The approximation of left-continuous t-norms

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Abstract

A discrete t-norm is a binary operation on a finite subset of the real unit interval fulfilling the same algebraic conditions as t-norms. We show that any left-continuous t-norm can, in a natural sense, be approximated by a discrete t-norm with an arbitrary precision.

1 Introduction

A triangular norm, or t-norm for short, is a binary operation on the real unit interval that is associative, commutative, neutral w.r.t. 1, and monotone in each argument [15]. In fuzzy logics [11], such an operation is canonically chosen to interpret the conjunction. For instance, the standard semantics of the logic MTL [7] is based on pairs of a t-norm and the corresponding residuum, the t-norm being used for the conjunction, the residuum for the implication. Not every t-norm has a residuum though; the necessary and sufficient condition is its left-continuity. We are interested here only in t-norms with this property; we write "l.-c." for "left-continuous".

L.-c. t-norms have been the subject of intensive research during the last decades. For an overview, see, e.g., [17]; a monograph devoted to the topic is [15]. In search of general structural features of t-norms, a number of construction methods have been proposed. For an overview of this particularly wide line of research, we refer to [20] and the references given there. In more recent times, t-norms have been studied more and more intensively from an algebraic viewpoint. In fact, t-norms give rise to MTL-algebras, which in turn are residuated lattices. For a general account of t-norms from an algebraic perspective, see again [20]; for residuated lattice in general, see [2]. Finally, the structure of left-continuous t-norms can be understood to a great extent by the decomposition along their filters [22, 23].

It is still to be admitted that the known theory of t-norm leaves space for further improvements. Only under the assumption of continuity a satisfactory general theory could be established; see, e.g., [15]. The present paper is meant as a contribution to the theory of t-norms from a quite particular point of view. In order to define its context, let us first observe that, when looking back, we might want to distinguish two styles of research. On the one hand, we have the many contributions where t-norms are considered as two-place real functions and aspects are examined that originate from real analysis or geometry. On the other hand, the algebraists aim at classifying t-norms up to isomorphism; recall that two t-norms are isomorphic if one results from the other one by an order automorphism of the real unit interval.

The present paper could be regarded borderline, but should most appropriately be counted to the first category. In fact, we are interested in the approximation of l.-c. t-norms. Approximation is in our context not exactly the same, but something inspired by the approximation of real functions by real functions of a specific simple type. Our proof methods, however, rely on algebraic results on t-norms. This is why a good part of this paper deals actually with totally ordered monoids and their congruences, and our results might not be insignificant also from an algebraic perspective.

The approximation of t-norms has been studied in several papers under the assumption of continuity; [15, Chapter 8] is devoted to this topic. To mention one of the most remarkable results, for any continuous t-norm there is a t-norm isomorphic to the product t-norm such that the two operations differ for each pair of arguments by less than a given parameter $\varepsilon > 0$. Moreover, the approximation of t-norms that are k-Lipschitz has been studied in [18].

To obtain similar results for t-norms that are only left-continuous is certainly difficult. Our concern is anyway somewhat different. Given the fact that l.-c. t-norms can have, as it seems, an arbitrarily complicated structure, we ask if it is nevertheless possible to identify them with a sequence of operations on a finite universe. This question has led to the idea of approximating l.-c. t-norms by discrete t-norms. These operations differ from ordinary t-norm in having only a finite subset of the real unit interval as their domain. We note that often, but not in the present paper, equidistancy of the elements of the domain is assumed. Otherwise, discrete t-norms are required to fulfil the same algebraic conditions as t-norms. Algebraically, discrete t-norms are finite, negative, commutative totally ordered monoids.

Discrete t-norms have been considered in several different contexts. The early paper [19], e.g., considers them in connection with linguistic modelling for expert systems. A general account is provided in the paper [5]. Furthermore, the number of discrete t-norms has been determined; in [4], up to a size of 14 the concrete numbers are indicated. We finally note that aggretation operations of more general type than t-norms have been considered on discrete scales as well; an example is the paper [16].

In this paper, we show the following. Let $\odot$ be a l.-c. t-norm. For a discrete t-norm $\varodot$ to approximate $\odot$ to the degree $\varepsilon > 0$, we require that the ε-neighbourhoods of the points in the domain of $\varodot$ cover $[0, 1]$ and that the difference between $a \odot b$ and $a \varodot b$ is smaller than ε for all a, b in the domain of $\varodot$. We shall show that for any ε, a discrete t-norm with this property exists. We then conclude that there is a sequence $\varodot_n$, $n \in \mathbb{N}$, of discrete t-norms that determine $\odot$ uniquely.

A variant of this method comes slightly closer to the results that have been obtained in the continuous case. Namely, we may alternatively use as approximating functions l.-c. t-norms of a specific type. In fact, any discrete t-norm can be extended in a canonical way to an ordinary t-norm [14]. Calling these t-norms *finitary*, we can show that there is a sequence of finitary t-norms approximating, in a certain sense, any given l.-c. t-norm.

The paper is structured as follows. Section 2 provides basic definitions and makes our aims explicit. To understand the proofs, some algebraic background is necessary; to this end, we provide some basic information on totally ordered monoids in Section 3 and their quotients by filters in Section 4. Before proving the main theorem, we explain in Section 5 our idea of approximation on the basis of easy examples. The main approximation theorem is the topic of Section 6. We present a variant of our approximation method, based on ordinary rather than finite t-norms, in Section 7. A few concluding remarks are contained in the last Section 8.

2 The approximation of left-continuous t-norms

We are concerned in this paper with the following type of binary operation on the real unit interval.

Definition 2.1. An operation $\odot\colon [0,1]^2 \to [0,1]$ is called a *triangular norm*, or *t-norm* for short, if the following conditions hold for all $a, b, c \in [0,1]$:

(T1) $(a \odot b) \odot c = a \odot (b \odot c)$ (*associativity*),

(T2) $a \odot b = b \odot a$ (*commutativity*),

(T3) $a \odot 1 = a$ (*neutrality of* 1),

(T4) $a \leq b$ implies $a \odot c \leq b \odot c$ (*monotonicity*).

A t-norm $\odot$ is called *left-continuous*, abbreviated *l.-c.*, if, for each $a \in [0,1]$, the function $(0,1] \to [0,1], \quad x \mapsto x \odot a$ is left-continuous.

Here, the left continuity of a function $\lambda\colon (0,1] \to [0,1]$ means $\lim_{x \nearrow a} \lambda(x) = \lambda(a)$ for each $a \in (0,1]$. Note that, since a t-norm $\odot$ is monotone in both its arguments, $\odot$ is left-continuous if and only if $a \odot \bigvee_\iota b_\iota = \bigvee_\iota (a \odot b_\iota)$ for any $a, b_\iota \in [0,1]$, $\iota \in I$.

Our aim is to approximate l.-c. t-norms by means of certain finite structures. In the area of t-norms, it has been common to consider binary operations fulfilling the same properties as t-norms but being defined only on a finite set of equidistant elements of the real unit interval. We will use such functions here as well; however, equidistance will not be required.

Definition 2.2. Let $\mathcal{L}$ be a finite subset of the real unit interval $[0,1]$ containing 0 and 1; let $\leq$ be the natural order on $\mathcal{L}$; and let $\odot$ be such that, for all $a, b, c \in \mathcal{L}$, the properties (T1)–(T4) hold. Then we call $\odot$ a *discrete t-norm*.

The domain of discrete t-norms is strictly included in the domain of a t-norm. Hence it is not a priori clear what we mean by an approximation of the latter by the former. We make the following definition.

Definition 2.3. Let $\odot$ be a l.-c. t-norm; let $\varodot$ be a discrete t-norm on $\mathcal{L} \subseteq [0,1]$; and let $\varepsilon > 0$. We say that $\varodot$ *approximates* $\odot$ *with the precision* ε if the following conditions hold:

(i) For any $a \in [0,1]$ there is a $b \in \mathcal{L}$ such that $|b - a| \leq \varepsilon$.

(ii) For any $a, b \in \mathcal{L}$, $|a \odot b \ - \ a \varodot b| \leq \varepsilon$.

Note what the approximation of a t-norm according to Definition 2.3 means and what it does not mean. The domain $\mathcal{L}$ of the approximating discrete t-norm is required to be dense within the real unit interval to a prescribed degree, and on this domain the approximating operation is supposed to deviate not more than a given threshold from the original values. Thus the original t-norm can be reconstructed on a certain subset with a definite precision; but the values in between are subject to bounds whose difference can exceed the threshold.

We have to show that our definition is appropriate in the following sense: we should be able to reconstruct a l.-c. t-norm provided we are given approximations with an arbitrary precision.

Proposition 2.4. *Assume that, for each $n \geq 1$, the discrete t-norm $\varodot_n$ approximates the l.-c. t-norm $\odot$ with the precision $\varepsilon_n > 0$, and $\lim_{n\to\infty} \varepsilon_n = 0$. Then $\odot$ is the only l.-c. t-norm with this property.*

Proof. Let $a, b \in [0,1]$. For each n, let $\mathcal{L}_n$ be the domain of $\varodot_n$ and let $a^{(n)}, b^{(n)} \in \mathcal{L}_n$ be largest such that $a^{(n)} \leq a$ and $b^{(n)} \leq b$, respectively. We conclude, using the left-continuity of $\odot$, that then $\lim_{n\to\infty} a^{(n)} \varodot_n b^{(n)} = a \odot b$. □

Proposition 2.4 depends on the left continuity of a t-norm. The following example shows that this assumption is essential.

Example 2.5. *Let us consider the t-norm proposed in* [15, Ex. 3.75]. *Let $C \subseteq [0, \frac{1}{2}]$ and let*

$$a \odot_C b \ = \ \begin{cases} 0 & \text{if } a+b<1, \quad \text{or } a+b=1 \text{ and } a \wedge b \in C \\ a \wedge b & \text{otherwise.} \end{cases}$$

Then $\odot_C$ is a t-norm, which coincides with the nilpotent minimum t-norm [8] *if $C = [0,1]$ and is otherwise not left-continuous.*

For each $n \geq 1$, let $\mathcal{L}_n = \{0, \frac{1}{n}, \ldots, 1\}$ and let $\varodot_n$ be the restriction of $\odot_C$ to $\mathcal{L}_n$. Then $\varodot_n$ is a discrete t-norm that approximates $\odot_C$ with the precision $\frac{1}{2n}$. But the sequence $\varodot_n$, $n \geq 1$, only depends on $C \cap \mathbb{Q}$. Consequently, $\varodot_n$ approximates for each n any other t-norm $\odot_{C'}$ such that $C \cap \mathbb{Q} = C' \cap \mathbb{Q}$ with the precision $\frac{1}{2n}$ as well.

3 Totally ordered monoids

In what follows, we need an algebraic framework to deal with t-norms.

Definition 3.1. An algebra $(\mathcal{L}; \odot, 1)$ is a *monoid* if (i) $\odot$ is an associative binary operation and (ii) 1 is neutral w.r.t. $\odot$. A total order $\leq$ on a monoid $\mathcal{L}$ is called *compatible* if, for any $a, b, c \in \mathcal{L}$, $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$. A structure $(\mathcal{L}; \leq, \odot, 1)$ such that $(\mathcal{L}; \odot, 1)$ is a monoid and $\leq$ is a compatible total order on $\mathcal{L}$ is a *totally ordered monoid*, or *tomonoid* for short.

A tomonoid $\mathcal{L}$ is called *commutative* if the monoidal operation $\odot$ is commutative. $\mathcal{L}$ is called *negative* if the neutral element 1 is the top element. $\mathcal{L}$ is called *quantic* if (i) $\mathcal{L}$ is almost complete and (ii) for any elements a, b_ι, $\iota \in I$, of $\mathcal{L}$ we have

$$a \odot \bigvee_\iota b_\iota \;=\; \bigvee_\iota (a \odot b_\iota) \quad \text{and} \quad (\bigvee_\iota b_\iota) \odot a \;=\; \bigvee_\iota (b_\iota \odot a).$$

We note that negative tomonoids are often also called *integral*. Furthermore, by a totally ordered set to be almost complete, we mean that all non-empty suprema exist. This means, informally, that the totally ordered set is complete up to the possible absence of a least element. There is a simple practical reason to require almost completeness rather than completeness: filters – which we define below – would otherwise lead outside the class of tomonoids that we consider here. Note finally that any quantic tomonoid $\mathcal{L}$ is residuated; for any $a, b \in \mathcal{L}$, there is a largest $c \in \mathcal{L}$ such that $a \odot c \leq b$ as well as a largest $d \in \mathcal{L}$ such that $d \odot a \leq b$.

Let $[0, 1]$ be the real unit interval, endowed with its natural order. Then obviously, a binary operation $\odot$ on $[0, 1]$ is a t-norm if and only if $([0, 1]; \leq, \odot, 1)$ is a commutative, negative tomonoid. We refer to this structure as the *tomonoid based on* $\odot$. Moreover, a t-norm $\odot$ is left-continuous if and only it the tomonoid based on $\odot$ is quantic. Consequently, our objective are quantic, negative, commutative tomonoids; we abbreviate these three properties with "q.n.c.".

Similarly, let $\mathcal{L}$ be a finite subset of $[0, 1]$ containing 0 and 1, endowed with its natural order. Then a binary operation $\circledcirc$ on $\mathcal{L}$ is a discrete t-norm if and only if $(\mathcal{L}; \leq, \circledcirc, 1)$ is a commutative, negative tomonoid. Again, we refer to this structure as the *tomonoid based on* $\circledcirc$. Note that, trivially, this tomonoid is also quantic, that is, a q.n.c. tomonoid as well.

The Cayley tomonoid

Each monoid can be identified with a monoid of mappings under composition. To this end, each element a of a monoid $(M; \odot, 1)$ is identified with the mapping $M \to M, \;\; x \mapsto x \odot a$, and the composition of mappings takes over the role of the monoidal operation. In [3], this is called the *regular representation* of monoids. It generalises the representation of groups by transformations groups, which goes back to A. Cayley.

A representation of q.n.c. tomonoid along these lines is, from a formal point of view, not essential for the results of this paper, yet a quite practical tool. Not only is the visual representation simplified compared to the commonly used three-dimensional graphs. What is important here is that we can represent the quotient structure of q.n.c. tomonoid in a transparent way, facilitating the understanding of our main result. We recall here only the basic facts; for more details, we may refer to our previous papers [22, 23].

Definition 3.2. Let $(R; \leq)$ be a totally ordered set, and let Φ be a set of order-preserving mappings from R to R. We denote by $\leq$ the pointwise order on Φ, by $\circ$ the composition of mappings, and by id_R the identity mapping on R. Assume that (i) $\leq$ is a total order on Φ, (ii) Φ is closed under $\circ$, and (iii) $id_R \in \Phi$. Then we call $(\Phi; \leq, \circ, id_R)$ a *composition tomonoid* on R.

It is easily checked that a composition tomonoid is in fact a tomonoid. Conversely, each tomonoid can be viewed as a composition tomonoid.

Proposition 3.3. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a tomonoid. For each $a \in \mathcal{L}$, put*

$$\lambda_a \colon \mathcal{L} \to \mathcal{L}, \quad x \mapsto x \odot a, \tag{1}$$

and let $\Lambda = \{\lambda_a \colon a \in \mathcal{L}\}$. Then $(\Lambda; \leq, \circ, id_{\mathcal{L}})$ is a composition tomonoid on $\mathcal{L}$, and

$$\pi \colon \mathcal{L} \to \Lambda, \quad a \mapsto \lambda_a \tag{2}$$

is an isomorphism of $(\mathcal{L}; \leq, \odot, 1)$ with $(\Lambda; \leq, \circ, id_{\mathcal{L}})$.

Moreover, $\mathcal{L}$ is commutative if and only if $\circ$ is commutative; $\mathcal{L}$ is negative if and only if $id_{\mathcal{L}}$ is the top element of Λ; and $\mathcal{L}$ is quantic if and only if (i) *$\mathcal{L}$ is almost complete and every $\lambda \in \Phi$ preserves arbitrary suprema and* (ii) *Λ is almost complete and suprema are calculated pointwise.*

Given a q.n.c. tomonoid $(\mathcal{L}; \leq, \odot, 1)$, we call the composition tomonoid $(\Lambda; \leq, \circ, id_{\mathcal{L}})$ associated with $\mathcal{L}$ according to Proposition 3.3 the *Cayley tomonoid* of $\mathcal{L}$.

Example 3.4. *By Proposition* 3.3*, each left-continuous t-norm can be identified with a monoid under composition of pairwise commuting, increasing, and left-continuous mappings from* $[0, 1]$ *to* $[0, 1]$*, which are moreover pairwise comparable and located below the identity. Recall that*

$$\begin{aligned} a \odot_1 b &= (a + b - 1) \vee 0, \\ a \odot_2 b &= a \cdot b, \\ a \odot_3 b &= a \wedge b \end{aligned}$$

defines the Łukasiewicz, product, and Gödel t-norm, respectively. The Cayley tomonoids of the t-norm monoids based on these t-norms are shown in Figure 1 *in a schematic way.*

A few additional properties and facts about tomonoids will be needed in the sequel.

We call a tomonoid $\mathcal{L}$ *finitely generated* if $\mathcal{L}$ is finitely generated as a monoid. The following lemma can be found in [6, Corollary 1.2]. It is, in addition, a special case of [13, Theorem 7.5].

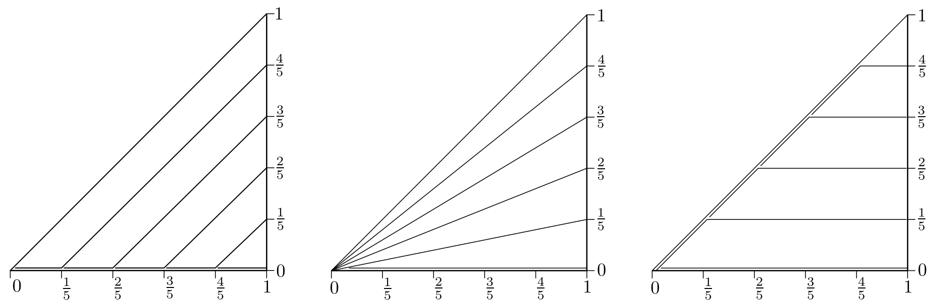

Figure 1: The Łukasiewicz, product, and Gödel t-norm.

Lemma 3.5. *Let* $(\mathcal{L}; \leq, \odot, 1)$ *be a negative, commutative tomonoid that is finitely generated. Then each subset of* $\mathcal{L}$ *possesses a largest element. In particular,* $\mathcal{L}$ *is quantic.*

We will call a q.n.c. tomonoid $(\mathcal{L}; \leq, \odot, 1)$ *Archimedean* if, for any $a \leq b < 1$, there is an $n \geq 1$ such that $b^{\odot n} \leq a$. Here, $b^{\odot n} = b \odot \ldots \odot b$ (n factors). We should note that the Archimedean property has in the particular context of t-norms often been defined slightly differently; e.g., in [15], the definition is similar but restricted to the non-zero elements. Thus, e.g., the product t-norm is Archimedean in the sense of [15], whereas the tomonoid based on the product t-norm is not Archimedean according to the definition employed here.

Furthermore, a q.n.c. tomonoid that possesses a least element will be called *bounded*.

Lemma 3.6. *Let* $(\mathcal{L}; \leq, \odot, 1)$ *be a q.n.c. tomonoid which is finitely generated, Archimedean, and bounded. Then* $\mathcal{L}$ *is finite.*

Proof. Let G be a finite set of generators of $\mathcal{L}$ and let 0 be the least element of $\mathcal{L}$. Then, for each $g \in G$, there is a $k \geq 1$ such that $g^{\odot k} = 0$. Hence $\mathcal{L}$ is finite. □

4 Quotients of tomonoids

The essential means that we need for the approximation of t-norms is the formation of tomonoid quotients. Namely, with each q.n.c. tomonoid, we may associate the chain of quotients arising from filters.

We note that the explanations of this section are bound to our particular context; for congruences of residuated lattices we refer to [2] or [10, Chapter 4]. For further details on congruences of tomonoids based on t-norms, we refer to [23].

In what follows, a *subtomonoid* of a tomonoid $\mathcal{L}$ is a submonoid of $\mathcal{L}$ endowed with the total order inherited from $\mathcal{L}$. Note that a subtomonoid of a negative, commutative tomonoid is again negative and commutative.

Definition 4.1. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. Then a *filter* of $\mathcal{L}$ is a subtomonoid $(F; \leq, \odot, 1)$ of $\mathcal{L}$ such that $f \in F$ and $g \geq f$ imply $g \in F$.

Note that each q.n.c. tomonoid possesses at least one filter. The filter $F = \{1\}$ will be called *trivial*, any other filter *non-trivial*. Moreover, the filter $F = \mathcal{L}$ is called *improper*, any other filter *proper*. Note furthermore that each filter of a q.n.c. tomonoid is again a q.n.c. tomonoid.

It is obvious from Definition 4.1 that, for any two filters of a q.n.c. tomonoid, one is included in the other one. That is, the set of filters is a chain under set-theoretical inclusion. This chain is bounded; the trivial filter is the smallest filter, the improper filter is the largest filter.

Quotients of tomonoids are defined as follows.

Definition 4.2. Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. An equivalence relation $\sim$ on $\mathcal{L}$ is called a *tomonoid congruence* if (i) $\sim$ is a congruence of $\mathcal{L}$ as a monoid and (ii) the $\sim$-classes are convex. We endow then the quotient $\langle \mathcal{L} \rangle_\sim$ with the total order given by

$$\langle a \rangle_\sim \leq \langle b \rangle_\sim \text{ if } a' \leq b' \text{ for some } a' \sim a \text{ and } b' \sim b$$

for $a, b \in \mathcal{L}$, with the induced operation $\odot$, and with the constant $\langle 1 \rangle_\sim$. The resulting structure $(\langle \mathcal{L} \rangle_\sim; \leq, \odot, \langle 1 \rangle_\sim)$ is called a *tomonoid quotient* of $\mathcal{L}$.

Because we work with a total order, each $\sim$-class is of the form (u, v), $(u, v]$, $[u, v)$, or $[u, v]$ for some $u, v \in \mathcal{L}$ such that $u < v$, or $\{u\}$ for some $u \in \mathcal{L}$. Note moreover that for two $\sim$-classes of a tomonoid quotient we have $\langle a \rangle_\sim < \langle b \rangle_\sim$ if and only if $a' < b'$ for all $a' \sim a$ and $b' \sim b$.

Each filter gives rise to a quotient. We will consider in the sequel actually only quotients arising in this way. We remark that q.n.c. tomonoids possess in general further quotients. For instance, a Rees quotient arises from a negative tomonoid $\mathcal{L}$ by identifying all elements below some given $q \in \mathcal{L}$; see, e.g., [6]. We may, however, also mention that the quotients induced by filters, which we consider here, are exactly those that preserve also the residual [2].

Definition 4.3. Let F be a filter of a q.n.c. tomonoid $\mathcal{L}$. For $a, b \in \mathcal{L}$, let

$$\begin{aligned} a \sim_F b \quad & \text{if } a = b, \\ & \text{or } a < b \text{ and there is a } f \in F \text{ such that } b \odot f \leq a, \\ & \text{or } b < a \text{ and there is a } f \in F \text{ such that } a \odot f \leq b. \end{aligned}$$

Then we call $\sim_F$ the *congruence induced by* F.

Given a q.n.c. tomonoid $\mathcal{L}$ and a filter F, it is easily verified that the congruence induced by F is a monoid congruence, and the quotient is in fact a tomonoid. We will denote this quotient by $\mathcal{L}/F$. Moreover, for $a \in \mathcal{L}$, we will write a/F instead of $\langle a \rangle_{\sim_F}$ and call this set an F-*class*. For $A \subseteq \mathcal{L}$, we will write $A/F = \{a/F \colon a \in A\}$.

It is clear that the formation of a quotient preserves negativity and commutativity. It might be considered as a less trivial fact that the same applies to quanticity; for a proof, see [23].

Lemma 4.4. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid, and let $(F; \leq, \odot, 1)$ be a filter of $\mathcal{L}$. Then $\mathcal{L}/F$ is again a q.n.c. tomonoid.*

Example 4.5. *In the sequel, to keep definitions of t-norms as short as possible, we will in general not provide full specifications, but assume commutativity to be used to cover all cases. In this sense, let*

$$a \odot_4 b \;=\; \begin{cases} 4ab - 3a - 3b + 3 & \text{if } a, b > \frac{3}{4}, \\ 4ab - 3a - 2b + 2 & \text{if } \frac{1}{2} < a \leq \frac{3}{4} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a - b + 1 & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } b > \frac{3}{4}, \\ 4ab - 3a & \text{if } a \leq \frac{1}{4} \text{ and } b > \frac{3}{4}, \\ 2ab - a - b + \frac{3}{4} & \text{if } \frac{1}{2} < a, b \leq \frac{3}{4}, \\ ab - \frac{1}{2}a - \frac{1}{4}b + \frac{1}{8} & \text{if } \frac{1}{4} < a \leq \frac{1}{2} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \\ 0 & \text{if } a \leq \frac{1}{4} \text{ and } \frac{1}{2} < b \leq \frac{3}{4}, \text{ or } a, b \leq \frac{1}{2}. \end{cases}$$

Then $\odot_4$ is a l.-c. t-norm, which, in a modified form, was defined by Hájek in [12]. *The t-norm monoid $([0, 1]; \leq, \odot_4, 1)$ possesses the filter $F = (\frac{3}{4}, 1]$, and the F-classes are $\{0\}$, $(0, \frac{1}{4}]$, $(\frac{1}{4}, \frac{1}{2}]$, $(\frac{1}{2}, \frac{3}{4}]$, and $(\frac{3}{4}, 1]$. The quotient by F is isomorphic to L_5, the five-element Łukasiewicz chain.*

On the basis of this example, Figure 2 *shows how the Cayley tomonoid of a quotient arises from the Cayley tomonoid of the original tomonoid.*

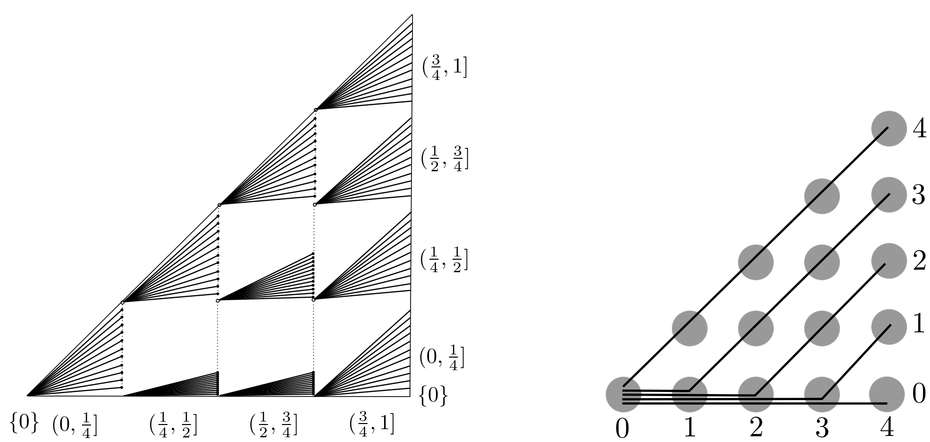

Figure 2: The tomonoid based on the t-norm $\odot_4$ possesses the proper, non-trivial filter $F = (\frac{3}{4}, 1]$. On the left, we see its Cayley tomonoid. On the right, we see the Cayley tomonoid of the five-element quotient L_5.

We conclude this section with a couple of lemmas related to filters and quotients.

Lemma 4.6. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid. Let F_1 and F_2 be filters of $\mathcal{L}$ such that $F_2 \subset F_1$. Moreover, let $R \subseteq \mathcal{L}$ be an F_1-class. If R/F_2 possesses a least element, then R itself possesses a least element.*

Proof. Let L_R be the least element of R/F_2, and let $r \in L_R$. By assumption, there is a $g \in F_1$ such that $g < f$ for all $f \in F_2$. Then $r \odot g$ is a lower bound of L_R and an element of R; consequently, $r \odot g$ is the least element of R. □

We next compile some consequences of the property of being finitely generated.

Lemma 4.7. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a finitely generated q.n.c. tomonoid. Then $\mathcal{L}$ has only finitely many filters.*

Proof. Let G be a finite subset of $\mathcal{L}$ generating $\mathcal{L}$. Then any $a \in \mathcal{L}$ is a product of elements of $\{g \in G \colon g \geq a\}$. Consequently, each filter F of $\mathcal{L}$ is generated by $F \cap G$, and it follows that there are only finitely many filters. □

Lemma 4.8. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a finitely generated q.n.c. tomonoid. Let F_1 and F_2 be filters of $\mathcal{L}$ such that $F_2 \subseteq F_1$. If both $\mathcal{L}/F_1$ and F_1/F_2 are finite, then also $\mathcal{L}/F_2$ is finite.*

Proof. Let $\mathcal{L}/F_1$ and F_1/F_2 be finite, and let $R \in \mathcal{L}/F_2$. We claim that, for some $k \geq 1$, we have $R^{\odot(k+1)} = R^{\odot k}$ in the tomonoid quotient $\mathcal{L}/F_2$; the lemma will then follow from the fact that $\mathcal{L}/F_2$ is finitely generated. There are two cases:

Case 1. Assume $R \subseteq F_1$. Since F_1/F_2 is by assumption finite, we have $R^{\odot(k+1)} = R^{\odot k}$ for some $k \geq 1$.

Case 2. Assume $R \subseteq S$ for some $S \in \mathcal{L}/F_1$ different from F_1. Then there is a $k \geq 1$ such that $S^{\odot(k+1)} = S^{\odot k}$ in the tomonoid quotient $\mathcal{L}/F_1$, because $\mathcal{L}/F_1$ is finite. Consequently, $R^{\odot k}$ and $R^{\odot(k+1)}$ are in the same F_1-class, namely $S^{\odot k}$. But $R < T$ for any $T \in F_1/F_2$, hence $R^{\odot(k+1)} = R^{\odot k} \odot R \leq U$ for any $U \in S^{\odot k}/F_2$. Consequently, $R^{\odot(k+1)}$ is the least element of $S^{\odot k}/F_2$, and it follows $R^{\odot(k+2)} = R^{\odot(k+1)}$. □

Our last lemma concerns the Archimedean property.

Lemma 4.9. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a q.n.c. tomonoid, and assume that $\mathcal{L}$ possesses a largest proper filter F. Then $\mathcal{L}/F$ is Archimedean.*

Proof. Let $R, S \in \mathcal{L}/F$ such that $R < S < F$. Let $r \in R$ and $s \in S$. Then $s \notin F$, and since the only filter properly containing F is $\mathcal{L}$, there is a $k \geq 1$ such that $s^{\odot k} \leq r$. It follows $S^{\odot k} \leq R$ and the assertion is shown. □

5 Some example cases

Before turning to the general theorem about the approximation of l.-c. t-norms, we consider a number of simple examples in order to provide a first impression of what we have in mind. Let us fix some $0 < \varepsilon < 1$.

First, consider the Łukasiewicz t-norm $\odot_1$. This case is particularly easy because the tomonoid $([0,1]; \leq, \odot_1, 1)$ is locally finite: any finite subset of $[0,1]$ generates a finite subtomonoid. More specifically, given ε, choose n large enough such that $\frac{1}{n} \leq \varepsilon$. Let $\mathcal{L} = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, and let $\varodot$ be the restriction of $\odot$ to $\mathcal{L}$. Then $\varodot$ obviously approximates $\odot$ with the precision $\frac{\varepsilon}{2}$. An illustration is provided in Figure 3 (left).

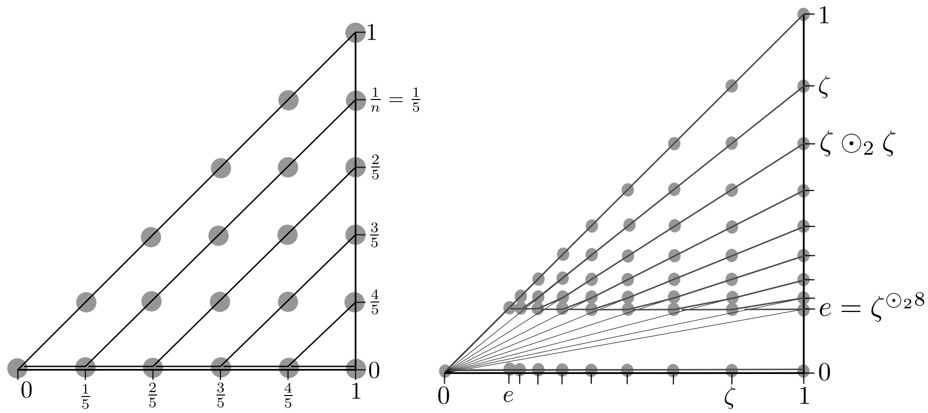

Figure 3: The Cayley tomonoids of a t-norm and of its approximating discrete t-norm. Left: Approximation of the Łukasiewicz t-norm. Right: Approximation of the product t-norm.

The tomonoid based on the Gödel t-norm $\odot_3$ is locally finite as well. Consequently, we can proceed exactly in the same way.

Let us next consider the product t-norm $\odot_2$. Note that the tomonoid based on $\odot_2$ is not locally finite. Let $\zeta = 1 - \varepsilon$, and let $k \geq 1$ be large enough such that $e = \zeta^{\odot_2 k} \leq \varepsilon$. Let $\mathcal{L} = \{0, \zeta^{\odot_2 k}, \zeta^{\odot_2 (k-1)}, \ldots, \zeta, 1\}$, and let $\varodot$ be defined as follows:

$$a \varodot b = \begin{cases} a \cdot b & \text{for } a, b > 0 \text{ such that } a \cdot b \geq e, \\ e & \text{for } a, b > 0 \text{ such that } a \cdot b < e, \\ 0 & \text{if } a = 0 \text{ or } b = 0. \end{cases}$$

Consider $(\mathcal{L}; \leq, \varodot, 1)$, the tomonoid based on $\varodot$. Interpreting the notion of an ordinal sum according to [1], $\mathcal{L}$ is the ordinal sum of the $k+1$-element Łukasiewicz chain and the two-element tomonoid. Obviously, the operation $\varodot$ deviates from the product of reals by less than $e \leq \varepsilon$, and it follows that $\varodot$ approximates $\odot$ with the precision ε. Cf. Figure 3 (right).

As the basic feature of this approximation, we may observe the following. The tomonoid based on the product t-norm possesses the filter $(0, 1]$, and the induced congruence has the two classes $\{0\}$ and $(0, 1]$. The latter class is a left-open real interval and contains the infinitely many powers $\zeta, \zeta^2, \zeta^3, \ldots$ for any $0 < \zeta < 1$. For the construction of the discrete t-norm $\varodot$ above, we have, so-to-say, "trimmed" the interval $(0, 1]$ and replaced it by $[e, 1]$, and all non-zero products smaller than e are mapped to e.

Let us finally consider a somewhat more involved example, which is taken from [22];

cf. Figure 4:

$$a \odot_5 b = \begin{cases} 2ab - a - b + 1 & \text{if } a, b > \frac{1}{2}, \\ \frac{1}{2}(2a)^{\frac{1}{2b-1}} & \text{if } a \leq \frac{1}{2} \text{ and } b > \frac{1}{2}, \\ 0 & \text{if } a, b \leq \frac{1}{2}. \end{cases}$$

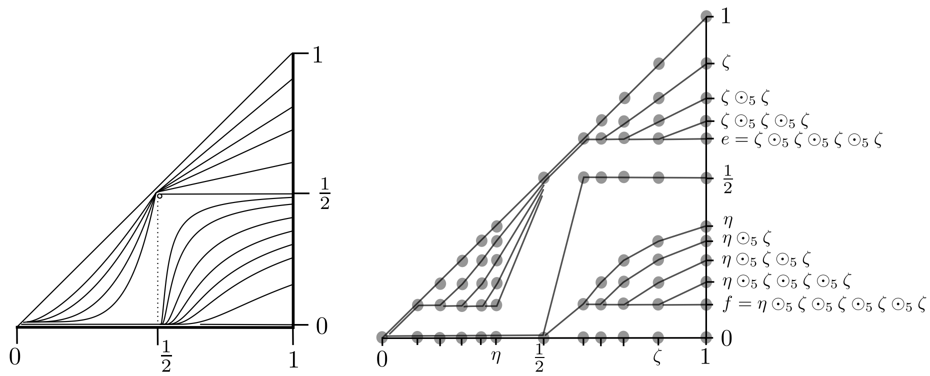

Figure 4: Left: The t-norm $\odot_5$. Right: An approximation of $\odot_5$.

Let $\zeta = 1 - \varepsilon$ and $\eta = \frac{1}{2} - \varepsilon$; let $k \geq 1$ be large enough such that $e = \zeta^{\odot_5 k} \leq \frac{1}{2} + \varepsilon$ and $f = \eta \odot_5 \zeta^{\odot_5 k} \leq \varepsilon$. Then put $\mathcal{L} = \{0, \eta \odot_5 \zeta^{\odot_5 k}, \eta \odot_5 \zeta^{\odot_5 (k-1)}, \ldots, \eta, \frac{1}{2}, \zeta^{\odot_5 k}, \zeta^{\odot_5 (k-1)}, \ldots, \zeta, 1\}$ and define

$$a \mathbin{\hat{\odot}} b = \begin{cases} (2ab - a - b + 1) \vee e & \text{if } a, b > \frac{1}{2}, \\ \frac{1}{2}(2a)^{\frac{1}{2b-1}} \vee f & \text{if } 0 < a \leq \frac{1}{2} \text{ and } b > \frac{1}{2}, \\ 0 & \text{if } a, b \leq \frac{1}{2}. \end{cases}$$

On $\mathcal{L}$, $\hat{\odot}$ then differs from $\odot$ not more than ε. The actual precision that we gain in this way depends on the maximal difference between neighbouring elements of $\mathcal{L}$.

Note that the tomonoid based on $\odot_5$ has a congruence with the classes $\{0\}$, $(0, \frac{1}{2})$, $\{\frac{1}{2}\}$, and $(\frac{1}{2}, 1]$. Once again applying our loose formulation, we "trimmed" the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$; we replaced the former by $[f, \eta]$ and the latter by $[e, 1]$.

This last example shows quite clearly the idea of the approximation theorem that follows. In the general case, we have to account for the nested structure of quotients. The process of "trimming" equivalence classes to closed intervals is repeated successively for each congruence.

6 Approximation by discrete t-norms

This section contains the main result of this paper.

For convenience, we will use the notion of approximation in a more general way. Let $(\mathcal{L}; \leq, \odot, 1)$ and $(\mathcal{L}'; \leq, \odot', 1)$ be q.n.c. tomonoids such that $\mathcal{L}' \subseteq \mathcal{L} \subseteq [0,1]$ and the total order is the natural one. Then we say that $\mathcal{L}'$ approximates $\mathcal{L}$ with the precision $\varepsilon > 0$ if (i) for any $a \in \mathcal{L}$ there are $b \in \mathcal{L}'$ such that $|b - a| \leq \varepsilon$ and (ii) for $a, b \in \mathcal{L}'$, we have $|a \odot' b \; - \; a \odot b| \leq \varepsilon$.

For q.n.c. tomonoids $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$, the following fact is then immediate. Assume that $\mathcal{L}''$ approximates $\mathcal{L}'$ with some precision $\varepsilon_1 > 0$, and $\mathcal{L}'$ approximates in turn $\mathcal{L}$ with the precision $\varepsilon_2 > 0$. Then $\mathcal{L}''$ approximates $\mathcal{L}$ with the precision $\varepsilon_1 + \varepsilon_2$.

Theorem 6.1. *Every l.-c. t-norm can be approximated by a discrete t-norm with an arbitrary precision.*

Proof. Let $\odot$ be a l.-c. t-norm, and let $0 < \varepsilon < 1$. Then $([0,1]; \leq, \odot, 1)$ is a q.n.c. tomonoid. Let G be a finite subset of $[0,1]$ containing 0 and 1 and such that neighbouring elements of G do not differ by more than ε. Let $(\mathcal{L}_1; \leq, \odot_1, 1)$ be the subtomonoid of $[0,1]$ generated by G. Obviously then, $\mathcal{L}_1$ approximates $([0,1]; \leq, \odot, 1)$ with the precision $\frac{\varepsilon}{2}$. Note that $\mathcal{L}_1$ is negative and commutative; by Lemma 3.5, $\mathcal{L}_1$ is in fact a q.n.c. tomonoid.

As $\mathcal{L}_1$ is finitely generated, $\mathcal{L}_1$ possesses by Lemma 4.7 finitely many filters. Let $\mathcal{L}_1 = F_0 \supset F_1 \supset \ldots \supset F_k = \{1\}$ be the filters of $\mathcal{L}_1$. If $k = 1$, $\mathcal{L}_1$ is Archimedean; since $\mathcal{L}_1$ contains by assumption 0, $\mathcal{L}_1$ is by Lemma 3.6 in this case finite, and we are done.

Assume $k \geq 2$. We shall construct a sequence $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of q.n.c. tomonoids with a successively restricted domain, such that $\mathcal{L}_i$ approximates $\mathcal{L}_{i-1}$, $i = 2, \ldots, k$, with the precision $\frac{\varepsilon}{2k}$. Moreover, the last tomonoid, $\mathcal{L}_k$, will be finite. It will then follow that $\mathcal{L}_k$ approximates $\mathcal{L}$ with the precision ε and the theorem will be proved.

F_1 is the largest proper filter, hence $\mathcal{L}_1/F_1$ is Archimedean by Lemma 4.9. Moreover, $\mathcal{L}_1/F_1$ is finitely generated and has the least element $0/F_1$, thus it is finite by Lemma 3.6.

We summarise that $\mathcal{L}_1$ is a finitely generated, bounded q.n.c. tomonoid possessing the filters $F_1 \supset \ldots \supset F_k = \{1\}$ such that $\mathcal{L}_1/F_1$ is finite. For the construction of $\mathcal{L}_2$, we consider the next largest filter to F_1, which is F_2. Note that F_1/F_2 is by Lemma 4.9 Archimedean. We distinguish two cases.

Case 1. Let F_1/F_2 have a least element. Then F_1/F_2 is finite by Lemma 3.6, and by Lemma 4.8, $\mathcal{L}_1/F_2$ is finite as well. We let $(\mathcal{L}_2; \leq, \odot_2, 1)$ coincide with $(\mathcal{L}_1; \leq, \odot_1, 1)$. Trivially then, $\mathcal{L}_2/F_2$ is finite and $\mathcal{L}_2$ approximates $\mathcal{L}_1$ with the precision 0.

Case 2. Let us assume that F_1/F_2 does not have a least element.

Each F_1-class R is the disjoint union of F_2-classes. As $\mathcal{L}_1/F_2$ is finitely generated, R/F_2 contains by Lemma 3.5 a greatest element, which we denote by U_R.

We next choose an element $L_{F_1} \in (F_1/F_2)\backslash\{F_2\}$ such that, for each $R \in \mathcal{L}_1/F_1$, the following holds: (1) if R/F_2 possesses a least element, $U_R \odot_1 L_{F_1}$ is this element; (2) if R/F_2 does not possess a least element, then for any $a \in R$ such that $a/F_2 < U_R \odot_1 L_{F_1}$, there is a $b \in U_R \odot_1 L_{F_1}$ such that $b - a \leq \frac{\varepsilon}{2k}$. Because $\mathcal{L}_1/F_1$ is finite, such a choice is possible.

For each $R \in (\mathcal{L}_1/F_1) \backslash \{F_1\}$, let $L_R = U_R \odot_1 L_{F_1}$. For each $R \in \mathcal{L}_1/F_1$, we claim that L_R possesses in $\mathcal{L}_1$ an infimum, and we put $l_R = \inf L_R$. Indeed, if L_R is not the least element of R/F_2, L_R is lower bounded, hence the infimum exists because $\mathcal{L}_1$ is quantic. And if L_R is the least F_2-class contained in R, then L_R possesses a least element by Lemma 4.6. We furthermore observe in both cases that $l_R \in R$.

Let $\mathcal{L}_2$ be the union of the sets $\{a \in R \colon a \geq l_R\}$, $R \in \mathcal{L}_1/F_1$. Note that F_2 and consequently all filters $F_3, \ldots, F_k$ are contained in $\mathcal{L}_2$, and also $0, 1 \in \mathcal{L}_2$.

We have to endow $\mathcal{L}_2$ with a tomonoid structure. We let the total order and the constant 1 be as in $\mathcal{L}_1$. We define the product of $a, b \in \mathcal{L}_2$ as follows:

$$a \odot_2 b \;=\; (a \odot_1 b) \vee l_R, \quad \text{where } R \in \mathcal{L}_1/F_1 \text{ is such that } a \odot_1 b \in R. \tag{3}$$

By construction, $a \odot_2 b$ and $a \odot_1 b$ differ not more than by $\frac{\varepsilon}{2k}$. Moreover, we easily check that $\odot_2$ is commutative, neutral w.r.t. 1, and in both arguments monotone.

We insert a remark before continuing. Let $R, S \in \mathcal{L}_1/F_1$ and $T = R \odot_1 S$. We claim that then, for any $a \in R$,

$$a \odot_1 l_S \leq l_T \tag{4}$$

and consequently, for any $a \geq l_R$, $a \odot_2 l_S = l_T$. In fact, in $\mathcal{L}_1/F_2$, we have $a/F_2 \odot_1 U_S \leq U_T$, hence $a/F_2 \odot_1 L_S = a/F_2 \odot_1 U_S \odot_1 L_{F_1} \leq U_T \odot_1 L_{F_1} = L_T$. Let $s \in L_S$; then $(a \odot_1 s)/F_2 \leq L_T$ and $l_S \leq s \odot_1 f$ for all $f \in F_2$, and it follows $a \odot_1 l_S \leq \inf_{f \in F_2} (a \odot_1 s \odot_1 f) = \inf\, (a \odot_1 s)/F_2 \leq \inf L_T = l_T$, and (4) is shown.

We now prove the associativity of $\odot_2$. Let $a, b, c \in \mathcal{L}_2$. Let $R, S \in \mathcal{L}_1/F_1$ be such that $a \odot_1 b \in R$ and $a \odot_1 b \odot_1 c \in S$. Then $(a \odot_2 b) \odot_2 c = (((a \odot_1 b) \vee l_R) \odot_1 c) \vee l_S = (a \odot_1 b \odot_1 c) \vee (l_R \odot_1 c) \vee l_S = (a \odot_1 b \odot_1 c) \vee l_S$, where we have made use of the monotonicity of $\odot_1$ and (4). It follows that $\odot_2$ is associative.

We have proved that $(\mathcal{L}_2; \leq, \odot_2, 1)$ is a negative, commutative tomonoid. Moreover, it follows from (4) that, in $\mathcal{L}_1$, each element of $\mathcal{L}_2$ is a product of elements of $G \cap \mathcal{L}_2$. Hence $G \cap \mathcal{L}_2$ generates $\mathcal{L}_2$, that is, $\mathcal{L}_2$ is finitely generated. By Lemma 3.5, $\mathcal{L}_2$ is quantic. Thus $\mathcal{L}_2$ is a q.n.c. tomonoid approximating $\mathcal{L}_1$ with the precision $\frac{\varepsilon}{2k}$.

$\mathcal{L}_2$ possesses the filter $F_1' = \mathcal{L}_2 \cap F_1 = \{a \in F_1 \colon a \geq l_{F_1}\}$, and for each F_1-class R of $\mathcal{L}_1$, $R \cap \mathcal{L}_2 = \{a \in R \colon a \geq l_R\}$ is an F_1'-class of $\mathcal{L}_2$. In particular, $\mathcal{L}_2/F_1'$ is finite. Moreover, F_1'/F_2 is Archimedean and has a least element, hence it is finite as well. We conclude by Lemma 4.8 that $\mathcal{L}_2/F_2$ is finite.

We summarise that, in Case 1 as well as in Case 2, $\mathcal{L}_2$ is a finitely generated, bounded q.n.c. tomonoid possessing the filters $F_2 \supset \ldots \supset F_k = \{1\}$ such that $\mathcal{L}_2/F_2$ is finite. If $k \geq 3$, we may repeat the construction $k-2$ further times, to end up with the q.n.c. tomonoid $\mathcal{L}_k$ possessing the filter F_k such that $\mathcal{L}_k/F_k$ is finite. But $F_k = \{1\}$, that is, $\mathcal{L}_k$ is itself finite. □

Remark 6.2. *We add that Theorem* 6.1 *states the existence of a discrete t-norm approximating a given l.-c. t-norm only. However, the proof actually indicates a method how to determine it. Note that, according to this method, the approximation is from above, as is obvious from* (3).

To find a good example illustrating Theorem 6.1 is not straightforward; t-norms with a suitable set of quotients are not common in the literature. An exception, however, exists. In several papers t-norms have been discussed that are special for the property of being left-continuous but having a set of discontinuity points that is dense in $[0,1]^2$. The following example is due to R. Mesiar [17] and further discussed in [21].

Example 6.3. *Let $\mathbb{Z}^- = \{a \in \mathbb{Z} \colon a \leq 0\}$, the set of negative integers, be endowed with the natural order, the usual addition, and the constant 0. Then $(\mathbb{Z}^-; \leq, +, 0)$ is a q.n.c. tomonoid. Let furthermore $(\mathbb{Z}^-)^\omega$ be the Cartesian product of countably infinitely many copies of $\mathbb{Z}^-$, that is, the set of sequences $(a_0, a_1, \ldots)$ in $\mathbb{Z}^-$. Let us endow $(\mathbb{Z}^-)^\omega$ with the lexicographic order, the componentwise addition, and the constant $\bar{0} = (0, \ldots)$. Then $((\mathbb{Z}^-)^\omega; \leq, +, \bar{0})$ is a q.n.c. tomonoid as well.*

This tomonoid is not bounded, but we can add a least element in the usual way. Let $W = (\mathbb{Z}^-)^\omega \cup \{z\}$, where z is a new element; extend the total order to W by making z the least element; and let $\tilde{+}$ be the extension of the componentwise addition such that z is absorbing for $\tilde{+}$, that is, $z \mathbin{\tilde{+}} w = w \mathbin{\tilde{+}} z = z$ for all $w \in W$. Then $(W; \leq, \tilde{+}, \bar{0})$ is a bounded q.n.c. tomonoid.

Note that W is order-isomorphic with the real unit interval. Let us fix an order isomorphism $\varphi \colon W \to [0,1]$ and let us define $\odot_6 \colon [0,1]^2 \to [0,1], \quad (a,b) \mapsto \varphi(\varphi^{-1}(a) \mathbin{\tilde{+}} \varphi^{-1}(b))$. Then $\odot_6$ is a l.-c. t-norm.

Let us see how $\odot_6$ can be approximated by a discrete t-norm. Let $n \geq 0$ and $m < 0$, and consider the subset

$$\begin{aligned} W' \ = \{ &(a_0, \ldots, a_n, 0, \ldots) \colon \ a_0, \ldots, a_n \geq m; \\ &\text{if } a_i = m \text{ for some } 0 \leq i \leq n-1, \text{ then } a_{i+1} = \ldots = a_n = 0 \} \\ &\cup \ \{z\} \end{aligned}$$

of W. Endow W' with the lexicographic order $\leq$, the truncated addition $+$, and the constant $\bar{0}$. By the truncated addition, we mean the operation absorbing for z and otherwise defined as follows:

$$\begin{aligned} &(a_0, \ldots, a_n, 0, \ldots) + (b_0, \ldots, b_n, 0, \ldots) \\ &= \begin{cases} (a_0 + b_0, \ \ldots, \ a_n + b_n, \ 0, \ \ldots) & \text{if } a_0 + b_0, \ldots, a_n + b_n > m \\ (a_0 + b_0, \ \ldots, \ a_{i-1} + b_{i-1}, \ m, \ 0, \ \ldots) & \text{if } a_0 + b_0, \ldots, a_{i-1} + b_{i-1} > m \\ & \text{and } a_i + b_i \leq m, \text{ where } 0 \leq i \leq n. \end{cases} \end{aligned}$$

Then $(W'; \leq, +, \bar{0})$ is a finite q.n.c. tomonoid, and under the isomorphism φ we get a discrete t-norm $\odot$. The latter apparently approximates $\odot_6$ with an arbitrary precision, provided that we choose n large enough and m small enough.

7 Approximation by finitary t-norms

In our last section, we raise the question if l.-c. t-norms can be approximated by other t-norms, that is, by t-norms of a specific simple type. Given the result of the previous section, it seems natural to ask if t-norms that arise from discrete t-norms can play the role of approximating functions. This is indeed possible, although the mode of approximation is somewhat non-standard.

Let us recall that any finite q.n.c. tomonoid can be isomorphically embedded into a tomonoid based on a t-norm. A canonical way of doing so can be found in [14]; cf. also [22].

Lemma 7.1. *Let $(\mathcal{L}; \leq, \odot, 1)$ be a finite negative, commutative tomonoid. Let 0 be its least element, and let*

$$\bar{\mathcal{L}} \;=\; \{(a,r)\colon a \in \mathcal{L}\backslash\{0\} \text{ and } r \in \mathbb{R} \text{ such that } -1 < r \leq 0\} \;\cup\; \{(0,0)\},$$

endowed with the lexicographical order. For $(a,r),(b,s) \in \bar{\mathcal{L}}$ define

$$(a,r) \odot (b,s) \;=\; (a,r) \wedge (b,s) \wedge (a \odot b, 0).$$

Then $(\bar{\mathcal{L}}; \leq, \odot, (0,0), (1,0))$ is a q.n.c. tomonoid. Moreover, the mapping $\vartheta\colon \mathcal{L} \to \bar{\mathcal{L}},\;\; a \mapsto (a,0)$ is an isomorphic embedding of tomonoids.

We draw the obvious conclusion.

Proposition 7.2. *Let $\odot\colon \mathcal{L}^2 \to \mathcal{L}$ be a discrete t-norm. Then $\odot$ can be extended to a l.-c. t-norm $\odot$ as follows:*

$$a \odot b \;=\; a \wedge b \wedge (\bar{a} \odot \bar{b}), \tag{5}$$

where $\bar{a}, \bar{b}$ are the smallest elements of $\mathcal{L}$ such that $a \leq \bar{a}$ and $b \leq \bar{b}$, respectively.

Proof. The universe of the tomonoid constructed in Lemma 7.1 is order-isomorphic to the real unit interval. □

Definition 7.3. A t-norm arising from a discrete t-norm according to Proposition 7.2 will be called *finitary*.

It is now natural to ask if finitary t-norms can approximate a l.-c. t-norm. An answer is not straightforward because, according to our procedure, the discrete t-norms approximate the t-norms from above rather from below. However, the following theorem proposes an alternative kind of approximation.

Theorem 7.4. *Let $\odot$ be a l.-c. t-norm. Then there are finitary t-norms $\odot_n$, $n \in \mathbb{N}$, such that, for $a, b \in (0,1]$, we have*

$$a \odot b \;=\; \lim_{a' \nearrow a,\; b' \nearrow b} \; \lim_{n \to \infty} \; a' \odot_n b'.$$

Proof. Let $\odot_n$, $n \in \mathbb{N}$ be discrete t-norms approximating $\odot$ with the precision ε_n, respectively, such that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $\odot_n$, $n \in \mathbb{N}$, be the finitary t-norms arising from $\odot_n$.

We assume that $\hexagon_n$ are constructed according to the proof of Theorem 6.1. By Remark 6.2, for each $n \in \mathbb{N}$ and a, b in the domain of $\hexagon_n$, we then have $a \odot b \leq a \hexagon_n b$. It follows that $a \odot b \leq a \odot_n b$ for any $a, b \in [0, 1]$. In fact, in the notation of (5), either $\bar{a} \hexagon_n \bar{b} < a \wedge b$ and then $a \odot_n b = \bar{a} \hexagon_n \bar{b} \geq \bar{a} \odot \bar{b} \geq a \odot b$; or otherwise $a \odot_n b = a \wedge b \geq a \odot b$.

Fix now a pair $a, b \in (0, 1]$. Let furthermore $a' < a$ and $b' < b$. From $a' \odot b' \leq a' \odot_n b'$ for each n we conclude $a' \odot b' \leq \lim_{n \to \infty} a' \odot_n b'$.

For each n, let now $a^{(n)}, b^{(n)}$ be the largest elements in the domain of $\hexagon_n$ such that $a^{(n)} \leq a$ and $b^{(n)} \leq b$. For sufficiently large n, we have $a' \leq a^{(n)} \leq a$ and $b' \leq b^{(n)} \leq b$ and hence $a' \odot_n b' \leq a^{(n)} \hexagon_n b^{(n)}$. But by (the proof of) Proposition 2.4, $\lim_{n \to \infty} a^{(n)} \hexagon_n b^{(n)} = a \odot b$, and we conclude $\lim_{n \to \infty} a' \odot_n b' \leq a \odot b$.

We have seen that $a' \odot b' \leq \lim_{n \to \infty} a' \odot_n b' \leq a \odot b$. By the left-continuity of $\odot$, the claim follows. □

8 Conclusion

We have dealt with the question if left-continuous triangular norms can be approximated by finitary means and we have given an affirmative answer. Namely, we may associate with an arbitrary l.-c. t-norm $\odot$ a discrete t-norm – which is a finite totally ordered monoid – that, in a natural sense, approximates $\odot$ to an arbitrary given degree of accuracy. Alternatively, we may also define a sequence of t-norms $\odot_n$ all of which are definable by finitary means and which, in a specific sense, converge to $\odot$. Given the fact that examples of l.-c. t-norms became over the years more and more involved, culminating in cases of dense sets of discontinuity points, these results might be considered of interest.

We may furthermore ask if we have contributed in this way to a deeper understanding of l.-c. t-norms. One certainly never knows; however, scepticism might be in place for a simple reason. Even if left-continuous t-norms are reducible to the finitary discrete t-norms, it must not be forgotten that we struggle with the theory of finite negative, commutative tomonoids as well.

Seen from an algebraic perspective, our work might nevertheless be found to contain some interesting aspects. The tomonoids that we determined successively in the proof of Theorem 6.1 arose on the basis of a construction that could be considered as a "generalised Rees quotient". Such a construction leads in general to technical difficulties, which we have avoided here by a certain finiteness condition. Still it would be interesting to explore if we can define tomonoid quotients on the basis of the ideas that we have used in this paper for the approximation of t-norms.

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