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Improved Form of a Fracture Mechanics Based Failure Probability Model for Brittle Materials

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1 Introduction

Batdorf and Crose [1] combined the statistical analysis of failure for brittle materials by Weibull [2] with an appropriate fracture criterion based on fracture mechanics theory and extended this notion to multiaxial stress states. If an appropriate form of crack distribution is chosen, the cumulative failure probability function proposed by Batdorf and Crose [1] reduces to the Weibull distribution for uniaxial tensile stress states. In this work, we will show that the approximation of an infinitesimally small volumetric element may have been prematurely employed by Batdorf and Crose [1] in obtaining failure probability for an arbitrary volumetric element ΔV. The widely used failure probability formula based on this approximation may present some errors under certain conditions. We will derive an alternative formula without the use of this unnecessary approximation.

2 Theoretical Derivation

Batdorf and Crose [1] introduced the solid angle Ω containing the normals to all orientations for which the component of the applied stress normal to the crack plane is larger than the critical stress, i.e., $\sigma_n > \sigma_{\rm cr}$. The solid angle Ω varies from zero to 4π for cracks contained inside a three-dimensional body, and it varies from zero to 2π for surface cracks based on the definition.

In general, a problem of crack propagation can only be one of the following two cases: (i) $\Omega/4\pi<1$, propagation of a crack depends on its orientation and there exists a range of orientation angle where a crack does not propagate and (ii) $\Omega/4\pi=1$, propagation of a crack is independent of its orientation.

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2.1 Case (I): $\Omega/4\pi<1$. If there is only one crack and its crack plane is randomly orientated, the probability of failure caused by this single crack [Eq. (1) in Batdorf and Crose [1]] is given by

$$P_f = \Omega(\Sigma,\sigma_{\rm cr})/4\pi, \tag{1}$$

where Σ is the applied stress, and $\sigma_{\rm cr}$ is the critical stress of the crack.

If the crack density is N, $N\Delta V$ represents the number of cracks inside the volumetric element ΔV. Therefore, the overall survival probability P_s is the multiplication of the survival probability of each crack.

$$P_s = [1-\Omega(\Sigma,\sigma_{\rm cr})/4\pi]^{N\Delta V}. \tag{2}$$

The overall failure probability of this volumetric element is given by

$$P_f = 1-[1-\Omega(\Sigma,\sigma_{\rm cr})/4\pi]^{N\Delta V}. \tag{3}$$

In Batdorf and Crose [1], the failure probability is given by

$$P_f = (N\Delta V)(\Omega/4\pi). \tag{4}$$

If we let an arbitrary volumetric element ΔV approach zero, then $N\Delta V$ becomes small. In this case, by neglecting the higher order terms in the Taylor expansion, Eq. (3) can be reduced to Eq. (4). However, this approximation is premature and unnecessary. The theoretical derivation of failure probability prediction formula by Batdorf and Crose [1] is based on Eq. (4). In this work, we will derive the failure probability for the total volume V based on Eq. (3) instead of Eq. (4).

As in Batdorf and Crose [1], we will introduce the crack density function $N(\sigma_{\rm cr})$ representing the number of cracks per unit volume with their critical stress less than or equal to $\sigma_{\rm cr}$. The survival probability of this volumetric element ΔV for any possible cracks under stress Σ, $P_s(\Delta V,\Sigma)$, is the product of survival probability for every specific size crack with its critical stress in the range between $\sigma_{\rm cr}^{\rm Min}$ and $\sigma_{\rm cr}^{\rm Max}$, where the values of the minimum critical stress $\sigma_{\rm cr}^{\rm Min}$ and the maximum critical stress $\sigma_{\rm cr}^{\rm Max}$ are determined by the actual stress status Σ and the fracture criterion.

$$P_s(\Delta V,\Sigma) = \prod_{m=1}^{M} P_s(\Delta V,\sigma_{\rm cr}^{m}) = \prod_{m=1}^{M} [1 - \Omega(\Sigma,\sigma_{\rm cr}^{m})/4\pi]^{\Delta V dN(\sigma_{\rm cr}^{m})/d\sigma_{\rm cr}\Delta\sigma_{\rm cr}}, \tag{5}$$

where the critical stress range is divided into M equal increments $\Delta\sigma_{\rm cr}$, and the critical stress in the m-th increment is denoted by $\sigma_{\rm cr}^{m}$. By applying logarithmic operation to Eq. (5), letting $\Delta\sigma_{\rm cr}$ approach zero, and reorganizing the resulting equations, we obtain

$$P_s(\Delta V,\Sigma) = \exp\left\{\Delta V \int_{\sigma_{cr}^{Min}}^{\sigma_{cr}^{Max}} \ln[1 - \Omega(\Sigma,\sigma_{cr})/4\pi] \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr}\right\}. \tag{6}$$

Therefore, the failure probability for the total volume V is given by

$$P_f^{I}(V) = 1 - \exp\left\{\int_V \int_{\sigma_{cr}^{Min}}^{\sigma_{cr}^{Max}} \ln[1 - \Omega(\Sigma,\sigma_{cr})/4\pi] \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} dV\right\}, \tag{7}$$

where a superscript I is introduced to indicate that this equation is suitable for calculating the failure probability caused by the cracks whose propagation is dependent on orientation, i.e., Case (I): $\Omega/4\pi < 1$. Only in the special cases where $\Omega/4\pi$ is very small, i.e., $\Omega/4\pi \ll 1$, can Eq. (7) be reduced to the equation obtained by Batdorf and Crose [1] (Eq. (11) in Batdorf and Crose [1]) by approximating $\ln[1-\Omega/4\pi]$ by $-\Omega/4\pi$.

2.2 Case (II): $\Omega/4\pi=1$. In this case, propagation of a crack is independent of its orientation and it is solely determined by the size of the crack. Therefore, the failure probability is the probability of finding at least one sufficiently large crack. If a sufficiently large single crack is contained inside a total volume V, the failure probability for an arbitrary volumetric element ΔV is given by $\Delta V/V$. The survival probability for ΔV is then $(1-\Delta V/V)$. Therefore, for a total volume V with K number of sufficient large cracks, the survival probability for the volumetric element ΔV is

$$P_s(\Delta V) = [1 - \Delta V/V]^K. \tag{8}$$

As in Case (I), we again introduce the crack density function $N(\sigma_{cr})$. The survival probability of ΔV for all cracks with size equal to or larger than the minimum critical size (with the maximum critical stress σ_{cr}^{M}) corresponding to a stress state Σ is the product of survival probability for each specific size crack as in Case (I).

$$P_s(\Delta V,\Sigma) = \prod_{m=1}^{M} P_s(\Delta V,\sigma_{cr}^{m}) = \prod_{m=1}^{M} [1 - \Delta V/V]^{V dN(\bar{\sigma}_{cr}^{m})/d\sigma_{cr}\Delta\sigma_{cr}} = [1 - \Delta V/V]^{\sum_{m=1}^{M} V dN(\sigma_{cr}^{m})/d\sigma_{cr}\Delta\sigma_{cr}}. \tag{9}$$

Total volume V is divided into n number of volumetric elements and σ_{cr}^{Mi} and Σ^i, respectively, denote the maximum critical stress and the stress level in the i-th volumetric element ΔV^i. If we let $\Delta\sigma_{cr}$ approach zero in the i-th volumetric element ΔV^i, we obtain

$$P_s(\Delta V^i,\Sigma^i) = [1 - \Delta V^i/V]^{V\int_0^{\sigma_{cr}^{Mi}} dN(\sigma_{cr})/d\sigma_{cr} d\sigma_{cr}}. \tag{10}$$

Furthermore, we assume that the total volume V is equally divided. This leads to the overall survival probability in the entire volume V given by

$$P_s(V) = \prod_{i=1}^{n} P_s(\Delta V^i) = [1 - \Delta V/V]^{V/\Delta V \sum_{i=1}^{n} \Delta V^i \int_0^{\sigma_{cr}^{Mi}} dN(\sigma_{cr})/d\sigma_{cr} d\sigma_{cr}}. \tag{11}$$

As the number of volumetric elements n approaches infinity, $\Delta V/V$ approaches zero. In this case [3],

$$\lim_{\Delta V/V \to 0} [1 - \Delta V/V]^{V/\Delta V} = e^{-1}. \tag{12}$$

Therefore, the failure probability for the total volume V, $P_f(V)$, is given by

$$P_f^{II}(V) = 1 - \exp\left(-\int_V \int_0^{\sigma_{cr}^{M}} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} dV\right), \tag{13}$$

where a superscript II is introduced to indicate that this equation is suitable for calculating the failure probability caused by the cracks whose propagation is not dependent on orientation, i.e., Case (II): $\Omega/4\pi=1$.

In Summary, the probability of failure caused by all types of cracks $P_f(V)$ is given by

$$P_f(V) = 1 - (1 - P_f^{I})(1 - P_f^{II}). \tag{14}$$

3 Examples

3.1 Example 1: Failure Probability Predictions for Uniform Stress States. We will examine the effect of differences in the above formulations on the failure probability predictions caused only by surface cracks given the same crack density parameters by using three simple stress states. Since only surface cracks are considered, $\Omega/2\pi$ is used instead of $\Omega/4\pi$, and surface integral instead of volumetric integral will be used in the above formulations. In order to evaluate the above equations and visualize the differences, we assume the following form of the crack density function $N(\sigma_{cr})$ as in Chao and Shetty [4]

$$N(\sigma_{cr}) = k\sigma_{cr}^{m} \tag{15}$$

where k and m, respectively, are scale and shape parameters. Here, we choose them as $m=4.917$ and $k=4.95 \times 10^{-11}\ \mathrm{mm}^{-2}\ \mathrm{MPa}^{-4.917}$. These numerical values are reasonable in representing glass surfaces sanded by 600 grit S_iC sand papers. In addition, we consider the case where the stress status is uniform, and the specimen surface area is $A=1\ \mathrm{mm}^2$.

To determine Ω, a fracture criterion is required. Here, we assume that a crack propagates when the stress normal to the crack surface σ_n reaches its critical value σ_{cr}, i.e.,

$$\sigma_n = \sigma_{cr}. \tag{16}$$

In the following examples, σ_1 and σ_2 are the two principal stresses on the specimen surface.

Example 1A. $\sigma_1=\sigma_2 \geq 0$

Since the normal stress in any direction is σ_1 (or σ_2), if the critical stress σ_{cr} for a crack is less than σ_1, i.e., $0 \leq \sigma_{cr} \leq \sigma_1$, the crack will propagate regardless of its orientation, i.e., $\Omega/2\pi=1$. Equation (13) is used by replacing the volume integral with the area integral. If $\sigma_{cr} > \sigma_1$ for a crack, the crack will not propagate regardless of its orientation. The formulation by Batdorf and Crose [1] becomes identical to Eq. (13) obtained in this work.

Example 1B. $\sigma_1 > \sigma_2 \geq 0$

If the critical stress σ_{cr} for a crack is between zero and σ_2, i.e., $0 \leq \sigma_{cr} \leq \sigma_2$, the stress normal to the crack plane is always larger than the critical stress, i.e., $\sigma_n > \sigma_{cr}$, regardless of its direction. This again leads to $\Omega/2\pi=1$, and Eq. (13) is used with the substituted area integral. If the critical stress σ_{cr} for a crack is between σ_2 and σ_1, i.e., $\sigma_2 < \sigma_{cr} \leq \sigma_1$, the crack propagation is influenced by the crack orientation, and the critical range of angle Ω needs to be calculated from the fracture criterion Eq. (16). Under this condition, we have

$$\Omega = 2\cos^{-1}\left(\frac{2\sigma_{cr} - \sigma_1 - \sigma_2}{\sigma_1 - \sigma_2}\right). \tag{17}$$

Since $\Omega/2\pi < 1$, Eq. (7) with the substituted area integral is used. The overall failure probability given by Eq. (14) leads to

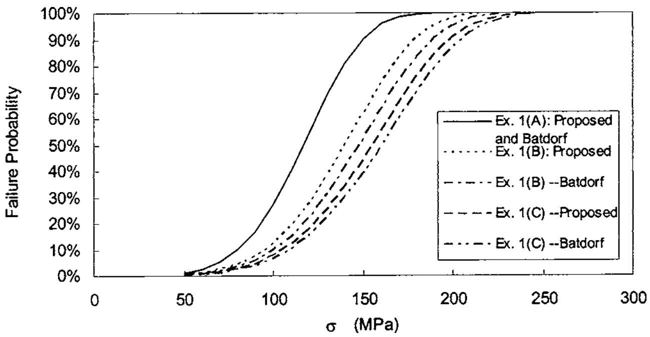

Fig. 1 Comparison of failure probability prediction

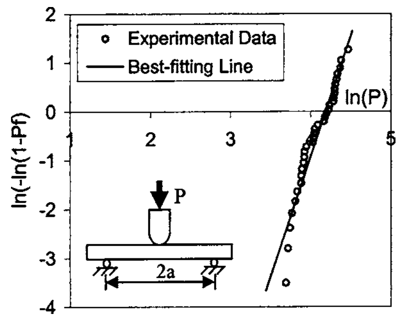

Fig. 2 Biaxial flexure test data

$$P_f = 1 - \exp\left(-A \int_0^{\sigma_2} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} + A \int_{\sigma_2}^{\sigma_1} \ln[1 - \Omega/2\pi] \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} \right). \quad (18)$$

Correspondingly, the overall failure probability based on Batdorf and Crose [1] can be obtained as follows:

$$P_r = 1 - \exp\left(-A \int_0^{\sigma_1} \frac{\Omega}{2\pi} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} d\sigma_{cr} \right). \quad (19)$$

Example 1C. $\sigma_1 \geqslant 0 > \sigma_2$

In this case, the critical stress σ_{cr} is between 0 and σ_1, and the crack propagation depends on the crack orientation. The range of critical angle Ω can be calculated from Eq. (17). For this case, any crack propagation depends on its orientation, i.e., $\Omega/2\pi < 1$. Therefore, Eq. (7) is employed with the substituted area integral.

In order to graphically present some results, we choose σ_1 and σ_2 as follows.

For Example 1A, $\sigma_1 = \sigma_2 = \sigma$ $(\sigma > 0)$

For Example 1B, $\sigma_1 = \sigma$, $\sigma_2 = \sigma/3$

For Example 1C, $\sigma_1 = \sigma$, $\sigma_2 = -\sigma/3$

Based on Eq. (15) and the above assumptions, the failure probability as a function of σ is calculated for both formulations as shown in Fig. 1.

3.2 Example 2: Crack Density Parameter Determination Based on Biaxial Flexure Tests. Crack density parameters are often determined by curve fitting of data from failure strength experiments. We will examine the effect of differences in the above formulations on the resulting statistical parameters determined from a set of experimental data. For this purpose, biaxial flexure tests were conducted using borosilicate glass specimens. The 1 mm thick glass disks with 15.9 mm diameter were supported at the edge by a ring of bearings and loaded on the top center through a tungsten carbide (WC) spherical ball indenter with a diameter of 10 mm. The surfaces of the glass disks were sanded on a rotating wheel with 600 grit SiC sandpaper under water coolant. The experiments were carried out on the Universal Testing Machine (Instron Model 4020, Canton, Mass.) at a crosshead speed of 0.01 mm/min. A total of 34 specimens were used to obtain the experimental failure probability distribution. The fracture initiation load P was recorded for each specimen and the cumulative probability of crack initiation was obtained.

The crack density function $N(\sigma_{cr})$ is assumed to be in the form of Eq. (15). By using this equation in the formulation by Batdorf and Crose [1] or Eq. (14) obtained in this work, we can derive the failure probability function P_f as follows:

$$P_f = 1 - \exp(-e^B P^m), \quad (20)$$

where

$$B = \ln\left[\frac{a^2 mk \mathrm{I}_D}{(\pi t^2)^m} \right]. \quad (21)$$

For the formulation based on Batdorf and Crose [1], I_D in Eq. (21) is given by

$$\mathrm{I}_D = \int_{r/a=0}^{1} \int_{\sigma_{cr}/P/\pi t^2=0}^{\sigma_1/P/\pi t^2} 2\pi \left(\frac{\Omega}{2\pi}\right) \left(\frac{\sigma_{cr}}{P/\pi t^2}\right)^{m-1} \left(\frac{r}{a}\right) d\left(\frac{\sigma_{cr}}{P/\pi t^2}\right) d\left(\frac{r}{a}\right). \quad (22)$$

For the formulation obtained in this work, I_D is given by

$$\mathrm{I}_D = \int_{r/a=0}^{1} 2\pi \left(\frac{r}{a}\right) d\left(\frac{r}{a}\right) \left[\int_{\sigma_{cr}/P/\pi t^2=0}^{\sigma_2/P/\pi t^2} \left(\frac{\sigma_{cr}}{P/\pi t^2}\right)^{m-1} d\left(\frac{\sigma_{cr}}{P/\pi t^2}\right) - \int_{\sigma_{cr}/P/\pi t^2=\sigma_2/P/\pi t^2}^{\sigma_1/P/\pi t^2} \ln\left(1 - \frac{\Omega}{2\pi}\right) \left(\frac{\sigma_{cr}}{P/\pi t^2}\right)^{m-1} d\left(\frac{\sigma_{cr}}{P/\pi t^2}\right) \right], \quad (23)$$

where a is the radius of the support ring, t is the thickness of the specimen, and Ω is the critical angle as defined before.

By taking logarithmic operation twice on Eq. (20) and best fitting the experimental data points with a straight line as in Fig. 2, we determine that the parameters m and B are 4.917 and −20.60, respectively. We can also obtain the crack density parameter k from Eq. (21). The determined crack density parameters m and k based on the two distinct formulations are summarized in Table 1.

4 Discussions

In Example 1A, there is no difference in terms of actual formulations between the work based on Batdorf and Crose [1] and the present work as shown in Fig. 1. In Examples 1B and 1C, the results shown in Fig. 1 reflect the differences caused by the two formulations. While these are not dramatic differences, the failure probability of the proposed formulation is higher than that of the formulation by Batdorf and Crose [1] given the same stress level and therefore it provides a more conservative estimate.

In Example 1B, if σ_{cr} falls between σ_2 and σ_1, the two formulations are significantly different as in Eqs. (18) and (19). When the value of σ_{cr} approaches that of σ_1, the value of $\Omega/2\pi$ approaches zero [see Eq. (17)]. In this case, $\ln(1-\Omega/2\pi)$ in Eq. (18) can be approximated by $-\Omega/2\pi$ in Eq. (19) by ignoring the higher order terms of the Taylor expansion. Because of this, the contribution to the overall failure probability by cracks with critical

Table 1 Crack density parameters determined by biaxial flexure tests

	m	k [$\mathrm{mm}^{-2}\,\mathrm{MPa}^{-4.917}$]
Batdorf and Crose Formulation	4.917	4.95×10^{-11}
Proposed Formulation	4.917	3.99×10^{-11}

stress close to σ_1 is very similar between the two formulations. When the value of σ_{cr} is larger than σ_2 but not close to σ_1, however, $\ln(1-\Omega/2\pi)$ cannot be approximated by $-\Omega/2\pi$ since $\Omega/2\pi$ is not small. As in Eq. (15), the number of cracks increases rapidly as critical stress σ_{cr} increases, indicating that the number of small cracks is much larger than the number of large cracks. Therefore, the crack density function, serving as a weight function in Eqs. (18) and (19), favors the small cracks which give a range of small $\Omega/2\pi$. In other words, the numerical difference between the two formulations in Example 1B can be attributed to the failure probability primarily due to cracks with critical stress larger than σ_2 but not close to σ_1.

Similarly in Example 1C, there are significant differences between $\ln(1-\Omega/2\pi)$ and $-\Omega/2\pi$ when the critical stress is larger than zero but not close to σ_1. However, the number of cracks corresponding to this range of critical stress is relatively small. Therefore, the contribution to the overall failure probability by such cracks is relatively small. On the contrary, when the critical stress is close to σ_1, $\ln(1-\Omega/2\pi)$ can be approximated by $-\Omega/2\pi$. The number of cracks corresponding to this range of critical stress is relatively large. Therefore, the contribution to the overall failure probability of such cracks is large, and the numerical differences between the two formulations remain small. By adding these small differences in two regions of critical stress, the numerical difference between the two formulations in Example 1C remains relatively small.

We have discussed the reason that there are not significant numerical differences between the two formulations in the results of Example 1 while the failure probability prediction formulas are significantly different. It is important to note that we chose a specific form of crack density function N expressed in Eq. (15). Given a more general form of crack density function, the numerical results based on these two formulations may be different.

In Example 2, the values of k are different for the two formulations while the values of m are the same. The difference is 24% using the result based on the proposed formulation as the baseline. This difference may lead to possible errors in failure probability prediction.

While the formulation by Batdorf and Crose [1] has been successfully used for many practical applications, the formulation is based on Eq. (4) where the premature assumption infinitesimally small volume element is implicitly employed. In order to improve accuracy of failure probability prediction, it is best to employ Eq. (14).

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