + 2F_{12}T_{11}T_{22}+2F_{13}T_{11}T_{33}+2F_{23}T_{22}T_{33}=1. \quad (12)$$

In order that the surface described by (12) be closed, that is to say topologically equivalent to a sphere in six dimensions, Tsai and Wu [11] note that it must satisfy the following conditions:

$$F_{12}^{-2}>\sigma_1^2\sigma_2^2, \quad F_{13}^{-2}>\sigma_1^2\sigma_3^2, \quad F_{23}^{-2}>\sigma_2^2\sigma_3^2. \quad (13)$$

These conditions insure that, for any two-dimensional stress state, (12) will be an ellipsoid in the stress space whose axes are the two nonzero normal stresses and the shear stress.

There are three special cases of (12) that are of interest, the case where the symmetry increases from orthotropic to transversely isotropic, the case where it increases from orthotropy to isotropy, and the case where it is isotropic and, in addition, satisfies the condition that it be independent of the hydrostatic state of stress. For transversely isotropic materials with an axis of symmetry in the x_1-direction the function f that must be considered is (5) rather than (4) and a similar analysis leads to an equation of the form (12) in which

$$\delta_2=\delta_3=\delta, \text{ say;} \quad \sigma_2=\sigma_3=\sigma, \text{ say;} \quad \sigma_{13}=\sigma_{12}=\tau, \text{say;}$$

$$F_{12}=F_{13}=F, \text{ say;} \quad F_{23}=\frac{1}{\sigma^2}-\frac{1}{2\sigma_{23}^2}. \quad (14)$$

For isotropic materials a similar argument leads to an equation of the form (12) in which

$$\delta_1=\delta_2=\delta_3=\delta, \quad \sigma_1=\sigma_2=\sigma_3=\sigma, \quad \sigma_{13}=\sigma_{23}=\sigma_{12}=\tau$$

$$F_{12}=F_{13}=F_{23}=\frac{1}{\sigma^2}-\frac{1}{2\tau^2}. \quad (15)$$

Finally, if it is required that the strength criterion be independent of the hydrostatic pressure as well as reflecting the full isotropic symmetry, an equation of the form (12) is obtained in which

$$\delta_1^{-1}=\delta_2^{-1}=\delta_3^{-1}=0, \quad \sigma_1=\sigma_2=\sigma_3=\sigma, \quad \sigma_{13}=\sigma_{23}=\sigma_{12}=\frac{\sigma}{\sqrt{3}},$$

$$F_{12}=F_{13}=F_{23}=-\frac{1}{2\sigma^2}. \quad (16)$$

Equation (12) with the coefficients (16) is the well-known von Mises yield criterion for isotropic metals.

Formulas for the Shear Strength on Planes That Are Not Planes of Material Symmetry

Let σ_{nm} denote the shear strength in the $\mathbf{n}$-direction on a plane whose normal is in the $\mathbf{m}$-direction, where $\mathbf{n}$ and $\mathbf{m}$ are orthogonal unit vectors,

$$\mathbf{n}\cdot\mathbf{m}=n_1m_1+n_2m_2+n_3m_3=0. \quad (17)$$

In keeping with the convention introduced previously, σ_{nm} is the lesser of two shear strengths, the one in the $\mathbf{n}$-direction on the plane whose normal is $\mathbf{m}$ and the one in the $\mathbf{m}$-direction on the plane whose normal is in the $\mathbf{n}$-direction. The problem considered here is that of determining σ_{nm} from $\mathbf{n}$, $\mathbf{m}$ and the strengths along the axes of material symmetry: $\sigma_1{}^+, \ldots, \sigma_1{}^-, \ldots, \sigma_{13}, \ldots, F_{12} \ldots$. The situation considered is that in which the orthotropic material is subjected to an ultimate stress state by the application of σ_{nm} only. That is to say the only stress at a point in the body is the stress σ_{nm} and, using the tensorial law for stress transformation, this stress state may be written relative to the axes of material symmetry as

$$T_{11}=2\sigma_{nm}n_1m_1, \quad T_{22}=2\sigma_{nm}n_2m_2, \quad T_{33}=2\sigma_{nm}n_3m_3$$

$$T_{12}=\sigma_{nm}(m_1n_2+n_1m_2), \quad T_{23}=\sigma_{nm}(m_2n_3+n_2m_3),$$

$$T_{13}=\sigma_{nm}(m_1n_3+n_1m_3). \quad (18)$$

A formula for σ_{nm} is obtained by substitution of (18) into the quadratic form of the strength criterion (12), thus

$$A\sigma_{nm}+B\sigma_{nm}^2=1 \quad (19)$$

or

$$\sigma_{nm}=-\frac{A}{2B}\pm\sqrt{\frac{1}{B}+\left(\frac{A}{2B}\right)^2} \quad (20)$$

where

$$A=2\left(\frac{n_1m_1}{\delta_1}+\frac{n_2m_2}{\delta_2}+\frac{n_3m_3}{\delta_3}\right) \quad (21)$$

and

$$B=4\left(\frac{n_1^2m_1^2}{\sigma_1^2}+\frac{4n_2^2m_2^2}{\sigma_2^2}+\frac{4n_3^2m_3^2}{\sigma_3^2}\right)+\frac{(m_1n_2+n_1m_2)^2}{\sigma_{12}^2}+\frac{(m_2n_3+n_2m_3)^2}{\sigma_{23}^2}+\frac{(m_1n_3+n_1m_3)^2}{\sigma_{13}^2}+8(F_{12}n_1m_1n_2m_2+F_{13}n_1m_1n_3m_3+F_{23}n_2m_2n_3m_3). \quad (22)$$

An interesting special case of the formula (19) occurs when one of the directions, $\mathbf{n}$ or $\mathbf{m}$, is an axis of material symmetry and the other direction lies entirely in the perpendicular plane, for example, the case where

$$m_1=m_2=0, \quad m_3=1; \quad n_1=\cos\theta, \quad n_2=\sin\theta, \quad n_3=0. \quad (23)$$

If, in this special situation, the notation τ_θ for σ_{nm} is introduced, then (19) yields the following formula for τ_θ;

$$\tau_\theta^2=\frac{\sigma_{13}^2\sigma_{23}^2}{\sigma_{13}^2\sin^2\theta+\sigma_{23}^2\cos^2\theta}. \quad (24)$$

This formula relates the shear strength on the θ plane to the shear strengths σ_{13} and σ_{23} and it shows that the square of the shear stress follows a Hankinson-type strength criterion. The only experimental data on shear strength as a function of angle to the grain located in the literature is the data on pine given by Ashkenazi [14]. These data are difficult to determine from the paper of Ashkenazi because they are presented in an extremely small figure as a percentage of the ultimate compressive strength. A least-squares fit of (24) to these data yields a value of σ_{13} as 32.6 percent and σ_{23} as 77.6 percent of the ultimate compressive strength. A plot of (24) with these values for σ_{13} and σ_{23} is shown in Fig. 1 along with the data from [14]. The theory represents the general trend of the data quite well.

Formulas for Strength on Planes That Are Not Planes of Material Symmetry

Let s_n denote the tensile or compressive strength in the **n**-direction where **n** does not, in general, coincide with any of the axes of material symmetry. The problem considered in this section is that of determining s_n from **n** and the strengths along the axes of material symmetry; $\sigma_1{}^+, \ldots, \sigma_1{}^-, \ldots, \sigma_{13}, \ldots$, and F_{12}, F_{13}, F_{23}. The situation considered is that in which the orthotropic material is subjected to an ultimate stress state by the application of s_n only. That is to say, the only stress at a point in the material body is the normal stress s_n and, again using the tensorial law for stress transformation, this stress state may be written relative to the axes of material symmetry as

$$T_{11} = s_n n_1^2, \quad T_{22} = s_n n_2^2, \quad T_{33} = s_n n_3^2,$$
$$T_{12} = s_n n_1 n_2, \quad T_{13} = s_n n_1 n_3, \quad T_{23} = s_n n_2 n_3. \tag{25}$$

A formula for s_n is obtained by substituting (25) into (12) and solving the resulting quadratic equation, thus

$$(s_n - \sigma_n{}^+)(s_n + \sigma_n{}^-) = 0, \tag{26}$$

where

$$\sigma_n^+ = \frac{2\delta_n \sigma_n}{\sqrt{\sigma_n^2 + 4\delta_n^2} + \sigma_n}, \quad \sigma_n^- = \frac{2\delta_n \sigma_n}{\sqrt{\sigma_n^2 + 4\delta_n^2} - \sigma_n}, \tag{27}$$

and δ_n and σ_n are given by

$$\frac{1}{\delta_n} = \frac{n_1^2}{\delta_1} + \frac{n_2^2}{\delta_2} + \frac{n_3^2}{\delta_3}, \tag{28}$$

$$\frac{1}{\sigma_n^2} = \frac{n_1^4}{\sigma_1^2} + \frac{n_2^4}{\sigma_2^2} + \frac{n_3^4}{\sigma_3^2} + \frac{n_1^2 n_2^2}{\sigma_{12}^2} + \frac{n_1^2 n_3^2}{\sigma_{13}^2} + \frac{n_2^2 n_3^2}{\sigma_{13}^2} + 2(F_{12} n_1^2 n_2^2 + F_{13} n_1^2 n_3^2 + F_{23} n_2^2 n_3^2). \tag{29}$$

In the special case when **n** lies completely in one plane of symmetry, for example,

$$n_1 = \cos\theta, \quad n_2 = \sin\theta, \quad n_3 = 0, \tag{30}$$

the formula for $\sigma_n{}^2$, (29), can be written as

$$\sigma_n^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 \sin^4\theta + \sigma_2^2 \cos^4\theta + \sigma_1^2 \sigma_2^2 \left(2F_{12} + \frac{1}{\sigma_{12}^2}\right) \sin^2\theta \cos^2\theta}. \tag{31}$$

In the case when F_{12} is given by

$$F_{12} = \frac{1}{\sigma_1 \sigma_2} - \frac{1}{2\sigma_{12}^2}, \tag{32}$$

the expression (31) is a perfect square of a formula similar to Hankinson's formula (1). Since much of the data on the strength of bone and wood can be represented by Hankinson's criterion (1) reasonably well, there is an implication that F_{12} is given approximately by (32). It will be shown below that the data on bone satisfies this condition roughly.

The results just obtained will now be compared and fitted to the data on bone strength reported by Reilly [6] and Reilly and Burstein [7]. This bone strength data is shown in Table 1. Reilly and Burstein assumed that bone was transversely isotropic, so their data for $\sigma_{30°}{}^+$, $\sigma_{30°}{}^-$, $\sigma_{60°}{}^+$ and $\sigma_{60°}{}^-$ was not all in one plane of symmetry, but it was in a variety of directions making angles of 30° and 60° with the long axis of the bone. Since Reilly and Burstein did not measure F_{12} their data cannot be compared with the formula (31) directly. However, the data of Reilly and Burstein can be used to estimate a value of F_{12} for both types of bone studied. In fact, two predictions of F_{12} are possible for each bone type because the tensile and compressive strengths were determined for both types of bone at angles to the grain of 30° and 60°. Substitution of the data from Table 1 into the formula (31) yields for the bovine Haversian bone

$$F_{12} = -\frac{1}{(147.2)^2} \quad \text{at} \quad \theta = 30°, \qquad F_{12} = -\frac{1}{(141.8)^2} \quad \text{at} \quad \theta = 60°, \tag{33}$$

Table 1 Table of bone strengths from reference [6]. All entries are stresses in units of MPa; the numbers in parentheses are the standard deviations

	Bovine Haversian Femur	Human Femur
σ_1^+	144 (6.2)	132 (16)
σ_1^-	254 (25.5)	187 (28.8)
σ_2^+	46 (7.1)	58 (5.5)
σ_2^-	146 (21.8)	132 (11.4)
σ_{12}	64 (6.9)	67 (3.7)
$\sigma_{30°}^+$	99 (7.5)	100 (8.8)
$\sigma_{30°}^-$	190 (12.7)	173 (13.8)
$\sigma_{60°}^+$	60 (4.1)	60.5 (4.8)
$\sigma_{60°}^-$	148 (2.5)	133 (15.0)

and for the human femoral bone

$$F_{12} = -\frac{1}{(158.5)^2} \quad \text{at} \quad \theta = 30°,$$
$$F_{12} = +\frac{1}{(240.1)^2} \quad \text{at} \quad \theta = 60°, \tag{34}$$

where F_{12} is in units of $(\text{MPa})^{-2}$, where MPa stands for mega pascals, one mega pascal is equal to one $\text{N/m}^2 \times 10^6$ or 145 psi. These results show considerable consistency for the bovine Haversian bone as both data points predict almost the same value of F_{12}, a value not inconsistent with the value $-(131\ \text{MPa})^{-2}$ estimated by (32). The same degree of consistency is not so apparent for the various predictions of F_{12} for human femoral bone. The value of F_{12} given by (32) is $-(160.9\ \text{MPa})^{-2}$. This number does coincide rather well with the prediction of F_{12} by the 30° data previously given, but it diverges considerably from the prediction of F_{12} based on the 60° data previously given, this latter value being of opposite sign. There are numerous potential rationales for this discrepancy, in addition to the fact that the quadratic strength theory may not be sufficiently flexible to represent the bone strength data. For example, bone may be orthotropic in strength rather than transversely isotropic as Reilly and Burstein [7] assumed when they took their specimens. It is also possible that, since bovine femurs are much larger than human femurs, the bovine test specimens are more homogeneous than the human and therefore have less scatter in the data.

The predicted tensile and compressive strengths, $\sigma_n{}^+$ and $\sigma_n{}^-$ for bovine and human femoral bone will be plotted as a function of grain angle for four cases. The case designated as Case 4 is Hankinson's empirical formula (1) where, for tensile strengths, the minus signs in the formulas are replaced by plus signs. The other three cases are all forms of the formulas (27) with different values of F_{12} being used in the determination of σ_n by (31). In the first case the value of F_{12} is determined from the experimental data at the 30° grain angle and, in the second case, at the 60° grain angle. In the third case F_{12} is determined from the experimental data in Table 1 using the formula (32). For each of these three cases σ_n is computed using the formula (31) and δ_n is computed from

$$\frac{1}{\delta_n} = \frac{\cos^2\theta}{\delta_1} + \frac{\sin^2\theta}{\delta_2} \tag{35}$$

which follows from (28) and (30). The numerical values of σ_n and δ_n

Table 2 Table of tensile and compressive strength of bovine Haversian femur. All entries are stresses in units of MPa

		σ_n			σ_n^+				σ_n^-			
θ	δ_n	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
0°	332.5	191.2	191.2	191.2	144.0	144.0	144.0	144.0	253.9	253.9	253.9	254.0
15°	262.9	171.6	172.7	175.5	124.5	125.1	126.4	126.0	236.5	238.5	243.6	242.0
30°	167.3	137.2	138.9	143.4	92.0	92.7	94.6	94.0	204.5	208.0	217.5	214.4
45°	111.7	110.4	111.7	114.8	68.6	69.0	70.1	69.7	177.7	180.7	188.1	185.4
60°	83.9	93.7	94.3	95.7	55.0	55.2	55.6	55.4	159.6	161.2	164.7	163.4
75°	71.0	84.8	84.9	85.3	48.1	48.1	48.3	48.2	149.4	149.7	150.7	150.3
90°	67.2	82.0	82.0	82.0	46.0	46.0	46.0	46.0	146.1	146.1	146.1	146.0

Table 3 Table of tensile and compressive strength of human femur. All entries are stresses in units of MPa

		σ_n			σ_n^+				σ_n^-			
θ	δ_n	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 4	Case 1	Case 2	Case 3	Case 4
0°	448.8	157.1	157.1	157.1	132.0	132.0	132.0	132.0	187.0	187.0	187.0	187.0
15°	366.8	149.4	138.7	149.1	122.0	114.9	121.8	121.6	182.9	167.4	182.4	181.9
30°	244.6	131.5	112.3	131.0	100.8	89.4	100.5	100.1	171.5	141.0	170.7	169.4
45°	168.2	112.8	96.6	112.4	81.1	72.8	81.0	80.6	156.8	128.2	156.1	154.7
60°	128.1	98.6	89.7	98.4	67.7	63.6	67.6	67.5	143.6	126.4	143.2	142.5
75°	109.1	90.2	87.7	90.2	60.3	59.3	60.3	60.3	134.9	129.8	134.9	134.7
90°	103.5	87.5	87.5	87.5	58.0	58.0	58.0	58.0	132.0	132.0	132.0	132.0

for various grain angles are given in Table 2 for bovine bone and in Table 3 for human bone. The tensile strength $\sigma_n{}^+$ and the compressive strength $\sigma_n{}^-$ are determined using (27) from the values of σ_n and δ_n. These numerical results are presented in Tables 2 and 3. The results are represented graphically in Figs. 2–4. The bovine tensile strength and the bovine compressive strength are plotted as a function of grain angle in Figs. 2 and 3, respectively. Fig. 2 shows that the bovine tensile strength curves for Case 1 (where F_{12} is given by $(33)_1$) and Case 2 (where F_{12} is given by $(33)_2$) are so close that they are indistinguishable. They are also very close for the bovine compressive strength as one can see from Fig. 3. For both bovine tensile and bovine compressive strengths, Case 3 (where F_{12} is determined from the data in Table 1 using the formula (32)) and Case 4 (Hankinson's formula) predict greater strengths. It is only for bovine compressive strength that the Cases 3 and 4 are distinguishable. For all other cases, including human femoral tensile and compressive, these two cases coincide. Thus, for practical purposes, a curve approximating the curve obtained from Hankinson's formula can be obtained from the strength theory by assuming F_{12} to be given by (32). For human femoral tensile and compressive strengths, Cases 3 and 4 lie below

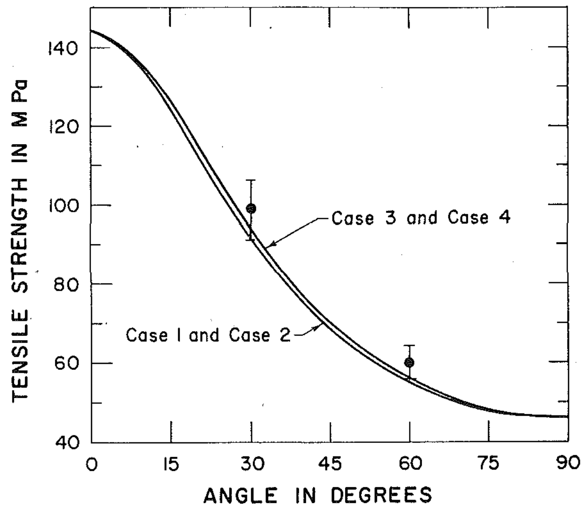

Fig. 2 The tensile strength of bovine bone as a function of angle to the grain. Equation $(27)_1$ is plotted using the experimental data on bovine bone from [6]

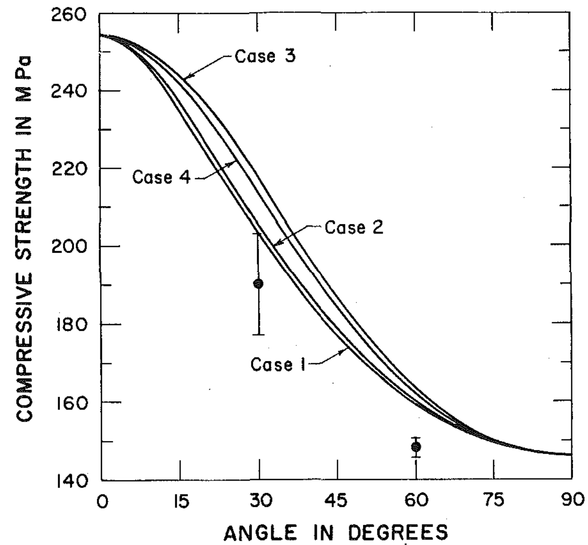

Fig. 3 The compressive strength of bovine bone as a function of angle to the grain. Equation $(27)_2$ is plotted using the experimental data on bovine bone from [6]

Case 1, but above Case 2 as shown in Fig. 4. The data from Table 1 for the tensile and compressive strengths at grain angles of 30° and 60° are shown in Figs. 2–4. Each data point is plotted as a dot with a vertical line passing through it. The vertical line begins one standard deviation below the datum and ends one standard deviation above it. The quadratic strength theory, Cases 1 and 2, is seen to give consistently a better representation of the data than Hankinson's formula, Case 4, and its approximation based on the quadratic strength theory, Case 3.

Conclusion

The initial objective of this study, namely, the derivation of Hankinson's formula from a complete strength theory, has been realized. From the quadratic strength theory for orthotropic materials, formulas for the shear strength, the tensile strength, and the compressive strength as functions of the angle to the grain have been obtained. The formulas for the tensile and compressive strength as a function of the angle to the grain have a greater flexibility than Hankinson's formula and they represent the data better. More importantly, they are related to a general theory of strength rather than being empirically based.

Certain conclusions that apply only to bone can be made. Most experimental studies of the mechanical properties of bone have indicated that it is satisfactory to consider bone to be a transversely isotropic material. If (14) is substituted into (8) the following strength criterion for transversely isotropic materials is obtained:

$$\left(\frac{1}{\sigma_1{}^+} - \frac{1}{\sigma_1{}^-}\right) T_{11} + \left(\frac{1}{\sigma^+} - \frac{1}{\sigma^-}\right)(T_{22} + T_{33}) + \frac{T_{11}^2}{\sigma_1{}^+\sigma_1{}^-} + \frac{(T_{22}+T_{33})^2}{\sigma^+\sigma^-} + \frac{1}{\tau^2}(T_{12}^2 + T_{13}^2) + \frac{1}{\sigma_{23}^2}(T_{23}^2 - T_{22}T_{33}) + 2F(T_{22}+T_{33})T_{11} = 1. \quad (36)$$

The analysis of this paper shows that this strength criterion is consistent with the Tsai-Wu theory and with the experimental data on bone. It has also been shown that (36) predicts a Hankinson-type formula for the shear strength dependence on grain angle and that (36) can improve upon the prediction of the original Hankinson formula (1) for ultimate strengths.

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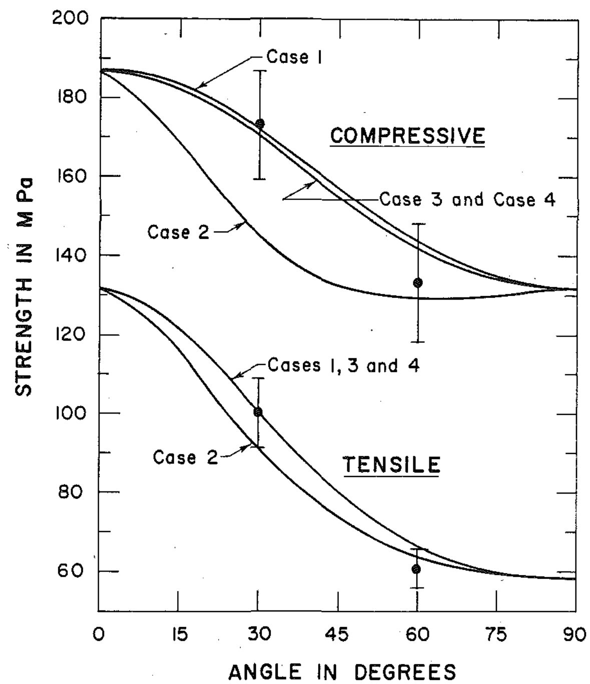

Fig. 4 The tensile and compressive strength of human femoral bone as a function of angle to the grain. Equation (27) is plotted using the experimental data on human femoral bone from [6]

APPENDIX

A Derivation of Hankinson's Formula

Hankinson's formula (1) can be derived from the first or linear approximation to the strength theory for orthotropic materials. This linear approximation can be determined by inspection from (7), thus

$$a_1 T_{11} + a_2 T_{22} + a_3 T_{33} = 1. \tag{37}$$

The coefficients a_1, a_2, and a_3 must be determined differently in each of the eight octants of the Cartesian normal stress space. In each octant a different plane will be determined and the resulting surface in the three-dimensional normal stress space will be an octahedron. If an orthotropic material is subjected to a tensile test in the x_1-direction and its ultimate strength is found to be σ_1^+, then since all the stresses in (37) excpet T_{11} are zero, it follows from (37) that a_1^{-1}, is σ_1^+. In a similar way the coefficients a_2^{-1} and a_3^{-1} are found to be σ_2^+ and σ_3^+ in the first octant. This argument must be modified for each octant of the three-dimensional Cartesian normal stress space. Thus, for example, in an octant in which T_{11} is negative the coefficient a_1^{-1} will be $-\sigma_1^-$. It follows then that in the octant where all the normal stresses are compressive (37) has the form:

$$\frac{T_{11}}{\sigma_1^-} + \frac{T_{22}}{\sigma_2^-} + \frac{T_{33}}{\sigma_3^-} = -1. \tag{38}$$

Let σ_θ^- denote the compressive strength in the direction making an angle θ with the x_1-axis in the x_1, x_2 plane. The problem considered here is then to express σ_θ^- in terms of σ_1^-, σ_2^-, σ_3^-, and θ. Observing that a uniaxial compressive stress in the θ-direction of magnitude σ_θ^- corresponds to the following stresses in the x_1, x_2, x_3 material coordinate system

$$T_{11} = -\sigma_\theta^- \cos^2\theta, \quad T_{22} = -\sigma_\theta^- \sin^2\theta,$$
$$T_{12} = -\sigma_\theta^- \sin\theta\cos\theta, \quad T_{33} = T_{23} = T_{13} = 0, \tag{39}$$

Hankinson's formula (1) follows from (38).

The fact that the linear approximation of the strength theory for orthotropic materials predicts Hankinson's formula and the fact that many investigators report that the experimental data on bone and wood strength follows Hankinson's formula should not be combined to conclude that the linear approximation of the strength theory for orthotropic materials will suffice for bone and wood. The linear approximation has many deficiencies, primary among them is the fact that it does not account for the shear strength nor the normal stress interaction. It has been shown in the text that the quadratic approximation yields formulas similar in predictions to Hankinson's which can be made to fit the same data.

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