New robust tests for the parameters of the Weibull distribution for complete and censored data

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Abstract

Using the likelihood depth, new consistent and robust tests for the parameters of the Weibull distribution are developed. Uncensored as well as type-I right-censored data are considered. Tests are given for the shape parameter and also the scale parameter of the Weibull distribution, where in each case the situation that the other parameter is known as well the situation that the other parameter is unknown is examined. In simulation studies the behavior in finite sample size and in contaminated data is analyzed and the new method is compared to existing ones. Here it is shown that the new tests based on likelihood depth are comparable to standard methods and robust against contamination. They are also robust in censored data in contrast to existing methods like the method of medians.

Keywords: Weibull distribution, censored data, data depth, likelihood depth, simplicial depth, testing, robustness against contamination

1 Introduction

The Weibull distribution is often used in survival analysis, especially in life-testing and reliability studies. It was introduced by Weibull (1951). The distribution function is one-dimensional and depends on two parameters. Its survival function has a rather simple form and it can be used to model constant as well as de- and increasing Hazard-functions.

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The textbook of Rinne (2009) gives a good overview of the developed test procedures for the parameters of the Weibull distribution, for complete and for censored data. Tests based on maximum likelihood procedures are presented there. Besides, many articles deal with tests and confidence intervals for the parameters of the Weibull distribution in censored and uncensored data, see for example Balakrishnan and Stehlik (2008), Chen (1997), Wong and Wong (1982) or Kahle (1996). But still these methods are not robust against contamination. Only He and Fung (1999) give outlier robust confidence intervals for the shape parameter, but only for data where at most 16 % of the largest and 34% of the smallest observations are censored.

In this work the likelihood depth shall be used to develop consistent and robust tests, as described in Denecke and Müller (2011b) for general depth notions. The concept of data depth is an approach to generalize the median and ranks to multivariate data and more complex situations. Algorithms for computation can be found e.g. in Rousseeuw and Ruts (1998). Notions of data depth were for example developed for multivariate data by Tukey (1975), Liu (1988, 1990) and Mosler (2002). Zou and Serfling (2000a,b) provided some general properties of depth notions. For recent approaches in data depth see e.g. Lin and Chen (2006), Li and Liu (2008), Romanazzi (2009), López-Pintado and Romo (2009), López-Pintado et al. (2010), Hu et al. (2009).

Any depth notion can be used to define simplicial depth as Liu (1988, 1990) did using the halfspace depth of Tukey (1975). Every simplicial depth is a U-statistic so that its asymptotic distribution is in principle known and asymptotic α-level tests can be derived as Müller (2005) did. However, many simplicial depth notions are degenerated U-statistics so that the spectral decomposition of a conditional expectation is needed to derive the asymptotic distribution which is then not a normal distribution. Such spectral decompositions were for example derived for polynomial regression, multiple regression, and orthogonal regression by Wellmann et al. (2009) and Wellmann and Müller (2010a,b), respectively. But there are also situations, when the simplicial depth is a non-degenerated U-statistic. This is the case, when the estimator maximizing the underlying depth notion is biased, as Denecke and Müller (2011b) showed. For these distributions they point out how consistent tests based on general depth notions can be defined. Thereby a test

$\varphi_N : \mathcal{Z}^N \to \{0, 1\}$ for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta \setminus \Theta_0$ is called consistent if

$$\lim_{N\to\infty} P_\theta(\varphi_N = 1) = 0 \text{ for all } \theta \in \text{int}(\Theta_0),$$
$$\lim_{N\to\infty} P_\theta(\varphi_N = 1) = 1 \text{ for all } \theta \in \text{int}(\Theta \setminus \Theta_0),$$

where $\text{int}(A)$ denotes the interior of a set A. The theory of Denecke and Müller (2011b) shall be used here to derive tests for the parameters of the Weibull distribution in complete and type-I right-censored data based on likelihood depth.

This work is organized as follows. In Section 2 a short introduction to likelihood depth and tests based on likelihood depth is given. Also we recall the important results about the consistency of the tests from Denecke and Müller (2011b). In Section 3 tests and confidence intervals for the shape and scale parameter of the Weibull distribution in uncensored data are developed. And in Section 4 the methods are used for type-I right-censored data. Thereby each of the two section starts with the tests for the shape parameter, when the scale is assumed to be known. Then the situation when the scale is unknown is examined. After this, the procedure is repeated for the scale parameter, i.e. we start with the shape to be known and then consider the shape to be unknown. Section 3, as well as Section 4, uses results from Denecke and Müller (2011c) about the robust and consistent estimation of the parameters of the Weibull distribution based on likelihood depth. The subsections of Section 3 and 4 include simulation studies to compare the new method to existing ones. Thereby in each setting the sample size is 100 and the number of repetitions 1000. For the simulations we used R (2010).

2 Tests based on likelihood depth

We will only repeat the main results of Denecke and Müller (2011b) that are used to construct consistent α-level tests for the parameters of the Weibull distribution.

Consider $Z_1, \dots, Z_N$ i.i.d. with continuous density f_θ, $\theta \in \Theta \subset \mathbb{R}^q$. The likelihood function at parameter θ and observation z will be denoted by $L(\theta, z) := f_\theta(z)$.

For an introduction to likelihood depth see e.g. Mizera and Müller (2004) or Müller (2005). As in this paper only one-dimensional parameters are considered, we define the tangential likelihood depth and simplicial likelihood depth, that will be used for testing, for $\theta \in \mathbb{R}$ only. The *(tangent) likelihood depth* of θ within $z_* := (z_1, \dots, z_N)^T$ is defined as

$$d_T(\theta, z_*) = \tfrac{1}{N} \min(\sharp\{n; \frac{\partial}{\partial\theta} \ln L(\theta, z_n) \geq 0\}, \sharp\{n; \frac{\partial}{\partial\theta} \ln L(\theta, z_n) \leq 0\}),$$

and the *simplicial likelihood depth* of θ within observations $z_* := (z_1, \dots, z_N)^T$ is defined as

$$d_S(\theta, z_*) := \tfrac{2}{N(N-1)} \sum_{1 \leq n_1 < n_2 \leq N} d_T(\theta, (z_{n_1}, z_{n_2})),$$

thus, it is the U-statistic belonging to the tangent likelihood depth, which is the symmetric kernel.

To simplify the presentation we introduce some abbreviations. The set, where $\frac{\partial}{\partial\theta} \ln L(\theta, z)$ is positive or zero (negative or zero), will be denoted by T^θ_{pos} (T^θ_{neg}), i.e.

$$T^\theta_{pos} := \{z \in \mathbb{R}^m; \tfrac{\partial}{\partial\theta} \ln L(\theta, z) \geq 0\}, \; T^\theta_{neg} := \{z \in \mathbb{R}^m; \tfrac{\partial}{\partial\theta} \ln L(\theta, z) \leq 0\}.$$

Define

$$p_{\theta,\theta'} := P_\theta(T^{\theta'}_{pos}) := P_\theta(Z \in T^{\theta'}_{pos}) = 1 - P_\theta(T^\theta_{neg}),$$

thus, the parameter with asymptotically maximum likelihood depth $s(\theta)$ for P_θ, $\theta \in \Theta$, is the solution of $p_{\theta,s(\theta)} = P_\theta(T^{s(\theta)}_{pos}) = \frac{1}{2}$.

Lemma 1 (see Denecke 2010)**.** *Let be $\theta \in \Theta$ with $p_{\theta,\theta} \neq \frac{1}{2}$ and $Z_* = (Z_1, \dots, Z_N)$, $Z_1, \dots, Z_N$ i.i.d., $Z_i \sim f_\theta$, $i = 1, \dots, N$. The test statistic, defined as*

$$T(\theta, z_*) := \sqrt{N} \frac{\frac{2}{N(N-1)} \sum_{1 \leq n_1 < n_2 \leq N} d_T(\theta, (z_{n_1}, z_{n_2})) - 2p_{\theta,\theta}(1 - p_{\theta,\theta})}{2\sqrt{(1 - p_{\theta,\theta})p_{\theta,\theta}(1 - 2p_{\theta,\theta})^2}},$$

satisfies $T(\theta, Z_{*,N}) \xrightarrow{\mathscr{D}} X \sim \mathcal{N}(0, 1)$. Then the test $\varphi(z_*) := 1_{\{\sup_{\theta \in \Theta_0} T(\theta, z_*) < \Phi^{-1}(\alpha)\}}(z_*)$ is an asymptotic α-level test for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \notin \Theta_0$.

In particular, under the conditions of Lemma 1 we have that

$\varphi^{0,\leq}_{\theta_0} := 1_{\{\sup_{\theta \leq \theta_0} T(\theta, \cdot) < \Phi^{-1}(\alpha)\}}$ for $H_0 : \theta \leq \theta_0$, $H_1 : \theta > \theta_0$,

$\varphi^{0,\geq}_{\theta_0} := 1_{\{\sup_{\theta \geq \theta_0} T(\theta, \cdot) < \Phi^{-1}(\alpha)\}}$ for $H_0 : \theta \geq \theta_0$, $H_1 : \theta < \theta_0$, and

$$\varphi_{\theta_0}^{0,=} := 1_{\{T(\theta_0,\cdot)<\Phi^{-1}(\alpha)\}} \text{ for } H_0 : \theta = \theta_0,\ H_1 : \theta \neq \theta_0$$

are asymptotic α-level tests.

However, if the maximum likelihood estimator overestimates the true parameter θ_0, i.e. $s(\theta_0) > \theta_0$, then the test $\varphi_{\theta_0}^{0,\geq}$ has a bad power, see Denecke and Müller (2011b). To improve the power, the rejection set $\{\sup_{\theta\geq\theta_0} T(\theta, z_*) < \Phi^{-1}(\alpha)\}$ is extended, by taking the supremum over a smaller set, i.e. $\theta \geq c_\alpha^1(\theta_0) > \theta_0$ instead of $\theta \geq \theta_0$. To obtain the best power, $c_\alpha^1(\theta_0)$ is chosen as the maximum value such that the resulting test is still an asymptotic α-level test. Hence $c_\alpha^1(\theta_0)$ is defined as $c_\alpha^1(\theta_0) := \max\{\theta; \lim_{N\to\infty} P_{\theta_0}\left(T(\theta, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \alpha\}$. Under the assumptions

$$\begin{aligned} &p_{(\cdot),\theta} = P_{(\cdot)}(T_{pos}^{\theta}) \text{ is strictly increasing from 0 to 1}, p_{\theta,(\cdot)} \text{ is strictly decreasing,} \\ &1/2 < p_{\theta,\theta} \leq 1/2 + 1/\sqrt{8}, \text{ and } \alpha < 0.5 \end{aligned} \tag{1}$$

for $\theta = \theta_0$, Denecke (2010) shows that $c_\alpha^1(\theta_0)$ is the value $\tilde{\theta}$, such that $1 - p_{\tilde{\theta},\tilde{\theta}} = p_{\theta_0,\tilde{\theta}}$. Additionally assume

$$c_\alpha^1 \text{ is an increasing function, } p_{\theta,\theta} \text{ is a continuous function of } \theta. \tag{2}$$

Proposition 1 (see Denecke 2010)**.** *Under the conditions (1) and (2) and* $p_{\theta,\theta} \neq \frac{1}{2}$ *for all* $\theta \in \Theta$*,*

$$\varphi_{\theta_0}^{0,\leq} \text{ for } H_0 : \theta \leq \theta_0,\ H_1 : \theta > \theta_0,$$

$$\varphi_{\theta_0}^{\geq} := 1_{\{\sup_{\theta\geq c_\alpha^1(\theta_0)} T(\theta,\cdot)<\Phi^{-1}(\alpha)\}} \text{ for } H_0 : \theta \geq \theta_0,\ H_1 : \theta < \theta_0, \text{ and}$$

$$\varphi_{\theta_0}^{=} := \max\{1_{\{T(\theta_0,\cdot)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}, 1_{\{T(c^1_{\frac{\alpha}{2}}(\theta_0),\cdot)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}\} \text{ for } H_0 : \theta = \theta_0,\ H_1 : \theta \neq \theta_0$$

are consistent tests with asymptotic level α*.*

Analogous results hold for $s(\theta_0) < \theta_0$. Here $\varphi_{\theta_0}^{0,\geq}$ is already a consistent test and does not have to be corrected. However, $\varphi_{\theta_0}^{0,\leq}$ and $\varphi_{\theta_0}^{0,=}$ must be corrected to $\varphi_{\theta_0}^{\leq} := 1_{\{\sup_{\theta\leq c_\alpha^2(\theta_0)} T(\theta,\cdot)<\Phi^{-1}(\alpha)\}}$ and $\varphi_{\theta_0}^{=} := \max\{1_{\{T(\theta_0,\cdot)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}, 1_{\{T(c^2_{\frac{\alpha}{2}}(\theta_0),\cdot)<\Phi^{-1}\left(\frac{\alpha}{2}\right)\}}\}$. Here $c_\alpha^2(\theta_0)$ is defined by $c_\alpha^2(\theta_0) := \min\{\theta; \lim_{N\to\infty} P_{\theta_0}\left(T(\theta, Z_{*,N}) < \Phi^{-1}(\alpha)\right) \leq \alpha\}$ and satisfies $c_\alpha^2(\theta_0) < s(\theta_0) < \theta_0$. Under conditions analogously to (1), $c_\alpha^2(\theta_0) =: \tilde{\theta}$ is given by $1 - p_{\tilde{\theta},\tilde{\theta}} = p_{\theta_0,\tilde{\theta}}$ and the tests based on likelihood depth are also consistent. For more details see Denecke (2010).

In Denecke and Müller (2011b) it is also discussed and proven that the tests based on simplicial likelihood depth have a high breakdown point in the sense of Ylvisaker (1977), He et al. (1990), Coakley and Hettmansperger (1994), Zhang (1996), Müller (1997). This is also supported by the results of the simulation studies of the next sections.

3 Tests and confidence intervals for the parameters of the Weibull distribution in uncensored data

The density of the Weibull distribution is given by $f_{a,b}(t) = \frac{a}{b}\left(\frac{t}{b}\right)^{a-1}\exp\left(-\left(\frac{t}{b}\right)^{a}\right)$, where $a > 0$ is the shape parameter and $b > 0$ the scale parameter. The corresponding distribution function is $F_{a,b}(t) = 1 - \exp\left(-\left(\frac{t}{b}\right)^{a}\right)$.

3.1 Tests for the shape parameter, known scale parameter

Assume the scale parameter $b_0 > 0$ to be known. Denecke and Müller (2011c) show that $T^a_{pos} = \{t \in \mathbb{R}; \frac{\partial}{\partial a}\ln f_{a,b_0}(t) \geq 0\} = [c_1^{\frac{1}{a}} b_0, c_2^{\frac{1}{a}} b_0]$, with $0.259 \approx c_1 < 1 < c_2 \approx 2.240$ being the solutions of $\ln c = \frac{1}{c-1}$. Thus $p_{a,a} = P_{a,b_0}(T^a_{pos}) = F_{a,b_0}(c_2^{\frac{1}{a}} b_0) - F_{a,b_0}(c_1^{\frac{1}{a}} b_0) = \exp(-c_1) - \exp(-c_2) \approx 0.665 > 0.5$. Further, the parameter with asymptotically maximum likelihood depth in Weibull data with shape a, $s(a)$, is given by the solution of $p_{a,s(a)} = \exp(-c_1^{\frac{a}{s(a)}}) - \exp(-c_2^{\frac{a}{s(a)}}) = 0.5$, thus $s(a) = \frac{1}{\kappa}a \approx 1.447a > a$.

Let be the test statistic $T(a, t_*)$ as defined in Lemma 1. Because $s(a) > a$, the tests for the hypothesis $H_0 : a \geq a_0$ and $H_0 : a = a_0$ have to be corrected. Therefore c^1_α has to be determined. It holds $p_{a_0,a} = \exp(-c_1^{\frac{a_0}{a}}) - \exp\left(-c_2^{\frac{a_0}{a}}\right)$. It is $p_{a_0,(\cdot)}$ strictly decreasing and $p_{(\cdot),a_0}$ is strictly increasing. Thus, the condition (1) is fulfilled and $\tilde{a} := c^1_\alpha(a_0)$ is given by $1 - p_{\tilde{a},\tilde{a}} = p_{a_0,\tilde{a}}$, so $\alpha < 0.5$,

$$c^1_\alpha(a_0) = k_0 \cdot a_0,$$

with $k_0 \approx 2.275$. Especially $c^1_\alpha(a_0)$ exists for all $a_0 > 0$, it is $c^1_\alpha(a_0) > a_0$ for all $a_0 > 0$ and $c^1_\alpha(\cdot)$ strictly increasing. As further also (2) is fulfilled, the tests based on simplicial likelihood depth are consistent tests with asymptotic level α: Let be $\alpha < 0.5$, then Proposition 1 gives:

The test $\varphi^{0,\leq}_{a_0}(t_*) = 1_{\{\sup_{a\leq a_0} T(a,t_*)<\Phi^{-1}(\alpha)\}}(t_*)$ is a consistent test with asymptotic level α for $H_0 : a \leq a_0$.

The test $\varphi^{\geq}_{a_0}$ for $H_0 : a \geq a_0$ is consistent with asymptotic level α, by rejecting H_0, if $\sup_{a\geq c^1_\alpha(a_0)} T(a, t_*) < \Phi^{-1}(\alpha)$.

A consistent test with asymptotic level α for $H_0 : a = a_0$ against $H_1 : a \neq a_0$ is given by $\varphi^{=}_{a_0}(t_*) = \max\left(1_{\{T(a_0,t_*)<\Phi^{-1}(\frac{\alpha}{2})\}}(t_*), 1_{\{T(c^1_{\frac{\alpha}{2}}(a_0),t_*)<\Phi^{-1}(\frac{\alpha}{2})\}}(t_*)\right)$. A confidence interval with asymptotic level $\gamma = 1 - \alpha$ for the shape parameter a is given by $\{a_0 > 0; \varphi^{=}_{a_0}(t_*) = 0\}$.

We compare this new test based on likelihood depth to a test based on the maximum likelihood estimator (MLE), which can be found e.g. in the textbook of Rinne (2009). The graphics in Figure 1 show the simulated power functions for $H_0 : a \leq a_0$ and $H_0 : a \geq a_0$ with $a_0 = 1$. We note that the level is not kept by both tests and that the power functions

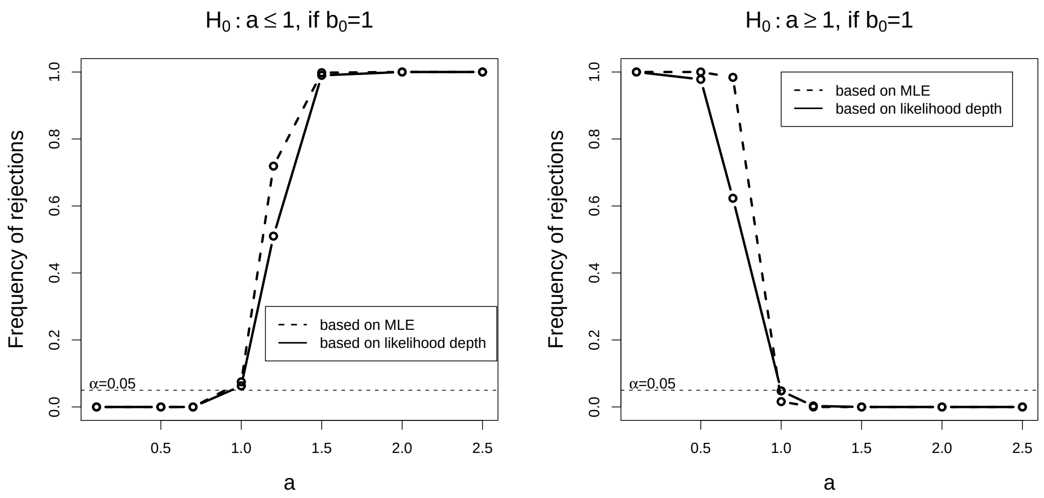

Figure 1: Simulated power of the tests for $H_0 : a \leq 1$ (left) and $H_0 : a \geq 1$ (right), b_0 the scale parameter known.

are very similar. Denecke (2010) shows some more simulation results. They demonstrate that the power does not change if different a_0 and b_0 are considered.

We also consider ε-contaminated data, as the new test is supposed to be robust against contamination. Thereby ε-contaminated data means that $(1 - \varepsilon)100\%$ of the data come from a Weibull distribution with the assumed parameters and $\varepsilon 100\%$ from a different distribution. As a contamination distribution we only consider Weibull distributions with different shape and/or scale. So, now examine data where some part is given by another distribution, here Wei(a_1, b_0). In Figure 2, the simulated power-functions for $H_0 : a \leq 1$ and $H_0 : a \geq 1$ are pictured, where the contaminated data has a shape parameter $a_1 = 0.5$ and the ratio of the contaminated data is 10%. For this contamination, the test based

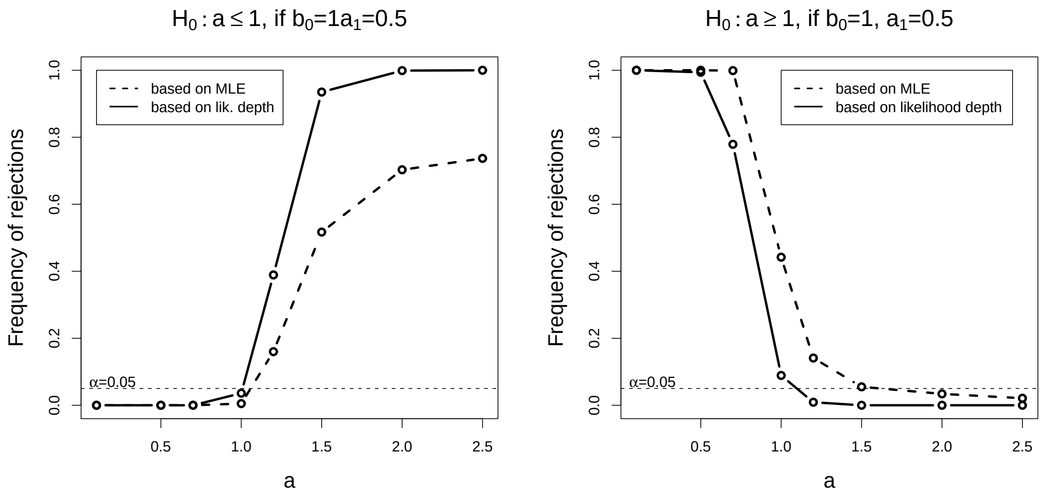

Figure 2: Simulated power of the tests for $H_0 : a \leq 1$ (left) and $H_0 : a \geq 1$ (right), b_0 known, for 0.1-contaminated data with Wei$(0.5, b_0)$.

on the maximum likelihood estimator for $H_0 : a \leq 1$ has a very bad power in contrast to the test based on likelihood depth. For $H_0 : a \geq 1$ both tests are infected by the contamination with a small shape parameter, as both do not keep the level anymore. The test based on the MLE behaves much worse than the test based on likelihood depth. When considering contamination distribution with a bigger shape (e.g. $a_1 = 10$), simulation studies, see Denecke (2010), picture that for $H_0 : a \leq a_0$ both tests do not keep the level, while the tests for $H_0 : a \geq a_0$ both tests are not really infected by the contamination.

Figure 3 displays the simulated power functions of the tests for $H_0 : a = a_0$ based on likelihood depth, based on the MLE and based on the method of medians (MoM) of He and Fung (1999), see also Boudt et al. (2011). All three tests have a very similar power for uncontaminated data (left-hand side of Figure 3), but only the test based on likelihood depth and the MoM test are robust against ε-contamination (right-hand side of Figure 3). When considering confidence intervals in the next step, also other contaminations are

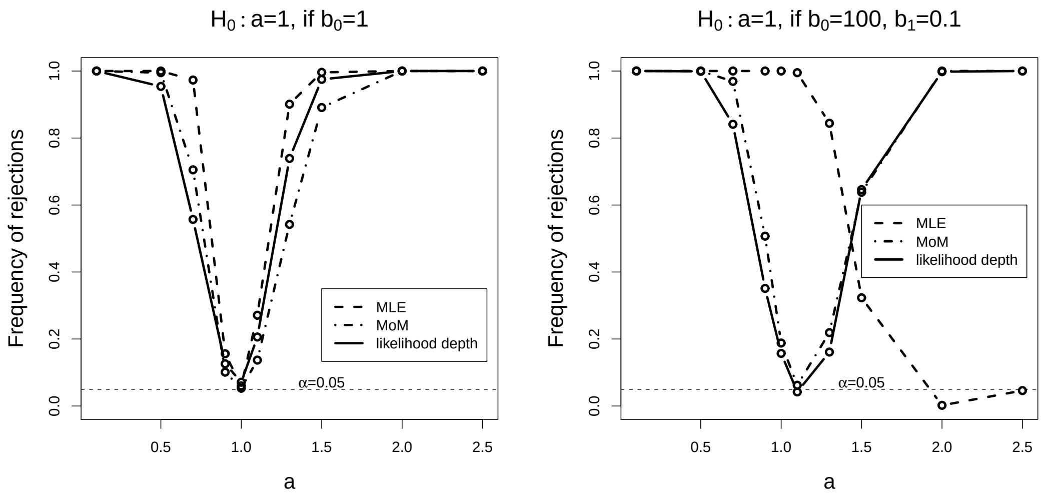

Figure 3: Simulation of the power-function of the tests for $H_0 : a = a_0$, known scale, left-hand side: uncontaminated data, right-hand side: 0.1-contaminated data from Wei(a, b_1).

simulated.

There are some more methods to determine confidence intervals for the shape parameter of the Weibull distribution. E.g. in Lawless (2003) a likelihood-ratio procedure is described. We compare the confidence intervals for the shape parameter based on likelihood depth (lik depth) to ones based on the method based on the MLE, the method based on likelihood-ratio statistics (LRS), and as a robust method we consider confidence intervals based on the MoM, see He and Fung (1999). Table 1 shows some results for ε-contaminated data.

					MLE		LRS		MoM		lik depth	
	a	b	a_1	b_1	cov.	length	cov.	l.	cov.	l.	cov.	l.
a)	2	0.1	0.5	10	$< 10^{-2}$	0.14	$< 10^{-2}$	0.28	0.85	0.85	0.85	1.04
b)	2	1	0.5	1	0.07	0.40	0.06	0.41	0.88	0.88	0.89	1.00
c)	1	10^2	1	10^6	0.05	0.23	0.13	0.34	0.88	0.44	0.90	0.52

Table 1: Coverage rate (cov.) and length (l.) of the confidence intervals for the shape parameter for ε-contaminated data from Wei(a_1, b_1), known scale parameter b_0.

The method of medians and the method based on likelihood depth are quite robust. The confidence intervals based on the maximum likelihood estimator and likelihood-ratio method are very bad. The method based on likelihood depth gives the best coverage rates in two of the cases regarded.

3.2 Tests for the shape parameter, unknown scale parameter

If the scale parameter b_0 is unknown and has to be estimated, the depth of a can only be calculated based on the biased maximum likelihood depth estimator for the scale $\tilde{b}_N$, the median of the data (see Denecke 2010). To calculate the test statistic, we plug $\tilde{b}_N$ into the simplicial depth instead of b_0. Thus, $d_S^{\tilde{b}_N}(T_*)$ is not a U-statistic any more. We can not use the theorem of Hoeffding to get the asymptotic distribution of the test statistic. Anyway, we develop how the quantities would look like, if we still could use the same theory as before and show in simulations studies that the power is still good for these disturbed cases. We determine $\tilde{p}_{shape}$ as the asymptotic value for the part of observations lying in $T_{pos}^{a_0,\tilde{b}} = [c_1^{\frac{1}{a_0}} \tilde{b}_N, c_2^{\frac{1}{a_0}} \tilde{b}_N]$, thus $\tilde{p}_{shape} = 2^{-c_1} - 2^{-c_2} \approx 0.624$.

Analog to the case, where the scale parameter is known, we define the test statistic as

$$\tilde{T}(a, t_*) := \sqrt{N} \frac{d_S^{\tilde{b}_N}(a, t_*) - 2\tilde{p}_{shape}(1 - \tilde{p}_{shape})}{2\sqrt{\tilde{p}_{shape}(1 - \tilde{p}_{shape})(1 - 2\tilde{p}_{shape})^2}}.$$

Thus, we test $H_0 : a \leq a_0$ with $\tilde{\varphi}_{a_0}^{0,\leq}(t_*) = 1_{\{\sup_{a \leq a_0} \tilde{T}(a,t_*) < \Phi^{-1}(\alpha)\}}(t_*)$.

As $\tilde{p}_{shape} \approx 0.624 > 0.5$, a correction for the test $H_0 : a \geq a_0$ is needed. Determining $\tilde{a} := \tilde{c}_\alpha^1(a_0)$ analog to the case of known scale parameter, i.e. as the solution of $1 - \tilde{p}_{\tilde{a},\tilde{a}} = \tilde{p}_{a_0,\tilde{a}}$ leads to $\tilde{c}_\alpha^1(a_0) = \tilde{k}_0 a_0$, with $\tilde{k}_0 \approx 1.835$. We test $H_0 : a \geq a_0$ against $H_1 : a < a_0$ with $\tilde{\varphi}_{a_0}^{\geq} := 1_{\{\sup_{a \geq \tilde{c}_\alpha^1(a_0)} \tilde{T}(a,t_*) \leq \Phi^{-1}(\alpha)\}}(t_*)$.

The power of the new test is, as before, compared to the power of the test based on the MLE. In Figure 4 the simulated power functions of the tests for $H_0 : a \leq a_0$ and $H_0 : a \geq a_0$ with $a_0 = 1$ are displayed. Compared to the plot in Figure 1, when b_0 is supposed to be known, no real changes can be detected. Still both tests do not keep the level.

Using the tests for $H_0 : a \leq a_0$ and $H_0 : a \geq a_0$, we can also define a test for the hypothesis $H_0 : a = a_0$ against $H_1 : a \neq a_0$ as $\tilde{\varphi}_{a_0}^{=} = \max(1_{\{\tilde{T}(a_0,\cdot) \leq \Phi^{-1}(\frac{\alpha}{2})\}}, 1_{\{\tilde{T}(\tilde{c}_{\frac{\alpha}{2}}^1(a_0),\cdot) \leq \Phi^{-1}(\frac{\alpha}{2})\}})$. Thus, we can also use the test for the shape parameter in situations, where the scale pa-

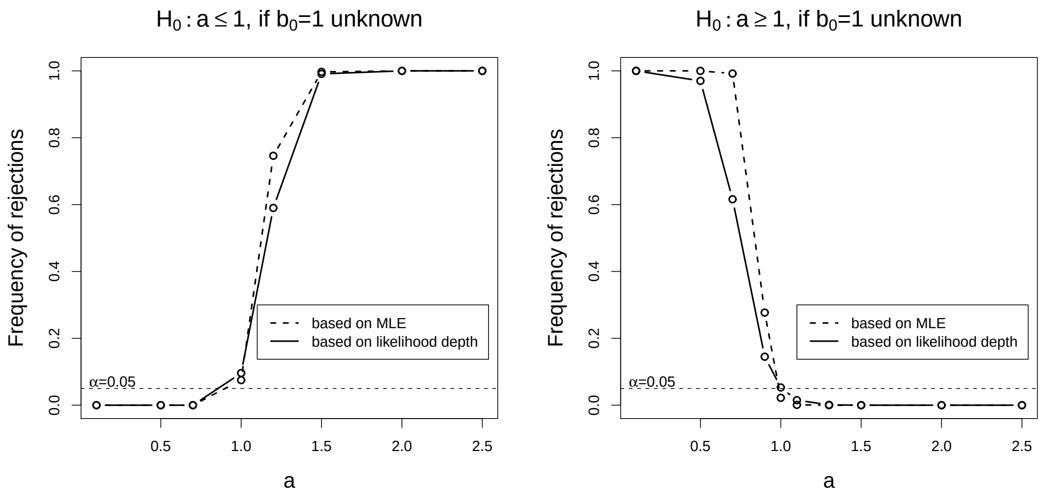

Figure 4: Power of the tests for the shape a_0, unknown scale parameter.

rameter is unknown. Again simulation studies, see Denecke (2010), show that this plug-in test has the same power as the test with known scale parameter.

3.3 Tests for the scale parameter, known shape parameter

Now assume the shape parameter $a_0 > 0$ to be known. In Denecke and Müller (2011c) it is proven that $T^b_{pos} = [b, \infty)$. Thus $p_{b,b} = P_{a_0,b}(T^b_{pos}) = 1 - F_{a_0,b}(b) = \exp(-1) < 0.5$. Further the parameter with asymptotic maximum likelihood depth $s(b)$ is the solution of $p_{b,s(b)} = P_{b,a_0}(T^{s(b)}_{pos}) = \exp\left(-\left(\frac{s(b)}{b}\right)^a\right) = 0.5$, so $s(b) = (\ln 2)^{\frac{1}{a_0}} b < b$. Therefore, the tests for the hypotheses $H_0 : b \leq b_0$ and $H_0 : b = b_0$ have to be corrected. It holds $p_{b_0,b} = \exp\left(-\left(\frac{b}{b_0}\right)^{a_0}\right)$, thus it is $p^{a_0}_{b_0,(\cdot)}$ strictly decreasing, $p^{a_0}_{(\cdot),b_0}$ is strictly increasing, and $1/2 < 1 - p_{b,b} = 1 - \exp(-1) \approx 0.623 < 1/2 + 1/\sqrt{8}$. So the correction $c^2_\alpha(b_0)$ can be determined as that value $\tilde{b}$ for that $p_{b_0,\tilde{b}} = p_{\tilde{b},\tilde{b}}$, see Denecke (2010), and we get

$$c^2_{\alpha,a_0}(b_0) = b_0(-\ln(1 - \exp(-1)))^{\frac{1}{a_0}} \approx b_0(0.4587)^{\frac{1}{a_0}},$$

if $\alpha < 0.5$. Especially, it is $c^2_{\alpha,a_0}(\cdot)$ strictly increasing and as $-\ln(1-\exp(-1)) \approx 0.46 < 1$, it holds $c^2_{\alpha,a_0}(b_0) < b_0$ for all $b_0 > 0$. We see that c^2_{α,a_0} is depending on a_0, so here we have to know a_0, while the test statistic and the test for $H_0 : b \geq b_0$ are independent of a_0. Let be $\alpha < 0.5$, according to Proposition 1 it holds:

The test $\varphi^{\geq}_{b_0}(t_*) := 1_{\{\sup_{b \geq b_0} T(b,t_*) < \Phi^{-1}(\alpha)\}}(t_*)$ is a consistent test with asymptotic level α for $H_0 : b \geq b_0$.

$\varphi^{\leq}_{b_0}(t_*) := 1_{\{\sup_{b \leq c^2_{\alpha,a_0}(b_0)} T(b,t_*) < \Phi^{-1}(\alpha)\}}(t_*)$ is consistent with asymptotic level α for the hypothesis $H_0 : b \leq b_0$.

It is $\varphi^{=}_{b_0}(t_*) := \max(1_{\{T(b,t_*) < \Phi^{-1}(\frac{\alpha}{2})\}}(t_*), 1_{\{T(c^2_{\frac{\alpha}{2},a_0}(b),t_*) < \Phi^{-1}(\frac{\alpha}{2})\}}(t_*))$ a consistent test with asymptotic level α for $H_0 : b = b_0$ against $H_0 : b \neq b_0$. A confidence interval with asymptotic level $\gamma = 1-\alpha$ for the scale parameter is given by $\{b_0 > 0; \varphi^{=}_{b_0}(t_*) = 0\}$.

We compare the power of this new test in a simulation study with a test for the scale parameter given in the textbook of Rinne (2009), based on the maximum likelihood estimator. Figure 5 shows the simulated power functions for the hypotheses $H_0 : b \geq b_0$ (left-hand side) and $H_0 : b \leq b_0$ (right-hand side) for $b_0 = 1$, where in the second case for both tests we assumed the shape $a_0 = 1$ to be known and in the first case only needed this information for the test based on the MLE. The test based on likelihood depth and

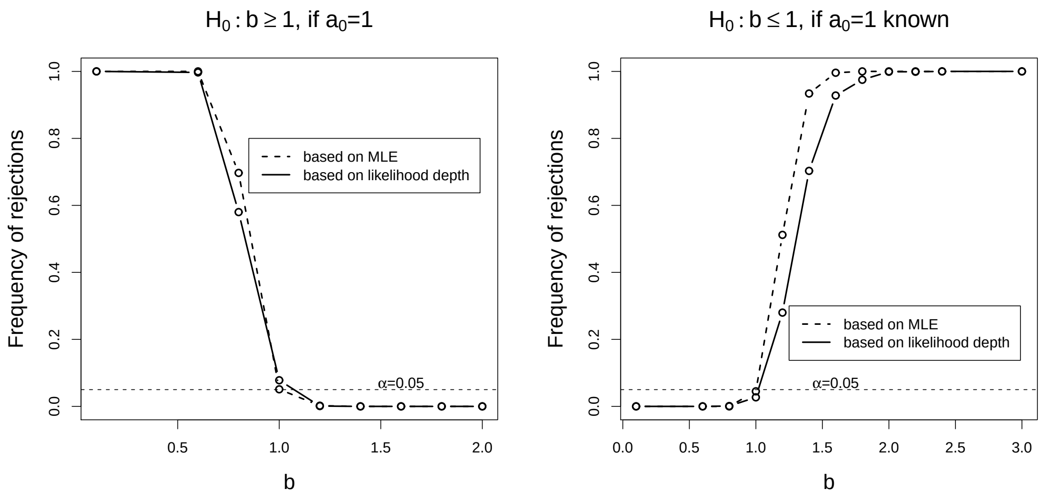

Figure 5: Simulated power for the tests $H_0 : b \geq b_0$ and $H_0 : b \leq b_0$ in uncontaminated data.

the test based on the MLE do not really differ in their power for uncontaminated data, especially when testing $H_0 : b \geq b_0$, the latter seems to give only slightly better results when considering the hypotheses $H_0 : b \leq b_0$. Further studies demonstrate that the shape parameter a_0 has an influence on the power of both tests, but still both tests behave very similar.

In a next step ε-contaminated data is considered, Figure 6 shows some simulation results. For this contaminations the power of the test based on likelihood depth is better

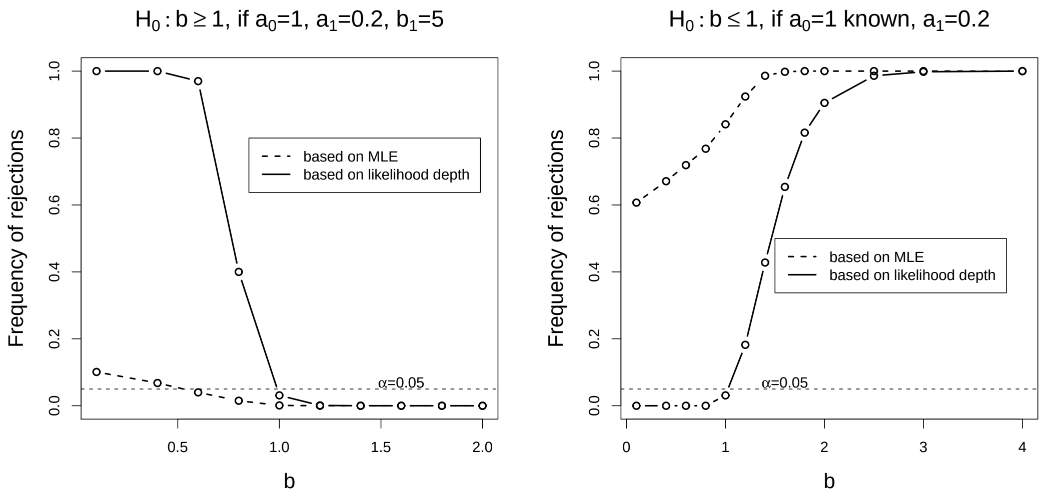

Figure 6: Simulated power for contaminated data, i.e. 10% of the data is coming from Wei(a_1, b_1), where on the right-hand side it is $b_1 = b$.

than the power of the test based on the MLE. Other studies, see Denecke (2010) showed that contamination distributions with a smaller shape lead to very bad power of the test based on the MLE, while the new test is not affected. Contamination with a smaller scale has no influence on the power of both tests for $H_0 : b \leq b_0$, but the tests for $H_0 : b \geq b_0$ do not keep the level anymore. For contamination with a bigger scale the tests behave vice versa.

A further simulation study shows that the new test for $H_0 : b = b_0$ based on likelihood depth does not keep the level and that its power is little worse than the power of the test based on the MLE, see Figure 7 on the left. Note that here we simulate tests with level $\alpha = 0.04$. For contaminated data, we display one result in Figure 7 on the right-hand side. It shows again that the new test is robust against ε-contamination in contrast to the test based on the MLE. Also the confidence intervals for the scale parameter based on the method of likelihood depth and confidence intervals based on the testing with the maximum likelihood estimator are compared. We calculate 96%-confidence intervals.

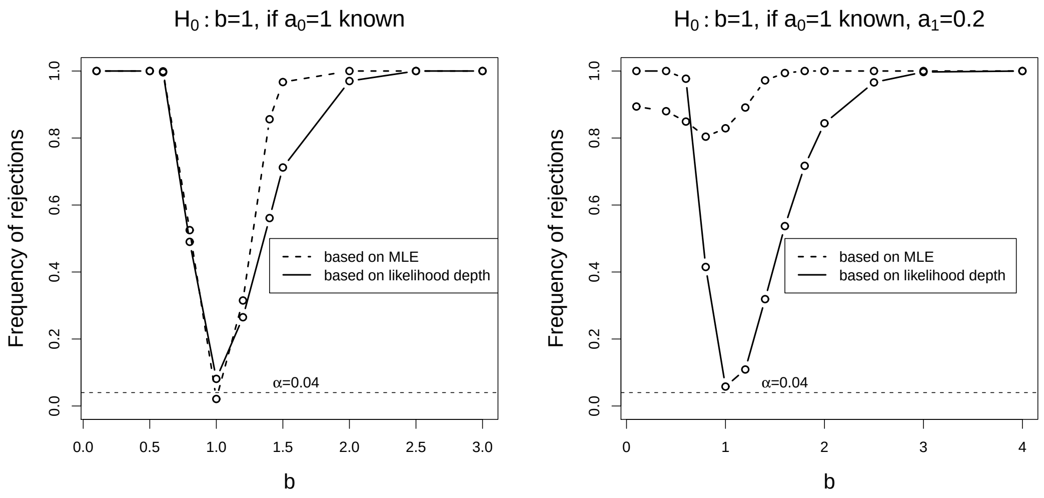

Figure 7: Simulated power for $H_0 : b = b_0$, known shape parameter, left-hand side: uncontaminated data, right-hand side: 0.1-contaminated data from $\text{Wei}(a_1, b)$.

The results for contaminated data are given in Table 2.

				MLE		likelihood depth	
a	b	a_1	b_1	coverage	length	coverage	length
1	1	0.5	1	0.796	0.496	0.941	0.592
1	1	5	1	0.976	0.445	0.805	0.360
1	1	1	10	0.008	0.847	0.857	0.683
5	5	1	10	0.005	1.120	0.952	0.650

Table 2: Coverage rate and length of the 96%-confidence intervals for the scale parameter for 0.1-contaminated data, with contamination distribution $\text{Wei}(a_1, b_1)$.

If we consider contaminated data, the coverage rate of the confidence intervals based on the MLE goes down to less than one percent in some cases, while the new method is robust against contamination.

3.4 Tests for the scale parameter, unknown shape parameter

If the shape parameter is unknown, we can not use Hoeffding's theorem to derive the asymptotic distribution of the test statistic anymore. As a_0 is unknown, we have to estimate it. Therefore we can use the estimation procedure based on likelihood depth introduced e.g. in Denecke (2010). We already mentioned that the test $\varphi_{b_0}^{\geq}$ for $H_0 : b \geq b_0$ against $H_1 : b < b_0$ is independent of a_0. Here we need not to do any extra work. For testing $H_0 : b \leq b_0$ and $H_0 : b = b_0$ we use the correction $c^2_{\alpha,a_0}(b_0)$ that is dependent on a_0, while the test statistic is independent of a_0. Considering a_0 unknown and estimated by $\hat{a}_{MLDE}$, we get a new plug-in test by using $\hat{a}_{MLDE} =: \hat{a}$, the estimator based on likelihood depth, instead of a_0. Then the correction becomes $\tilde{c}^2_{\alpha,\hat{a}}(b_0) = (-\ln(1-\exp(-1)))^{\frac{1}{\hat{a}}} b_0$. Simulation studies, see Denecke (2010), show that the power does not really change, if the shape parameter is unknown and has to be estimated in case of uncontaminated data. We get comparable results as in the case, where it is supposed to be known.

4 Tests for the parameters of the Weibull distribution in type-I right-censored data

In this section we study type-I right-censored data with fixed censor time c_0. Consider variables $Z_1, \ldots, Z_N$, $Z_i = \min(T_i, c_0)$, where T_i is the real lifetime and c_0 is the censor time, and the indicator variables Δ_i, $i = 1, \ldots, N$, that indicate if T_i is censored or not, i.e. $\Delta_i := \begin{cases} 1, & T_i < c_0 \\ 0, & T_i \geq c_0 \end{cases}$. The realizations of (Z_i, Δ_i) are described by (z_i, δ_i), $i = 1, \ldots, N$. Assume

$$c_0 > \text{med}(z_1, \ldots, z_N), \tag{3}$$

that means less than half the data is censored. The likelihood-function of a data (z_n, δ_n) is given in type-I right-censored data by

$$L(a, b, z_n, \delta_n) = f_{a,b}(z_n)^{\delta_n} S_{a,b}(z_n)^{1-\delta_n}, \; n = 1, \ldots, N,$$

where $S_{a,b}(z) = \exp\left(-\left(\frac{z}{b}\right)^a\right)$ is the survival function of the Weibull distribution.

4.1 Tests for the shape parameter, known scale parameter

Assume the scale parameter $b_0 > 0$ to be known. The set, where the derivative of the log-likelihood-function is positive or zero, i.e. $T^a_{pos} = \{(z, \delta); \frac{\partial}{\partial a} \ln L(a, b, z, \delta) \geq 0\}$ is determined in Denecke (2010) as $T^a_{pos} = [c_1^{\frac{1}{a}} b_0, c_2^{\frac{1}{a}} b_0] \cap (0, c_0)$ for $b_0 < c_0$. If $b_0 \geq c_0$, it holds $T^a_{pos} = [\min(c_0, c_1^{\frac{1}{a}} b_0), c_0]$. Thus, we have

$$p_{a,c_0} := P_{a,b_0,c_0}(T^a_{pos}) = \begin{cases} e^{-c_1} - e^{-c_2}, & b_0 < c_0 \wedge c_2^{\frac{1}{a}} b_0 < c_0 \\ e^{-c_1} - e^{\left(-\left(\frac{c_0}{b_0}\right)^a\right)}, & b_0 < c_0 \wedge c_2^{\frac{1}{a}} b_0 \geq c_0 \\ e^{-c_1}, & b_0 \geq c_0 \end{cases}.$$

Further, we have for $c_0 > b_0$, $p_{a_0,a,c_0} := P_{a_0,b_0}(T^{a,b_0}_{pos}) = e^{\left(-c_1^{\frac{a_0}{a}}\right)} - e^{\left(-\left(\frac{\min(c_0, c_2^{\frac{1}{a}} b_0)}{b_0}\right)^{a_0}\right)}$, and if $b_0 \geq c_0$, it is $p_{a_0,a,c_0} = \exp\left(-c_1^{\frac{a_0}{a}}\right)$. So, $p_{a_0,(\cdot),c_0}$ is strictly decreasing and $p_{(\cdot),a,c_0}$ strictly increasing in both cases.

For the correction of the tests, we have to check, whether it holds that the parameter with asymptotically maximum depth $s(a)$ is greater or smaller than the real parameter a. Denecke (2010) shows, if b_0 is known and $c_2^{\frac{1}{s(a)}} b_0 \geq c_0 > b_0$, then both cases can appear: If $a \leq \frac{\ln\left(-\ln\left(e^{-c_1} - \frac{1}{2}\right)\right)}{\ln(c_0/b_0)} \approx \frac{0.265}{\ln(c_0/b_0)}$, then $s(a) \leq a$, with $s(a) = a$, if $a = \frac{0.265}{\ln(c_0/b_0)}$, else $s(a) > a$ holds, where again $s(a)$ is such that $p_{a,s(a),c_0} = 0.5$. So we have situations, where the tests $H_0 : a \geq a_0$ and $H_0 : a = a_0$ have to be corrected by $c^1_\alpha(a_0)$ and situations, where we have to introduce the correction $c^2_\alpha(a_0)$ for the tests $H_0 : a \leq a_0$ and $H_0 : a = a_0$, depending on the value of a_0.

Lemma 2. *(a) If $b_0 < c_2^{\frac{1}{a_0}} b_0 < c_0$, it holds $c^1_\alpha(a_0) = k_0 \cdot a_0$, with $k_0 \approx 2.275$ as in the uncensored case.*

(b) If $b_0 < c_0 \leq c_2^{\frac{1}{a_0}} b_0$ and $a_0 < \frac{\ln(-\ln(\exp(-c_1) - \frac{1}{2}))}{\ln(c_0/b_0)} \approx \frac{0.265}{\ln(c_0/b_0)}$, $c^2_\alpha(a_0)$ can be determined as the solution for a of $e^{-c_1^{\frac{a_0}{a}}} - e^{-\left(\frac{c_0}{b_0}\right)^{a_0}} = 1 - e^{-c_1} + e^{-\left(\frac{c_0}{b_0}\right)^a}$. Further it holds $c^2_\alpha(a_0) < a_0$.

(c) If $b_0 < c_0 \leq c_2^{\frac{1}{a_0}} b_0$ and $a_0 > \frac{0.265}{\ln(c_0/b_0)}$, it is $c^1_\alpha(a_0)$ the solution for a of $e^{-c_1^{\frac{a_0}{a}}} - e^{-\left(\frac{\min(c_0, c_2^{\frac{1}{a}} b_0)}{b_0}\right)^{a_0}} = 1 - e^{-c_1} + e^{-\left(\frac{c_0}{b_0}\right)^a}$. Further, it holds $c^1_\alpha(a_0) > a_0$.

(d) If $b_0 \geq c_0$, then $s(a) > a$, but $c^1_\alpha(a_0)$ does not exist.

The proof can be found in Denecke (2010), the main argument used is again that $c^1_\alpha(a_0)$ and $c^2_\alpha(a_0)$ are given by $1 - p_{c^1_\alpha(a_0),c_0} = p_{a_0,c^1_\alpha(a_0),c_0}$, in the different situations.

Let be $\alpha < 0.5$.

(a) If $b_0 < c_2^{\frac{1}{a_0}} b_0 < c_0$, then the test $\varphi_{a_0}^{\leq} = 1_{\{\sup_{a \leq a_0} T(a,\cdot) < \Phi^{-1}(\alpha)\}}$ is a consistent test with asymptotic level α for $H_0 : a \leq a_0$. Further, let be $c_\alpha^1(a_0) = k_0 \cdot a_0$, with $k_0 \approx 2.275$ and a consistent asymptotic α-level test for $H_0 : a \geq a_0$ is given by $\varphi_{a_0}^{\geq} := 1_{\{\sup_{a \geq c_\alpha^1(a_0)} T(a,\cdot) < \Phi^{-1}(\alpha)\}}$. And $\varphi_{a_0}^{=} := \max(1_{\{T(a_0,\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}}, 1_{\{T(c_{\frac{\alpha}{2}}^1(a_0),\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}})$, is a consistent asymptotic α-level test for $H_0 : a = a_0$.

(b) If $b_0 < c_0 \leq c_2^{\frac{1}{a_0}} b_0$ and $a_0 > \frac{0.265}{\ln(c_0/b_0)}$, then the test $\varphi_{a_0}^{\leq} = 1_{\{\sup_{a \leq a_0} T(a,\cdot) < \Phi^{-1}(\alpha)\}}$ is a test with asymptotic level α for $H_0 : a \leq a_0$, $\varphi_{a_0}^{\geq} = 1_{\{\sup_{a \geq c_\alpha^1(a_0)} T(a,\cdot) < \Phi^{-1}(\alpha)\}}$, with $c_\alpha^1(a_0)$ being the solution for $a > a_0$ of $\exp\left(-c_1^{\frac{a_0}{a}}\right) - \exp\left(-\left(\frac{\min(c_0, c_2^{\frac{1}{a}} b_0)}{b_0}\right)^{a_0}\right) = 1 - \exp(-c_1) + \exp\left(-\left(\frac{c_0}{b_0}\right)^a\right)$, is a test with asymptotic level α for $H_0 : a \geq a_0$. And $\varphi_{a_0}^{=} := \max(1_{\{T(a_0,\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}}, 1_{\{T(c_{\frac{\alpha}{2}}^1(a_0),\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}})$, is an asymptotic α-level test for $H_0 : a = a_0$.

(c) If $b_0 < c_0 \leq c_2^{\frac{1}{a_0}} b_0$ and $a_0 < \frac{0.265}{\ln(c_0/b_0)}$, then the test $\varphi_{a_0}^{\leq} = 1_{\{\sup_{a \leq c_\alpha^2(a_0)} T(a,\cdot) < \Phi^{-1}(\alpha)\}}$, with $c_\alpha^2(a_0)$ being the solution of $\exp(-c_1^{\frac{a_0}{a}}) - \exp\left(-\left(\frac{c_0}{b_0}\right)^{a_0}\right) = 1 - \exp(-c_1) + \exp\left(-\left(\frac{c_0}{b_0}\right)^a\right)$, for $a < a_0$, is a test with asymptotic level α for $H_0 : a \leq a_0$ against $H_1 : a > a_0$, $\varphi_{a_0}^{\geq} := 1_{\{\sup_{a \geq a_0} T(a,\cdot) < \Phi^{-1}(\alpha)\}}$ is a test with asymptotic level α for $H_0 : a \geq a_0$, and $\varphi_{a_0}^{=} := \max(1_{\{T(c_{\frac{\alpha}{2}}^2(a_0),\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}}, 1_{\{T(a_0,\cdot) < \Phi^{-1}(\frac{\alpha}{2})\}})$, is an asymptotic α-level test for $H_0 : a = a_0$ against $H_1 : a \neq a_0$.

Consequently, in all cases a confidence interval for a is given by $\{a_0; \varphi_{a_0}^{=}(z_*) = 0\}$.

In the situation of (a) we prove consistency using Proposition 1, while in the case of (b) and (c) we can not prove that the resulting tests are consistent, as the proof needs c_α^1 and c_α^2 being strictly increasing. But in case of $b_0 < c_0 < c_2^{\frac{1}{a_0}} b_0$, this is not easy to see. We only give one example here where we fix b_0 and c_0 and determine $c_\alpha^1(a_0)$ for $a_0 \geq \frac{0.265}{\ln(c_0/b_0)}$, resp. $c_\alpha^2(a_0)$ for $a_0 \leq \frac{0.265}{\ln(c_0/b_0)}$. The results for $b_0 = 1, c_0 = 2$ are displayed in Figure 8.

The graphics provide the assumption, that c_α^1 and c_α^2 are strictly increasing, at least for $c_0 = 2, b_0 = 1$, for a second example see Denecke (2010). The remaining assumptions of Proposition 1 are true, as $p_{(\cdot),a,c_0}$ is strictly increasing, $p_{a,(\cdot),c_0}$ strictly decreasing, $\frac{1}{2} < p_{a_0,a_0,c_0} < \frac{1}{2} + \frac{1}{\sqrt{8}}$ and $c_\alpha^1(a_0) > a_0$ resp. $\frac{1}{2} < 1 - p_{a_0,a_0,c_0} < \frac{1}{2} + \frac{1}{\sqrt{8}}$ and $c_\alpha^2(a_0) < a_0$ hold.

The simulated power function of this new test is compared to the simulated power function of the test based on the MLE for the hypotheses $H_0 : a \leq a_0$ and $H_0 : a \geq a_0$ in

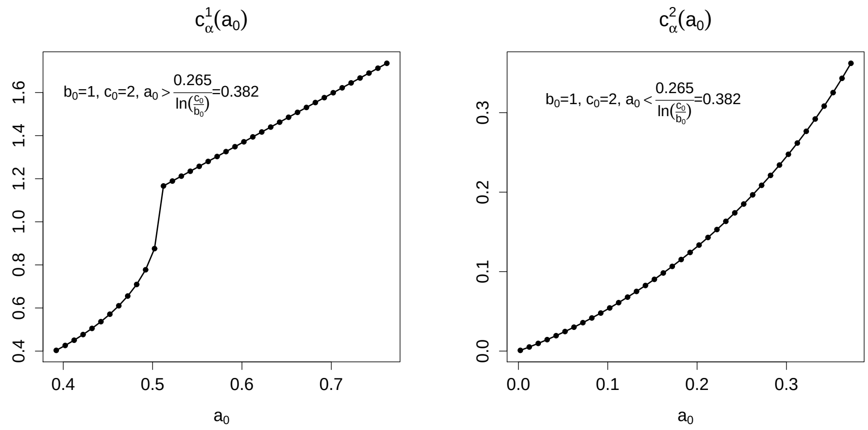

Figure 8: Development of c_α^1 and c_α^2 for $b_0 = 1$, $c_0 = 2$.

20% right-censored data for $a_0 = 1$. The results are displayed in Figure 9.

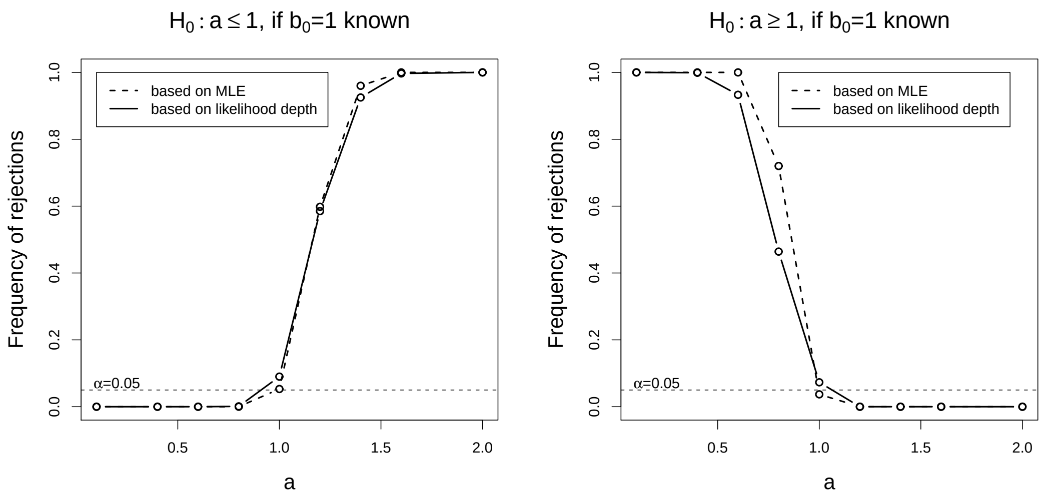

Figure 9: Simulated power of the tests for the scale in 20% right-censored data with known scale parameter b_0.

If one fifth of the data is censored, the new test does not keep the level. The differences between the new test and the test based on the maximum likelihood estimator seem not to be very large for uncontaminated data, especially when considering $H_0 : a \leq a_0$ (see left-hand side). We consider also contaminated data, when simulating confidence intervals. There we will see that the new test is robust in contrast to the test based on the MLE.

We simulate 95%-confidence intervals for the shape parameter with the new test (lik depth) in ε-contaminated data, see Table 3. The contamination distribution is Wei(a_1, b) and $\varepsilon = 0.1$, the rate of censored data 20%. The confidence intervals are compared to the ones based on the maximum likelihood estimator (MLE) and on the method of medians (MoM). In further simulation studies, see Denecke (2010), it is shown that for uncontaminated data the coverage rate of the MoM goes even down to less than 25 % in censored data, so that this method seems not practical at all. The method of the MLE produces covering rates that go down to five percent in contaminated data. The new method is robust and can also be used in censored data in contrast to the MoM.

			MLE		MoM		lik depth	
a	b	a_1	cov.	length	cov.	length	cov.	length
1	1	0.2	0.427	0.437	0.819	0.527	0.844	0.472
1	1	2	0.997	0.603	0.429	0.630	0.774	0.570
5	1	0.2	0.005	1.086	0.824	2.595	0.805	2.383

Table 3: Coverage rate (cov.) and length of the confidence intervals for the shape parameter, contaminated (10% with Wei(a_1, b_1)) and censored (20%) data, known scale b_0.

4.2 Tests for the shape parameter, unknown scale parameter

If the scale parameter is unknown, again the theory as in the case, where the scale b_0 is known cannot be used anymore. We plug $\tilde{b}_N = \text{med}(y_*)$ into the depth-function of a and calculate the likelihood depth as

$$\begin{aligned} d_T^{\tilde{b}_N}(a, z_*) \;=\; \frac{1}{N} \min \Big(& \sharp\{n; \delta_n = 1 \text{ and } c_1^{\frac{1}{a}} \tilde{b}_N \leq y_n \leq c_2^{\frac{1}{a}} \tilde{b}_N\}, \\ & \sharp\{n; \delta_n = 1 \text{ and } (y_n \geq c_2^{\frac{1}{a}} \tilde{b}_N \text{ or } y_n \leq c_1^{\frac{1}{a}} \tilde{b}_N)\} + (N-k) \Big). \end{aligned}$$

Since, it holds

$$\frac{1}{N} \sharp\{n; \delta_n = 1 \text{ and } t_{01}^{a,\tilde{b}_N} \leq y_n \leq t_{02}^{a,\tilde{b}_N}\} \to -\exp\left(- \left(\frac{\min(c_0, c_2^{\frac{1}{a}} b_0 (\ln 2)^{\frac{1}{a}})}{b_0} \right)^{a_0} \right) + 2^{-c_1^{\frac{a_0}{a}}}$$

as N tends to infinity, we use $\tilde{p}_{a,c_0} := \begin{cases} 2^{-c_1} - 2^{-c_2}, & c_2^{\frac{1}{a}}\tilde{b}_N < c_0 \\ 2^{-c_1} - 2^{-\left(\frac{c_0}{\tilde{b}_N}\right)^a}, & c_2^{\frac{1}{a}}\tilde{b}_N \geq c_0 \end{cases}$, and define the test statistic as $\widetilde{T}(a, z_*) := \sqrt{N}\frac{d_S^{\tilde{b}_N}(a,z_*)-2\tilde{p}_{a,c_0}(1-\tilde{p}_{a,c_0})}{2\sqrt{\tilde{p}_{a,c_0}(1-\tilde{p}_{a,c_0})(1-2\tilde{p}_{a,c_0})^2}}$. We work with this, as if we could use the same theory as in the case of b_0 known in the last subsection. $\tilde{c}_\alpha^2(a_0)$ is given by the solution for $0 < a < a_0$ of $2^{-c_1^{\frac{a_0}{a}}} - 2^{-\left(\frac{c_0}{\tilde{b}_N}\right)^{a_0}} = 1 - 2^{-c_1} + 2^{-\left(\frac{c_0}{\tilde{b}_N}\right)^a}$, analog to the case of known scale parameter. Define a test for $H_0 : a \leq a_0$ against $H_1 : a > a_0$ for unknown scale parameter as

$$\tilde{\varphi}_{a_0}^{\leq}(z_*) := \begin{cases} 1_{\{\sup_{a\leq\tilde{c}_\alpha^2(a_0)} \tilde{T}(a,z_*)<\Phi^{-1}(\alpha)\}}(z_*), & c_2^{\frac{1}{a_0}}\tilde{b}_N \geq c_0 \text{ and } a_0 < \frac{k_1}{\ln\left(\frac{c_0}{\tilde{b}_N}\right)} \\ 1_{\{\sup_{a\leq a_0} \tilde{T}(a,z_*)<\Phi^{-1}(\alpha)\}}(z_*), & \text{else} \end{cases},$$

where $k_1 = \ln\left(-\frac{\ln\left(2^{-c_1}-2^{-1}\right)}{\ln(2)}\right) \approx 0.455$. If $c_2^{\frac{1}{a_0}}\tilde{b}_N \geq c_0$ and $a_0 > \frac{k_1}{\ln\left(\frac{c_0}{\tilde{b}_N}\right)}$, it is $\tilde{c}_\alpha^1(a_0)$ given by $1 - 2^{-c_1} + 2^{-\left(\frac{\min(c_0, c_2^{\frac{1}{\tilde{c}_\alpha^1(a_0)}}\tilde{b}_N)}{\tilde{b}_N}\right)^{\tilde{c}_\alpha^1(a_0)}} = 2^{-c_1^{\frac{a_0}{\tilde{c}_\alpha^1(a_0)}}} - 2^{-\left(\frac{\min(c_0, c_2^{\frac{1}{\tilde{c}_\alpha^1(a_0)}}\tilde{b}_N)}{\tilde{b}_N}\right)^{a_0}}$. Further, the test for $H_0 : a \geq a_0$ against $H_1 : a < a_0$ is defined as

$$\tilde{\varphi}_{a_0}^{\geq}(z_*) = \begin{cases} 1_{\{\sup_{a\geq\tilde{k}_0 a_0} \tilde{T}(a,z_*)<\Phi^{-1}(\alpha)\}}(z_*), & c_2^{\frac{1}{a_0}}\tilde{b}_N < c_0 \\ 1_{\{\sup_{a\geq a_0} \tilde{T}(a,z_*)<\Phi^{-1}(\alpha)\}}(z_*), & c_2^{\frac{1}{a_0}}\tilde{b}_N \geq c_0 \text{ and } a_0 < \frac{k_1}{\ln\frac{c_0}{\tilde{b}_N}} \\ 1_{\{\sup_{a\geq\tilde{c}_\alpha^1(a_0)} \tilde{T}(a,z_*)<\Phi^{-1}(\alpha)\}}(z_*), & c_2^{\frac{1}{a_0}}\tilde{b}_N \geq c_0 \text{ and } a_0 > \frac{k_1}{\ln\frac{c_0}{\tilde{b}_N}} \end{cases},$$

where $k_1 \approx 0.455$.

Using the tests for $H_0 : a \geq a_0$ and $H_0 : a \leq a_0$, a test for $H_0 : a = a_0$ against $H_1 : a \neq a_0$ is defined as $\tilde{\varphi}_{a_0}^{=} := \max(1_{\{\tilde{T}(a_0,\cdot)<\Phi^{-1}(\frac{\alpha}{2})\}}, 1_{\{\tilde{T}(a,\cdot)<\Phi^{-1}(\frac{\alpha}{2})\}})$, with

$$a = \begin{cases} \tilde{k}_0 a_0, & c_2^{\frac{1}{a_0}}\tilde{b}_N < c_0 \\ \tilde{c}_\alpha^1(a_0), & c_2^{\frac{1}{a_0}}\tilde{b}_N \geq c_0 \text{ and } a_0 > \frac{k_1}{\ln\frac{c_0}{\tilde{b}_N}} \\ \tilde{c}_\alpha^2(a_0), & c_2^{\frac{1}{a_0}}\tilde{b}_N \geq c_0 \text{ and } a_0 < \frac{k_1}{\ln\frac{c_0}{\tilde{b}_N}} \end{cases}$$

and $\tilde{k}_0 \approx 1.835$, $k_1 \approx 0.455$. Hence, confidence intervals for the shape parameter of the Weibull distribution in type-I right-censored data with unknown scale parameter are given by $\{a_0 > 0; \tilde{\varphi}_{a_0}^{=}(z_*) = 0\}$.

Again simulation studies, see Denecke (2010), show that as in the case of uncensored data, that the estimation of the scale parameter has no influence on the power of the tests.

4.3 Tests for the scale parameter, known shape parameter

Now consider type-I right-censored data and the shape parameter to be known. The number of uncensored data is denoted with k, according to (3) it holds $k > \frac{N}{2}$. In Denecke and Müller (2011c) it is shown that $T^b_{pos} = \begin{cases} [b\,,\infty), & b < c_0 \\ [c_0\,,\infty), & b \geq c_0 \end{cases}$. Thus, it holds $p_{b,c_0} := P_{a_0,b}(T^b_{pos}) = \begin{cases} \exp(-1) = p_{scale}, & b < c_0 \\ \exp\left(-\left(\frac{c_0}{b}\right)^{a_0}\right), & b \geq c_0 \end{cases}$. If $b < c_0$, this is just the same as in the case of uncensored data. If $b \geq c_0$, the simplicial likelihood depth is constant, but p_{b,c_0} is growing with b up to $\exp(0) = 1$. Thus, the test statistic is growing to infinity. Hence, testing hypotheses for $b_0 > c_0$ does not make sense. Moreover, when testing $H_0 : b \geq b_0$ for $b_0 \leq c_0$, we only consider the supremum of the test statistics over $b \in \{b; b_0 \leq b \leq c_0\}$. As for $b \leq c_0$ the test statistic is the same as in the uncensored case and also $P_{a_0,b_0}(T^b_{pos}) = P_{a_0,b_0}(T \geq b) = \exp\left(-\left(\frac{b}{b_0}\right)^{a_0}\right)$, for $b \leq b_0 \leq c_0$, is the same as in the uncensored case, we can use the results from there and get the same (consistent) tests.

Figure 10 shows some results for the simulation of the power functions for the different hypotheses in 20% censored data. As a comparison we consider the Wald test, see e.g. the textbook of Lawless (2003).

For $H_0 : b \geq b_0$ the power-functions of the new test and the Wald test do not really differ. For $H_0 : b \leq b_0$ the power of the Wald test is only slightly better.

We use the test for $H_0 : b = b_0$ to give confidence intervals for the scale parameter of the Weibull distribution: $\{b_0 > 0; \varphi^{=}_{b_0}(z_*) = 0\}$, where $\varphi^{=}_{b_0}$ denotes the test for $H_0 : b = b_0$.

We compare the confidence intervals based on likelihood depth for censored data with confidence intervals given by Wald-type confidence procedures, see e.g. Lawless (2003). Table 4 shows the results of some simulations of confidence intervals in ε-contaminated data, $\varepsilon = 0.2$ and 20% right-censored data.

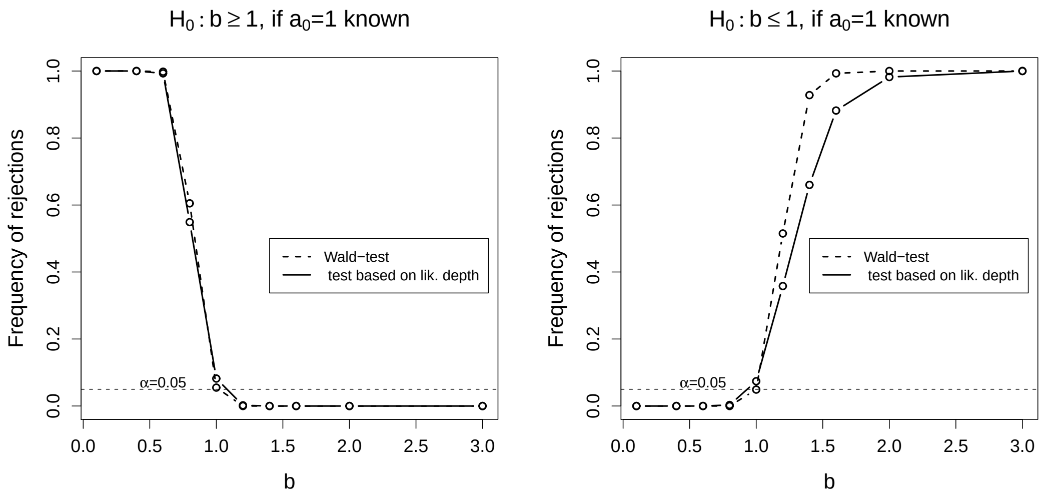

Figure 10: Simulated power-function of the tests for the scale parameter with 20% right-censored data.

				Wald-type		likelihood depth	
a	b	a_1	b_1	coverage	length	coverage	length
1	1	0.2	1	0.888	0.459	0.953	0.755
1	1	0.2	10	0.684	0.524	0.962	0.823
1	1	1	10	0.039	0.698	0.617	0.953
1	2	0.2	10	0.778	1.004	0.977	1.575
1	2	1	100	0.003	1.722	0.587	2.354

Table 4: Coverage rate and length of the 95%-confidence intervals for the scale parameter for 20% right-censored and ε-contaminated data, $\varepsilon = 0.2$.

In Table 4 we see that the covering rates of the confidence intervals based on the new test are much more stable than the ones based on the Wald test.

4.4 Tests for the scale parameter, unknown shape parameter

Up to know, for the type-I right-censored data, we only considered tests in the situation of known shape parameter a_0. If it is unknown, we use an estimation based on likelihood depth for it, see e.g. Denecke (2010), and plug it into the the correction, c^2_{α,a_0}, of the tests

$\varphi_{b_0}^{\leq}$ and $\varphi_{b_0}^{=}$ instead of a_0. As in the uncensored case, the test statistic is independent of a_0. For this "plug-in" tests we can not prove the consistency. Denecke (2010) shows in simulation studies, that the estimation of a_0 has no influence on the power of the tests.

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