A note on intermittency for the fractional heat equation

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July 31, 2014

Abstract

The goal of the present note is to study intermittency properties for the solution to the fractional heat equation

$$\frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\beta/2} u(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x \in \mathbb{R}^d$$

with initial condition bounded above and below, where $\beta \in (0,2]$ and the noise W behaves in time like a fractional Brownian motion of index $H > 1/2$, and has a spatial covariance given by the Riesz kernel of index $\alpha \in (0,d)$. As a by-product, we obtain that the necessary and sufficient condition for the existence of the solution is $\alpha < \beta$.

MSC 2010: Primary 60H15; secondary 37H15, 60H07

Keywords: fractional heat equation; fractional Brownian motion; Malliavin calculus; intermittency

1 Introduction

In this article we consider the fractional heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\beta/2} u(t,x) + u(t,x)\dot{W}(t,x), & t>0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1}$$

where $\beta \in (0,2]$, $(-\Delta)^{\beta/2}$ denotes the fractional power of the Laplacian, and u_0 is a deterministic function such that

$$a \le u_0(x) \le b \quad \text{for all} \quad x \in \mathbb{R}^d \tag{2}$$

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for some constants $b \geq a > 0$. We let $W = \{W(\varphi); \varphi \in \mathcal{H}\}$ be a zero-mean Gaussian process with covariance

$$E(W(\varphi)W(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

Here $\mathcal{H}$ is a Hilbert space defined as the completion of the space $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ of infinitely differentiable functions with compact support on $\mathbb{R}_+ \times \mathbb{R}^d$, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \varphi(t,x)\psi(s,y)|t-s|^{2H-2}|x-y|^{-\alpha} dt\, dx\, ds\, dy, \qquad (3)$$

where $\alpha_H = H(2H-1)$, $H \in (1/2, 1)$ and $\alpha \in (0, d)$. We denote by $\dot{W}$ the formal derivative of W. The noise W is spatially homogeneous with spatial covariance given by the Riesz kernel $f(x) = |x|^{-\alpha}$ and behaves in time like a fractional Brownian motion of index H. We refer to [2, 3, 5] for more details.

Let $G(t,x)$ be the fundamental solution of $\frac{\partial u}{\partial t} + (-\Delta)^{\beta/2} u = 0$ and

$$w(t,x) = \int_{\mathbb{R}^d} u_0(y) G(t, x-y) dy$$

be the solution of the equation $\frac{\partial u}{\partial t} + (-\Delta)^{\beta/2} u = 0$ with initial condition $u(0,x) = u_0(x)$. Note that

$$G(t,\cdot) \text{ is the density of } X_t \qquad (4)$$

where $X = (X_t)_{t \geq 0}$ is a symmetric Lévy process with values in $\mathbb{R}^d$. If $\beta = 2$, then X coincides with a Brownian motion $B = (B_t)_{t \geq 0}$ in $\mathbb{R}^d$ with variance 2. If $\beta < 2$, then X is a β-stable Lévy process given by $X_t = B_{S_t}$, where $(S_t)_{t \geq 0}$ is a $(\beta/2)$-stable subordinator with Lévy measure

$$\nu(dx) = \frac{\beta/2}{\Gamma(1-\beta/2)} x^{-\beta/2-1} 1_{\{x>0\}} dx.$$

(See for instance the explanation on page 62 of [19] on how to construct a new Lévy process from a Wiener process using subordination, and in particular Example 4.38 of [19]). Due to (2) and (4), it follows that for all $t > 0$ and $x \in \mathbb{R}^d$,

$$a \leq w(t,x) \leq b. \qquad (5)$$

There is a rich literature dedicated to the case $H = 1/2$, when the noise W is white in time. We refer to [10, 13] for some general properties, and to [12, 9, 7] for intermittency properties of the solution to the heat equation with this type of noise. Different methods have to be used for $H > 1/2$, since in this case the noise is not a semi-martingale in time. The stochastic heat equation driven by a fractional noise in time with index $H \in (\frac{1}{4}, \frac{1}{2})$ was studied in [15], assuming that the noise has a γ-continuous spatial covariance function.

In the present article, we follow the approach of [16, 5] for defining the concept of solution. We say that a process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a *mild solution* of (1) if it is square-integrable, adapted with respect to the filtration induced by W, and satisfies:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta s, \delta y),$$

where the stochastic integral is interpreted as the divergence operator of W (see ([18]). Using Malliavin calculus techniques, it can be shown that the mild solution (if it exists) is unique and has the Wiener chaos decomposition:

$$u(t,x) = \sum_{n \geq 0} I_n(f_n(\cdot, t, x)) \tag{6}$$

where I_n denotes the multiple Wiener integral (with respect to W) of order n, and the kernel $f_n(\cdot, t, x)$ is given by:

$$f_n(t_1, x_1, \ldots, t_n, x_n, t, x) = \\ G(t-t_n, x-x_n) \ldots G(t_2 - t_1, x_2 - x_1) w(t_1, x_1) 1_{\{0 < t_1 < \ldots < t_n < t\}}$$

(see page 303 of [16]). By convention, $f_0(t,x) = w(t,x)$ and I_0 is the identity map on $\mathbb{R}$.

The necessary and sufficient condition for the existence of the mild solution is that the series in (6) converges in $L^2(\Omega)$, i.e.

$$S(t,x) := \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t,x) < \infty, \tag{7}$$

where

$$\alpha_n(t,x) = n! E|I_n(f_n(\cdot, t, x))|^2 = (n!)^2 \|\tilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$$

and $\tilde{f}_n(\cdot, t, x)$ is the symmetrization of $f_n(\cdot, t, x)$ in the n variables $(t_1, x_1), \ldots, (t_n, x_n)$. If the solution u exists, then $E|u(t,x)|^2 = S(t,x)$. We refer to Section 4.1 of [16] and Section 2 of [5] for the details. Note that if $u_0(x) = u_0$ for all $x \in \mathbb{R}^d$, then the law of $u(t,x)$ does not depend on x, and hence $\alpha_n(t,x) = \alpha_n(t)$.

The goal of the present work is to give an upper bound for the p-th moment of the solution of (1) (for $p \geq 2$), and a lower bound for its second moment. In particular, this will show that, if $u_0(x)$ does not depend on x, then the solution u of (1) is *weakly ρ-intermittent*, in a sense which has been recently introduced in [4], i.e. $\gamma_\rho(2) > 0$ and $\gamma_\rho(p) < \infty$ for all $p \geq 2$, where

$$\gamma_\rho(p) = \limsup_{t \to \infty} \frac{1}{t^\rho} \log E|u(t,x)|^p$$

is a modified Lyapunov exponent (which does not depend on x), and

$$\rho = \frac{2H\beta - \alpha}{\beta - \alpha}. \tag{8}$$

As a by-product, we obtain that the necessary and sufficient condition for the existence of the solution is $\alpha < \beta$. Note that this condition is equivalent to

$$I_\beta(\mu) := \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta/2} \mu(d\xi) < \infty \tag{9}$$

with $\mu(d\xi) = c_{\alpha,d}|\xi|^{-d+\alpha}d\xi$, which is encountered in the study of equations with white noise in time. When $\beta = 2$, (9) is called *Dalang's condition* (see [10]). (If $\beta/2 = k$ was a positive integer, (9) would coincide with condition (3.3) of [11].)

In the case $\beta = 2$, a lower bound for the p-the moment of the solution has been obtained in the recent preprint [14], for the equation interpreted in the Skorohod sense (as in the present paper), and also in the Stratonovich sense. The method of [14] is based on a Feynman-Kac (FK) type representation for the moments of the solution. A similar approach may work in the case $\beta < 2$, as this type of FK representations might still hold for the solution of the fractional heat equation, under some additional constraints on the parameters H and α of the noise. (This problem was considered in [8] for a noise with spatial covariance $f(x) = \prod_{i=1}^n |x_i|^{2H_i-2}$ with $H_i \in (\frac{1}{2}, 1)$.) We do not investigate this problem here.

2 The result

The goal of the present article is to prove the following result.

Theorem 2.1. *The necessary and sufficient condition for equation* (1) *to have a mild solution is $\alpha < \beta$. If the solution $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ exists, then for any $p \geq 2$, for any $x \in \mathbb{R}^d$ and for any $t > 0$ such that $pt^{2H-\alpha/\beta} > t_1$*

$$E|u(t,x)|^p \leq b^p \exp(C_1 p^{(2\beta-\alpha)/(\beta-\alpha)} t^\rho)$$

and for any $x \in \mathbb{R}^d$ and for any $t > t_2$,

$$E|u(t,x)|^2 \geq a^2 \exp(C_2 t^\rho),$$

where ρ is given by (8)*, a, b are the constants given by* (2)*, and t_1, t_2, C_1, C_2 are some positive constants depending on d, α, β and H.*

We suspect that the inequalities given by Theorem 2.1 cannot be improved, except for possibly different constants C_1 and C_2. This problem is not investigated in the present article.

Before giving the proof, we recall from [5] that

$$\alpha_n(t,x) = \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(\mathbf{t}, \mathbf{s}) d\mathbf{t} d\mathbf{s} \tag{10}$$

where

$$\psi_n(\mathbf{t}, \mathbf{s}) = \int_{\mathbb{R}^{2nd}} \prod_{j=1}^n |x_j - y_j|^{-\alpha} \tilde{f}_n(t_1, x_1, \ldots, t_n, x_n, t, x) \tilde{f}_n(s_1, y_1, \ldots, s_n, y_n, t, x) d\mathbf{x} d\mathbf{y}$$

and we denote $\mathbf{t} = (t_1, \ldots, t_n)$, $\mathbf{s} = (s_1, \ldots, s_n)$ with $t_i, s_i \in [0,t]$ and $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ with $x_i, y_i \in \mathbb{R}^d$.

Note that the Fourier transform of $G(t,\cdot)$ is given by:

$$\mathcal{F}G(t,\cdot)(\xi) := \int_{\mathbb{R}^d} e^{-i\xi\cdot x} G(t,x)dx = \exp(-t|\xi|^\beta), \quad \xi \in \mathbb{R}^d \tag{11}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. Recall that for any $\varphi, \psi \in L^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\psi(y)|x-y|^{-\alpha}dxdy = c_{\alpha,d}\int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}|\xi|^{-d+\alpha}d\xi \tag{12}$$

where $\mathcal{F}\varphi$ is the Fourier transform of φ, $c_{\alpha,d} = (2\pi)^{-d}C_{\alpha,d}$ and $C_{\alpha,d}$ is the constant given by (21) (see Appendix A). This identity can be extended to functions $\varphi, \psi \in L^1(\mathbb{R}^{nd})$:

$$\int_{\mathbb{R}^{nd}}\int_{\mathbb{R}^{nd}} \varphi(\mathbf{x})\psi(\mathbf{y})\prod_{j=1}^n |x_j-y_j|^{-\alpha}d\mathbf{x}d\mathbf{y} = \tag{13}$$

$$c_{\alpha,d}^n \int_{\mathbb{R}^{nd}} \mathcal{F}\varphi(\xi_1,\ldots,\xi_n)\overline{\mathcal{F}\psi(\xi_1,\ldots,\xi_n)}\prod_{j=1}^n |\xi_j|^{-d+\alpha}d\xi_1\ldots\xi_n.$$

We will use the following elementary inequality.

Lemma 2.2. *For any $t>0$ and $\eta \in \mathbb{R}^d$*

$$\int_{\mathbb{R}^d} e^{-t|\xi|^\beta}|\xi-\eta|^{-d+\alpha}d\xi \le K_{d,\alpha,\beta}t^{-\alpha/\beta}$$

where

$$K_{d,\alpha,\beta} := \sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{1+|\xi-\eta|^\beta}|\xi|^{-d+\alpha}d\xi.$$

Proof: Using the change of variable $z = t^{1/\beta}(\eta-\xi)$, we have:

$$\int_{\mathbb{R}^d} e^{-t|\xi|^\beta}|\xi-\eta|^{-d+\alpha}d\xi = t^{-\alpha/\beta}\int_{\mathbb{R}^d} e^{-|z-t^{1/\beta}\eta|^\beta}|z|^{-d+\alpha}dz.$$

The result follows using the inequality $e^{-x} \le 1/(1+x)$ for $x>0$. $\Box$

Proof of Theorem 2.1: *Step 1. (Sufficiency and upper bound for the second moment)* Suppose that $\alpha < \beta$. We will prove that the series (7) converges, by providing upper bounds for $\psi_n(\mathbf{t},\mathbf{s})$ and $\alpha_n(t,x)$.

By the Cauchy-Schwarz inequality, $\psi_n(\mathbf{t},\mathbf{s}) \le \psi_n(\mathbf{t},\mathbf{t})^{1/2}\psi_n(\mathbf{s},\mathbf{s})^{1/2}$. So it is enough to consider the case $\mathbf{t}=\mathbf{s}$. Let $u_j = t_{\sigma(j+1)} - t_{\sigma(j)}$ where σ is a permutation of $\{1,\ldots,n\}$ such that $t_{\sigma(1)} < \ldots < t_{\sigma(n)}$ and $t_{\sigma(n+1)} = t$. Using (5), (11) and (13), and arguing as in the proof of Lemma 3.2 of [3], we obtain:

$$\psi_n(\mathbf{t},\mathbf{t}) \le b^2 c_{\alpha,d}^n \int_{\mathbb{R}^d} d\eta_1 \exp(-u_1|\eta_1|^\beta)|\eta_1|^{-d+\alpha}\int_{\mathbb{R}^d} d\eta_2 \exp(-u_2|\eta_2|^\beta)|\eta_2-\eta_1|^{-d+\alpha}$$

$$\dots \int_{\mathbb{R}^d} d\eta_n \exp(-u_n|\eta_n|^\beta)|\eta_n - \eta_{n-1}|^{-d+\alpha}.$$

(Since Lemma 3.2 of [3] refers to the wave equation, the argument has to be adjusted by replacing the Fourier transform $\mathcal{F}G_{\rm w}(t,\cdot)(\xi) = \sin(t|\xi|)/|\xi|$ of the fundamental solution $G_{\rm w}$ of the wave equation by (11).) By Lemma 2.2, it follows that:

$$\psi_n(\mathbf{t},\mathbf{t}) \leq b^2 c_{\alpha,d}^n K_{d,\alpha,\beta}^n (u_1 \dots u_n)^{-\alpha/\beta}.$$

By inequality (26) (Appendix A), $K_{d,\alpha,\beta} \leq I_{d,\alpha,\beta}$, where

$$I_{d,\alpha,\beta} := \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{\beta/2} |\xi|^{-d+\alpha} d\xi = \frac{(2\pi)^d c_d \Gamma((\beta-\alpha)/2)\Gamma(\alpha/2)}{2\Gamma(\beta/2)}$$

(see relation (24) and Remark A.3, Appendix A). Hence,

$$\psi_n(\mathbf{t},\mathbf{s}) \leq b^2 C_{d,\alpha,\beta}^n [\beta(\mathbf{t})\beta(\mathbf{s})]^{-\alpha/(2\beta)}$$

where $\beta(\mathbf{t}) = u_1 \dots u_n$, $\beta(\mathbf{s})$ is defined similarly, and $C_{d,\alpha,\beta} > 0$ is a constant depending on d, α, β. Similarly to the proof of Proposition 3.6 of [5], we have:

$$\alpha_n(t,x) \leq b^2 C_{d,\alpha,\beta,H}^n (n!)^{\alpha/\beta} t^{n(2H-\alpha/\beta)}, \tag{14}$$

where $C_{d,\alpha,\beta,H} > 0$ is a constant depending on d, α, β, H. (The only change compared to the proof mentioned above is the fact that 2β was equal to 4 in [5]. These authors worked with a different parametrization: their $d - \alpha$ is denoted here by α.)

Since $\alpha < \beta$, it follows that the series (7) converges and

$$E|u(t,x)|^2 = \sum_{n\geq 0} \frac{1}{n!}\alpha_n(t,x) \leq b^2 \sum_{n\geq 0} \frac{C_{d,\alpha,\beta,H}^n}{(n!)^{1-\alpha/\beta}} t^{n(2H-\alpha/\beta)} \leq b^2 \exp(C_0 t^\rho),$$

for all $t > t_0$, where $C_0 > 0$ and $t_0 > 0$ are constants depending in d, α, β, H. We used the fact that for any $a > 0$ and $x > 0$,

$$\sum_{n\geq 0} \frac{x^n}{(n!)^a} \leq \exp(c_0 x^{1/a}) \quad \text{for all} \quad x > x_0, \tag{15}$$

where $x_0 > 0$ and $c_0 > 0$ are some constants depending on a (see e.g. Lemma A.1 of [4]).

Step 2. (Upper bound for the p-th moment) Note that $u(t,x) = \sum_{n\geq 0} J_n(t,x)$ in $L^2(\Omega)$, where $J_n(t,x)$ lies in the n-th order Wiener chaos $\mathcal{H}_n$ associated to the Gaussian process W (see [18]). Hence,

$$E|u(t,x)|^2 = \sum_{n\geq 0} E|J_n(t,x)|^2 = \sum_{n\geq 0} \frac{1}{n!}\alpha_n(t,x).$$

We denote by $\|\cdot\|_p$ the $L^p(\Omega)$-norm. We use the fact that for a *fixed* Wiener chaos $\mathcal{H}_n$, the $\|\cdot\|_p$ are equivalent, for all $p \geq 2$ (see the last line of page 62 of [18] with $q = p$ and $p = 2$). Hence,

$$\begin{aligned}\|J_n(t,x)\|_p &\leq (p-1)^{n/2}\|J_n(t,x)\|_2 = (p-1)^{n/2}\left(\frac{1}{n!}\alpha_n(t,x)\right)^{1/2} \\ &\leq b[(p-1)C_{d,\alpha,\beta,H}]^{n/2}\frac{1}{(n!)^{(\beta-\alpha)/(2\beta)}}t^{n(2H\beta-\alpha)/(2\beta)}\end{aligned}$$

using (14) for the last inequality. Using Minkowski's inequality for integrals (see Appendix A.1 of [20]) and inequality (15), we obtain that:

$$\|u(t,x)\|_p \leq \sum_{n\geq 0}\|J_n(t,x)\|_p \leq b\exp(C_1(p-1)^{\beta/(\beta-\alpha)}t^{\rho})$$

if $pt^{2H-\alpha/\beta} > t_1$, where the constants $C_1 > 0$ and $t_1 > 0$ depend on d, α, β, H.

Step 3. (Necessity and lower bound for the second moment) Suppose that equation (1) has a mild solution u, i.e. the series (7) converges. In particular,

$$\begin{aligned}\infty > \alpha_1(t,x) &\geq a^2\alpha_H\int_{[0,t]^2}\int_{\mathbb{R}^{2d}}|r-s|^{2H-s}|y-z|^{-\alpha}G(s,y)G(r,z)dydzdrds \\ &= a^2\alpha_H c_{\alpha,d}\int_{\mathbb{R}^d}\left(\int_0^t\int_0^t|r-s|^{2H-2}e^{-(r+s)|\xi|^{\beta}}drds\right)|\xi|^{-d+\alpha}d\xi \\ &\geq a^2\alpha_H c_{\alpha,d}c_H\int_{\mathbb{R}^d}\left(\frac{1}{1/t+|\xi|^{\beta}}\right)^{2H}|\xi|^{-d+\alpha}d\xi,\end{aligned}$$

where we used (12) for the equality and Theorem 3.1 of [2] for the last inequality. From here, we infer that

$$\alpha < 2H\beta. \tag{16}$$

(In particular, this implies that $\alpha < 2\beta$ since $H < 1$.)

Note that one can replace $\psi_n(\mathbf{t},\mathbf{s})$ by $\psi_n(t\mathbf{e}-\mathbf{t}, t\mathbf{e}-\mathbf{s})$ in the definition (10) of $\alpha_n(t,x)$, where $\mathbf{e} = (1,\ldots,1) \in \mathbb{R}^n$. By Lemma 2.2 of [1], we have:

$$\psi_n(t\mathbf{e}-\mathbf{t}, t\mathbf{e}-\mathbf{s}) = E\left[w(t-t^*, x+X^1_{t^*})w(t-s^*, x+X^2_{s^*})\prod_{j=1}^n|X^1_{t_j}-X^2_{s_j}|^{-\alpha}\right],$$

where $t^* = \max\{t_1,\ldots,t_n\}$, $s^* = \max\{s_1,\ldots,s_n\}$ and X^1, X^2 are two independent copies of the Lévy process $X = (X_t)_{t\geq 0}$ mentioned in the Introduction. (Lemma 2.2 of [1] was proved for $\beta = 2$. The same proof is valid for $\beta < 2$, the only change required for this case being to replace the fundamental solution $p_t(x) = (2\pi t)^{-d/2}\exp(-|x|^2/(2t))$ of the heat equation by $G(t,x)$ given by (4).)

Due to (5), it follows that

$$a^2M_n(t) \leq \alpha_n(t,x) \leq b^2M_n(t) \tag{17}$$

where

$$M_n(t) := E\left[\alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \prod_{j=1}^n |X^1_{t_j} - X^2_{s_j}|^{-\alpha} d\mathbf{t} d\mathbf{s}\right] = E(L(t)^n)$$

and $L(t)$ is a random variable defined by:

$$L(t) := \alpha_H \int_0^t \int_0^t |r-s|^{2H-2} |X^1_r - X^2_s|^{-\alpha} dr ds.$$

To prove that $L(t)$ is finite a.s., we show that its mean is finite. Note that $X^1_r - X^2_s \stackrel{d}{=} X_{r+s} \stackrel{d}{=} (r+s)^{1/\beta} X_1$, and hence

$$E[L(t)] = \alpha_H C_{d,\alpha,\beta} \int_0^t \int_0^t |r-s|^{2H-2} (r+s)^{-\alpha/\beta} dr ds,$$

where

$$C_{d,\alpha,\beta} := E|X_1|^{-\alpha} = \frac{c_d C_{\alpha,d}}{\beta} \Gamma(\alpha/\beta).$$

(see (28), Appendix A). Due to (16), it follows that $E[L(t)] < \infty$.

By (17), we have:

$$a^2 E(e^{L(t)}) \le E|u(t,x)|^2 = \sum_{n \ge 0} \frac{1}{n!} \alpha_n(t,x) \le b^2 E(e^{L(t)}). \tag{18}$$

We consider also the random variable

$$\zeta(t) := \int_0^t \int_0^t |X^1_r - X^2_s|^{-\alpha} dr ds.$$

Since $|r-s|^{2H-2} \ge (2t)^{2H-2}$ for any $r, s \in [0,t]$, $L(t) \ge \beta_H t^{2H-2} \zeta(t)$, where $\beta_H = \alpha_H 2^{2H-2}$. Hence $\zeta(t)$ is finite a.s.

By the self-similarity (of index $1/\beta$) of the processes X^1 and X^2, it follows that for any $t > 0$ and $c > 0$,

$$\zeta(t) \stackrel{d}{=} c^{(2\beta-\alpha)/\beta} \zeta(t/c).$$

In particular, for $c = t^{-(2H-2)\beta/(2\beta-\alpha)}$, we obtain that

$$t^{2H-2}\zeta(t) \stackrel{d}{=} \zeta(t^\delta), \quad \text{with} \quad \delta = \frac{2H\beta - \alpha}{2\beta - \alpha}$$

and for $c = t$, we obtain that $\zeta(t) \stackrel{d}{=} t^{(2\beta-\alpha)/\beta} \zeta(1)$. Hence,

$$E(e^{L(t)}) \ge E(e^{\beta_H t^{2H-2} \zeta(t)}) = E(e^{\beta_H \zeta(t^\delta)}). \tag{19}$$

The asymptotic behavior of the moments of $\zeta(t)$ was investigated in [6], under the condition $\alpha < 2\beta$. More precisely, under this condition, by relation (2.3) of [6], we know that:

$$\lim_{n\to\infty} \frac{1}{n} \log\left\{\frac{1}{(n!)^{\alpha/\beta}} E[\zeta(1)^n]\right\} = \log\left(\frac{2\beta}{2\beta-\alpha}\right)^{(2\beta-\alpha)/\beta} + \log\gamma,$$

where $\gamma > 0$ is a constant depending on d, α, β. Hence, there exists some $n_1 \geq 1$ such that for all $n \geq n_1$, $E[\zeta(1)^n] \geq c^n (n!)^{\alpha/\beta}$, where $c > 0$ is a constant depending on d, α, β. Consequently, for any $t > 0$,

$$E[\zeta(t)^n] \geq c^n t^{n(2\beta-\alpha)/\beta} (n!)^{\alpha/\beta} \quad \text{for all } n \geq n_1.$$

Hence, for any $\theta > 0$,

$$E(e^{\theta\zeta(t)}) = \sum_{n\geq 0} \frac{1}{n!}\theta^n E[\zeta(t)^n] \geq \sum_{n\geq n_1} \frac{1}{(n!)^{1-\alpha/\beta}} \theta^n c^n t^{n(2\beta-\alpha)/\beta}. \tag{20}$$

Using (18), (19) and (20), we obtain that:

$$\infty > E|u(t,x)|^2 \geq a^2 E(e^{L(t)}) \geq a^2 E\left(e^{\beta_H \zeta(t^\delta)}\right) \geq a^2 \sum_{n\geq n_1} \frac{\beta_H^n c^n t^{n(2H\beta-\alpha)/\beta}}{(n!)^{1-\alpha/\beta}}.$$

This implies that $\alpha < \beta$. For any $x > 0$ and $h \in (0,1)$, we note that

$$E_h(x) := \sum_{n\geq 0} \frac{x^n}{(n!)^h} \geq \left(\sum_{n\geq 0} \frac{(x^{1/h})^n}{n!}\right)^h = \exp(hx^{1/h}).$$

We denote $x_t = \theta c t^{(2\beta-\alpha)/\beta}$ and $h = 1 - \alpha/\beta$. Writing the last sum in (20) as the sum for all terms $n \geq 0$, minus the sum S_t with terms $n \leq n_1$, we see that for all $\theta > 0$, and for all $t \geq t_0$,

$$\begin{aligned} E(e^{\theta\zeta(t)}) &\geq E_h(x_t) - S_t \geq \exp(hx_t^{1/h}) - S_t \geq \frac{1}{2}\exp(hx_t^{1/h}) \\ &\geq \exp(c_0 \theta^{\beta/(\beta-\alpha)} t^{(2\beta-\alpha)/(\beta-\alpha)}), \end{aligned}$$

where $c_0 = hc^{1/h}$ and $t_0 > 0$ is a constant depending on θ, α, β. Using this last inequality with $\theta = \beta_H$ and t^δ instead of t, we obtain that:

$$E|u(t,x)|^2 \geq a^2 E\left(e^{\beta_H \zeta(t^\delta)}\right) \geq a^2 \exp(C_2 t^\rho),$$

where $C_2 = c_0 \beta_H^{\beta/(\beta-\alpha)}$ depends on d, α, β, H. $\square$

A Some useful identities

In this section, we give a result which was used in the proof of Theorem 2.1 for finding an upper bound for $\psi_n(\mathbf{t},\mathbf{t})$. This result may be known, but we were not able to find a reference. We state it in a general context.

Following Definition 5.1 of [17], we say that a function $f:\mathbb{R}^d \to [0,\infty]$ is a kernel of *positive type* if it is locally integrable and its Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ is a function g which is non-negative almost everywhere. Here we denote by $\mathcal{S}'(\mathbb{R}^d)$ the dual of the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$.

The Riesz kernel defined by $f(x)=|x|^{-\alpha}$ for $x\in\mathbb{R}^d\backslash\{0\}$ and $f(0)=\infty$ (with $\alpha\in(0,d)$), is a kernel of positive type. Its Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ is given by $g(\xi)=C_{\alpha,d}|\xi|^{-(d-\alpha)}$ where

$$C_{\alpha,d}=\pi^{-d/2}2^{-\alpha}\frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \tag{21}$$

(see Lemma 1, page 117 of [20]).

Let f be a continuous symmetric kernel of positive type such that $f(x)<\infty$ if and only if $x\neq 0$. By Lemma 5.6 of [17], for any Borel probability measures μ and ν on $\mathbb{R}^d$, we have:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\mu(dx)\nu(dy)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\mu(\xi)\overline{\mathcal{F}\nu(\xi)}g(\xi)d\xi,$$

where $\mathcal{F}\mu,\mathcal{F}\nu$ denote the Fourier transforms of μ,ν. In particular, if $\mu(dx)=\varphi(x)dx$ and $\nu(dy)=\psi(y)dy$ for some density functions φ,ψ in $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi(x)\psi(y)dxdy=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}g(\xi)d\xi. \tag{22}$$

This relation holds for arbitrary non-negative functions $\varphi,\psi\in L^1(\mathbb{R}^d)$. (To see this, we consider the normalized functions $\varphi/\|\varphi\|_1$ and $\psi/\|\psi\|_1$, where $\|\cdot\|_1$ denotes the $L^1(\mathbb{R}^d)$-norm.) Using the decomposition $\varphi=\varphi^+-\varphi^-$ with non-negative functions φ^+,φ^-, we see that (22) holds for any functions $\varphi,\psi\in L^1(\mathbb{R}^d)$. In fact, (22) holds for any functions $\varphi,\psi\in L^1_{\mathbb{C}}(\mathbb{R}^d)$, replacing $\psi(y)$ by its conjugate $\overline{\psi(y)}$ on the left-hand side. (To see this, we write $\varphi=\varphi_1+i\varphi_2$ where φ_1,φ_2 are the real and imaginary parts of φ.)

We consider the Bessel kernel (in $\mathbb{R}^d$) of order $\beta>0$:

$$G_{d,\beta}(x)=\frac{1}{\Gamma(\beta/2)}\int_0^\infty u^{\beta/2-1}e^{-u}\frac{1}{(4\pi u)^{d/2}}e^{-|x|^2/(4u)}du.$$

Note that $G_{d,\beta}$ is a density function (see Remark A.3 below) and

$$\mathcal{F}G_{d,\beta}(\xi)=\left(\frac{1}{1+|\xi|^2}\right)^{\beta/2}, \quad \xi\in\mathbb{R}^d. \tag{23}$$

Moreover, $G_{d,\alpha}*G_{d,\beta}=G_{d,\alpha+\beta}$ for any $\alpha,\beta>0$ (see pages 130-135 of [20]).

The following result is an extension of relations (3.4) and (3.5) of [11] to the case of arbitrary $\beta>0$.

Lemma A.1. *Let f be a continuous symmetric kernel of positive type such that $f(x) < \infty$ if and only if $x \neq 0$. Let $\mu(d\xi) = (2\pi)^{-d} g(\xi) d\xi$, where g is the Fourier transform of f in $\mathcal{S}'(\mathbb{R}^d)$. Let $\beta > 0$ be arbitrary. Then*

$$\int_{\mathbb{R}^d} G_{d,\beta}(x) f(x) dx = \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta/2} \mu(d\xi) := I_\beta(\mu). \tag{24}$$

If $I_\beta(\mu) < \infty$, then, for any $a \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{ia\cdot x} G_{d,\beta}(x) f(x) dx = \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi - a|^2} \right)^{\beta/2} \mu(d\xi). \tag{25}$$

Proof: Relation (24) follows from (22) with $\varphi = \psi = G_{d,\beta/2}$. On the left-hand side (LHS), we use the fact that $G_{d,\beta/2} * G_{d,\beta/2} = G_{d,\beta}$. On the right-hand side (RHS), we use (23) (with $\beta/2$ instead of β).

To prove (25), we apply (22) to the complex-valued functions:

$$\varphi(x) = \psi(x) = e^{ia\cdot x} G_{d,\beta/2}(x).$$

The term on the LHS is

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ia\cdot(x-y)} G_{d,\beta/2}(x) G_{d,\beta/2}(y) f(x-y) dx dy = \int_{\mathbb{R}^d} e^{ia\cdot x} f(x) G_{d,\beta}(x) dx,$$

using Fubini's theorem. The application of Fubini's theorem is justified since

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |e^{ia\cdot(x-y)} G_{d,\beta/2}(x) G_{d,\beta/2}(y) f(x-y)| dx dy = \int_{\mathbb{R}^d} G_{d,\beta}(x) f(x) dx < \infty.$$

For the term on the RHS, we use the fact that

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i(\xi - a)\cdot x} G_{d,\beta/2}(x) dx = \mathcal{F}G_{d,\beta/2}(\xi - a) = \left(\frac{1}{1+|\xi - a|^2} \right)^{\beta/4}.$$

□

Corollary A.2. *Let (f, μ) be as in Lemma A.1 and $\beta > 0$ be arbitrary. Assume that $I_\beta(\mu) < \infty$. Then*

$$\sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi - a|^2} \right)^{\beta/2} \mu(d\xi) = I_\beta(\mu).$$

Consequently,

$$\sup_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1+|\xi - a|^\beta} \mu(d\xi) \leq I_\beta(\mu). \tag{26}$$

Proof: The fact that $I_\beta(\mu)$ is smaller than the supremum is obvious. To prove the other inequality, we take absolute values on both sides of (25) and we use the fact that $|\int \cdots| \leq \int |\cdots|$. For the last statement, we use the fact that $(1+|\xi - a|^2)^{\beta/2} \leq 1 + |\xi - a|^\beta$, since $\beta/2 \in (0,1]$ and the following inequality holds: $(a+b)^p \leq a^p + b^p$ for any $a, b > 0$ and $p \in (0,1]$. □

Remark A.3. The Bessel kernel $G_{d,\beta}(x)$ arises in statistics as the density of the random vector X given by the following hierarchical model:

$$X|U = u \sim N_d(0, 2uI) \qquad U \sim \text{Gamma}(\beta/2, 1)$$

where $N_d(0, 2uI)$ denotes the d-dimensional normal distribution with covariance matrix $2uI$, I being the identity matrix. Hence, the term on the LHS of (24) is

$$\int_{\mathbb{R}^d} G_{d,\beta}(x) f(x) dx = E[f(X)] = \frac{1}{\Gamma(\beta/2)} \int_0^\infty u^{\beta/2-1} e^{-u} E[f(X)|U = u] du.$$

This can be computed explicitly if $f(x) = |x|^{-\alpha}$ with $\alpha \in (0, d)$. First, note that if $Z \sim N_d(0, 2tI)$, then its negative moment of order $-\alpha$ is:

$$E(|Z|^{-\alpha}) = \frac{1}{2} C_{\alpha,d} c_d \Gamma(\alpha/2) t^{-\alpha/2} \tag{27}$$

where $c_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$. To see this, we use the fact that $\mathcal{F}f(\xi) = C_{\alpha,d}|\xi|^{-d+\alpha} d\xi$ in $\mathcal{S}'(\mathbb{R}^d)$. Hence,

$$E(|Z|^{-\alpha}) \quad = \quad \int_{\mathbb{R}^d} |x|^{-\alpha} \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)} dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d+\alpha} e^{-t|\xi|^2} d\xi$$

and (27) follows by passing to the polar coordinates. We obtain that

$$\int_{\mathbb{R}^d} G_{d,\beta} |x|^{-\alpha} dx = \frac{c_{\alpha,d} c_d \Gamma(\alpha/2)}{2\Gamma(\beta/2)} \int_0^\infty u^{(\beta-\alpha)/2-1} e^{-u} du = \frac{C_{\alpha,d} c_d \Gamma((\beta-\alpha)/2)\Gamma(\alpha/2)}{2\Gamma(\beta/2)}.$$

(Note that the integral is finite if and only if $\alpha < \beta$.)

Remark A.4. A relation similar to (27) for stable random variables was used in the proof of Theorem 2.1 (Step 3). More precisely, if X is a d-dimensional random variable with a symmetric stable distribution with index $\beta \in (0, 2)$ (i.e. $E(e^{-i\xi \cdot X}) = e^{-|\xi|^\beta}$ for all $\xi \in \mathbb{R}^d$), then

$$E(|X|^{-\alpha}) = \frac{1}{\beta} C_{\alpha,d} c_d \Gamma(\alpha/\beta). \tag{28}$$

We include the proof of (28) for the sake of completeness. We denote by f_X the density of X. Recall that $\mathcal{F}f(\xi) = C_{\alpha,d}|\xi|^{-d+\alpha} d\xi$ in $\mathcal{S}'(\mathbb{R}^d)$ i.e. for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |x|^{-\alpha} \varphi(x) dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d+\alpha} \mathcal{F}\varphi(\xi) d\xi.$$

Using a regularization technique, one can show that the previous relation also holds for $\varphi = f_X$, since $\mathcal{F}f_X(\xi) \to 0$ rapidly as $|\xi| \to \infty$ and f_X is bounded and infinitely differentiable (see page 13 of [21]). Hence,

$$E(|X|^{-\alpha}) = \int_{\mathbb{R}^d} |x|^{-(d-\alpha)} f_X(x) dx = C_{\alpha,d} \int_{\mathbb{R}^d} |\xi|^{-d+\alpha} e^{-|\xi|^\beta} d\xi.$$

We now pass to the polar coordinates $\xi = rz$ with $r > 0$ and $z \in S_d$, where S_d is the unit sphere in $\mathbb{R}^d$. Let c_d be the area of S_d. We have

$$E(|X|^{-\alpha}) = C_{\alpha,d} c_d \int_0^\infty r^{-d+\alpha} e^{-r^\beta} r^{d-1} dr$$

and relation (28) follows using the change of variable $s = r^\beta$.

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