On two conjectures by K. Kashihara on prime numbers

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Abstract. We settle two conjectures posed by K. Kashihara in his book [2]. The first conjecture states that $\prod_{i=1}^{n}\left(1-\frac{1}{p_i}\right) < \frac{1}{p_{n+1}-p_n}$ for all n; while the second one that the sequence of general term $\sum_{i=1}^{n} p_i^2 \Big/ \left(\sum_{i=1}^{n} p_i\right)^2$ is convergent. Here p_n denotes the nth prime. We will prove that the first conjecture is false for sufficiently large n. The second conjecture is true, the limit being zero.

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1 Introduction

Let p_n denote the nth prime number. In his book [2] K. Kashihara posed several conjectures and open problems. On page 45 it is conjectured the following inequality:

$$p_{n+1} - p_n < \prod_{i=1}^{n} \frac{1}{1-\frac{1}{p_i}}, \quad (n = 1, 2, \ldots) \tag{1}$$

A numerical evidence suggests that this inequality may be true for all values of n. However, as we will see, for large values of n, relation (1) cannot hold.

Another conjecture (see page 46) states that the sequence (x_n) of general term

$$x_n = \frac{\sum_{i=1}^{n} p_i^2}{\left(\sum_{i=1}^{n} p_i\right)^2} \quad (n \geq 1) \tag{2}$$

is convergent, having a limit between 1,4 and 1,5. Though this sequence is indeed convergent, we will see that its limit is $\rho = 0$.

2 Proofs of theorems

An old theorem of F. Mertens (see e.g. [3], p.259) states that

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)\sim\frac{c}{\log x}\text{ as }x\to\infty,\tag{3}$$

where $c=e^{-\gamma}$ (e and γ being the two Euler constants). Inequality (1) can be written also as

$$\prod_{p\le p_n}\left(1-\frac{1}{p}\right)<\frac{1}{p_{n+1}-p_n}\tag{4}$$

Since the first term of (4) is $\sim\dfrac{c}{\log p_n}$, if (4) would be true, then for all $\varepsilon>0$ (fixed) and $n\ge n_0$ we would obtain that $\dfrac{1}{p_{n+1}-p_n}>\prod\limits_{p\le p_n}\left(1-\dfrac{1}{p}\right)>\dfrac{c-\varepsilon}{\log p_n}$. Let $\varepsilon=\dfrac{c}{2}>0$. Then $\dfrac{c}{2}\cdot\dfrac{1}{\log p_n}<\dfrac{1}{p_{n+1}-p_n}$, so $b_n=\dfrac{p_{n+1}-p_n}{\log p_n}<\dfrac{2}{c}=K$. This means that the sequence of general term (b_n) is bounded above. On the other hand, a well-known theorem by E. Westzynthius (see [3], p. 256) states that $\lim\limits_{n\to\infty}\sup b_n=+\infty$, i.e. the sequence (b_n) is unbounded. This finishes the proof of the first part.

For the proof of convergence of (x_n) given by (2), we shall apply the result

$$\sum_{p\le x}p^{\alpha}\sim\frac{x^{1+\alpha}}{(1+\alpha)\log x}\text{ as }x\to\infty\ (\alpha\ge 0)\tag{5}$$

due to T. Salát and S. Znám (see [3], p. 257). We note that for $\alpha=1$, relation (5) was discovered first by E. Landau. Now, letting $\alpha=1$, resp. $\alpha=2$ in (5), we can write:

$$\sum_{p\le p_n}p\sim\frac{p_n^2}{2\log p_n}\text{ as }n\to\infty;\tag{6}$$

and

$$\sum_{p\le p_n}p^2\sim\frac{p_n^3}{3\log p_n}\text{ as }n\to\infty.\tag{7}$$

Thus, $x_n=\left[\left(\sum\limits_{p\le p_n}p^2\right)\cdot\dfrac{3\log p_n}{p_n^3}\cdot\dfrac{p_n^4}{\left(\sum\limits_{p\le p_n}p\right)^2\cdot 4\log^2 p_n}\right]\cdot\dfrac{4}{3}\cdot\dfrac{\log p_n}{p_n}$.

By (6) and (7), the limit of term $[\ldots]$ is 1. Since $\dfrac{4}{3}\cdot\dfrac{\log p_n}{p_n}\to 0$, we get $\lim\limits_{n\to\infty}x_n=0$. This finishes the proof of the second part.

Remarks

1) An extension of (5) is due to M. Kalecki [1]:

Let $f:(0,+\infty)\to\mathbb{R}$ be an arbitrary function having the following properties:

a) $f(x)>0$; *b)* $f(x)$ is a non-decreasing function; *c)* for each $n>0$, $\varphi(n)=\lim\limits_{x\to\infty}\dfrac{f(nx)}{f(x)}$ exists.

Put $s=\log\varphi(e)$. Then

$$\sum_{p\le x} f(p)\sim\frac{f(x)\cdot x}{\log x}\cdot\frac{1}{s+1}\ \text{as } x\to\infty. \tag{8}$$

For $f(x)=x^\alpha$ ($\alpha\ge 0$) we get $\varphi(n)=n^\alpha$, so $s=\alpha$ and relation (5) is reobtained. We note that for $\alpha=0$, relation (5) implies the "prime number theorem" ([3])

$$\pi(x)\sim\frac{x}{\log x}\ \text{as } x\to\infty,$$

where $\pi(x)=\sum\limits_{p\le x}1$ = number of primes $\le x$.

2) By letting $f(x)=(g(x))^\alpha$, where g satisfies conditions $a)-c)$ a general sequence of terms $x_n=\sum\limits_{i=1}^{n}(g(p_i))^\alpha\Big/\left(\sum\limits_{i=1}^{n}g(p_i)\right)^\alpha$ may be studied (via (8)) in a similar manner. We omit the details.

References

[1] M. Kalecki, *On certain sums extended over primes or prime factors*, Prace Mat. **8**(1963/64), 121–127

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[3] J. Sándor et al., *Handbook of number theory I*, Springer Verlag, 2005