On the role of viscosity in ideal Hadley circulation models

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[1] A comparison is made between inviscid and viscous solutions of an axially symmetric nonlinear Shallow Water Model (SWM) of the Hadley circulation on the spherical rotating Earth that includes vertical advection of momentum and Rayleigh friction. The results of the 1D SWM are compared with those of 2D (latitude-height) nonlinear axially symmetric models. It is shown that solutions obtained from inviscid models do not predict correctly the main features of the circulation outside the Tropics. Specifically, the latitudes of the subtropical jets depend strongly on viscosity and on the pole-to-equator temperature difference. The differences from inviscid theory are attributed to the increased rate of energy dissipation in the viscous atmosphere. At the Tropics, however, the inviscid solutions of the SWM predict well the latitude and strength of the easterly jet and the weak temperature gradient. **Citation:** Adam, O., and N. Paldor (2010), On the role of viscosity in ideal Hadley circulation models, *Geophys. Res. Lett.*, *37*, L16801, doi:10.1029/2010GL043745.

1. Introduction

[2] Solutions of axially symmetric inviscid models of the upper branches of the Hadley circulation capture many of the main features of the zonally averaged circulation, despite the vast simplifications employed in their formulations. Outside the Tropics, heat and momentum fluxes are dominated by macro-turbulence, making axially symmetric models less relevant in these latitudes. Most notable of these models is *Held and Hou*'s [1980, hereafter HH] model, which approximates the circulation in the upper branches of the Hadley cell in the inviscid limit. The solutions of HH follow a 2-region paradigm made up of a tropical, uniform-*M* region, and an extratropical, radiative-equilibrium region. The Angular Momentum Conserving (AMC) solutions of HH were extended to off-equatorial heating by *Lindzen and Hou* [1988, hereafter LH]. When compared with the zonally averaged atmosphere and with axially symmetric 2D models, the AMC solutions capture the observed Hadley circulation width, the relative strength of the subtropical jets, and the relative strength of the winter and summer cells. However, the AMC solutions fail in predicting the non-uniform distribution of angular momentum at the Tropics [cf. *Schneider*, 2006], and in the case of off-equatorial heating, they predict stronger-than-observed tropical temperature gradients and easterlies that are also fixed at the equator, and a winter jet that is farther poleward than the summer jet (opposite to their observed relative latitudes). In addition, the AMC solutions for off-equatorial heating predict a nonlinear amplification of the winter circulation even for small displacements of the heating off the equator, in contrast to the observed atmosphere [*Dima and Wallace*, 2003].

[3] As was shown by *Adam and Paldor* [2009, hereafter AP09] and *Adam and Paldor* [2010, hereafter AP10] a Shallow Water Model (SWM) captures the essential features of the idealized axisymmetric models used by HH and LH. In addition, when Vertical Advection of Momentum (VAM) is assumed into the rising branch of the Hadley circulation from an underlying passive surface layer a tropical mass/heat acquiring, uniform-temperature and non-uniform-*M* region forms in addition to the subtropical uniform-*M* and extratropical radiative-equilibrium regions that exist in the AMC solutions. When VAM is "turned off" the VAM solutions of AP09 and AP10 become exactly the AMC solutions of HH and LH, respectively, by enforcing Hide's theorem (choosing those states for which the zonal velocity vanishes at the latitudes separating the winter and summer cells). The VAM solutions predict realistic temperature gradients and easterlies at the Tropics but (as in the AMC solutions) predict stronger than observed circulation strength amplification with increasing latitude of maximal heating and incorrect relative latitudes of the subtropical jets.

[4] The latitudes of the subtropical jets in the AMC and VAM solutions depend on the latitude of maximal heating and on the thermal Rossby number (equation (1f)). The jet latitudes observed in the numerical results of 2D axially symmetric models [e.g., LH; *Fang and Tung*, 1999, hereafter FT] differ from those predicted by AMC and VAM solutions and exhibit a strong dependence on the pole-to-equator temperature difference (PETD). This suggests that the simple parametric dependence of the AMC and VAM solutions does not fully describe "nearly-inviscid" axially symmetric steady states.

[5] In the 2D axially symmetric models such as the ones used by HH, LH, FT and *Walker and Schneider* [2005, hereafter WS], the "nearly-inviscid" limit was assumed to be reached by using the lowest viscosity allowed by the numerical scheme for which stable solutions were obtained. The underlying assumption in all these studies is that in the limit of vanishing viscosity the mean properties of the circulation become insensitive to the value of the viscosity (this is the "nearly-inviscid" assumption) [*Schneider*, 1977]. This assumption (which also implies that inviscid solutions can be used to predict the mean features of the "nearly-inviscid" circulation), however, has never been validated by a direct comparison of viscous 2D axially symmetric models with inviscid models (i.e., with the viscosity coefficient set equal to zero).

[6] A direct comparison of inviscid and nearly-inviscid results is possible using a shallow water framework, which mimics the upper branch of the Hadley circulation. *Polvani*

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and Sobel [2002, hereafter PS] studied the inviscid limit in a SWM with Rayleigh friction (but without vertical advection of momentum) on the equatorial *f*- and β-planes. PS derived expressions for the circulation strength and width by assuming a Weak Temperature Gradient (WTG) throughout the circulation cell. The effect of viscosity in the WTG solutions of PS is to widen and strengthen the circulation while damping the subtropical jet amplitude, in accordance with results of viscous 2D axially symmetric models. Unlike the VAM and AMC solutions that are purely inviscid the WTG solutions are the only solutions that permit non-vanishing viscosity. These solutions, however, do not include vertical advection of momentum and may not be suitable for studying Hadley circulations outside the Tropics.

[7] This work examines the effect of viscosity on the Hadley circulation using a SWM that, like PS, includes Rayleigh friction. The present SWM adds the following physical elements to PS: 1) Vertical advection of momentum; 2) Spherical geometry; 3) Off-equatorial and annually varying heating. The dependence of the solutions on the PETD is also examined. The numerical solutions in the viscous case are compared with the inviscid solutions of AP09 and AP10.

2. Model Equations

[8] Following AP09 and AP10, the nondimensional equations of the axially symmetric SWM are:

$$\dot{V} = -VV' - \frac{\mu}{1-\mu^2}\left(\alpha_\tau M^2 + V^2\right) + \left(1-\mu^2\right)\left(\alpha_\tau \mu - \alpha_g h'\right) - \left(I\frac{Q}{h} + \gamma\right)V \tag{1a}$$

$$\dot{M} = -VM' - \left(I\frac{Q}{h} + \gamma\right)\left(M - 1 + \mu^2\right) \tag{1b}$$

$$\dot{h} = -(hV)' + Q \tag{1c}$$

where h is the height, μ is sin(latitude), V is the meridional flux and M is the component of absolute angular momentum in the direction parallel to Earth's axis of rotation. The dot and prime denote derivatives with respect to time and μ, respectively. The source, $Q = h_f - h$, is a relaxation to a prescribed "radiative-equilibrium" height, h_f, associated with differential solar heating:

$$h_f = 1 + h_1\left(1 - 2(\mu - \mu_0)^2\right) \tag{1d}$$

where h_1 is the ratio between half the pole-to-equator height difference, H_1, and the scale height, H_0, and μ_0 is sin(latitude) of maximal heating. This form of the radiative-equilibrium height follows the upper branch of the radiative forcing used by HH in the $\mu_0 = 0$ case, by LH for the $\mu_0 \neq 0$ case and by FT for the seasonally varying μ_0 case. A constant atmospheric relaxation time, τ, is used to scale time, t. Viscosity is introduced via the parameter $\gamma = \alpha\tau$ where α denotes the dimensional Rayleigh friction coefficient. The flag, I, is zero for $Q \leq 0$ and 1 for $Q > 0$. The problem is completely prescribed by the 5 nondimensional parameters μ_0, γ, and:

$$\alpha_\tau = (\Omega\tau)^2; \alpha_g = \frac{gH_0\tau^2}{a^2}; h_1 = \frac{H_1}{H_0} \tag{1e}$$

where g, a and Ω are Earth's gravitational acceleration, radius and rotational frequency, respectively. The thermal Rossby number given by

$$R_T = \frac{2\alpha_g h_1}{\alpha_\tau} = \frac{2gH_1}{\Omega^2 a^2} \tag{1f}$$

is the counterpart of the thermal Rossby number of HH and LH. Following FT and LH we use $R_T = 0.113$ in this work. A detailed description of the scaling, model parameters and numerical scheme can be found in AP09 and AP10.

3. Numerical Results

[9] Steady states of the SWM with $\gamma \neq 0$ were obtained by numerical integration of the time dependent equations until a steady state is achieved. The numerical scheme used to integrate the equations is a staggered grid leap-frog scheme identical to that described by AP09 and AP10 but with 360 grid points.

[10] Unlike 2D axially symmetric models, the times at which steady states are achieved in the SWM for steady forcing are significantly longer than several seasons. In the SWM, steady states corresponding to γ = 1, 0.1, 0.01 and 0.001 are achieved after 1, 5, 10, and 80 years, respectively. Steady states corresponding to $\gamma < 10^{-5}$ have not been included in this work as these involve unphysically long times (>100 years) and numerical instabilities that result from the discontinuity in the angular momentum at the Hadley cell boundaries.

[11] Figure 1 displays the dependence of the latitudes of the winter and summer subtropical (μ_w, μ_s) and easterly (μ_E) jets on the latitude of maximal heating for various values of γ and for a relative PETD of 1/6 and 1/3 (i.e., h_1 = 1/12 and 1/6, respectively). In the inviscid case (γ = 0 (Figure 1a)), the relative latitudes of the subtropical jets (taken from the analytical solutions of AP10) are opposite to their observed relative locations. As seen in Figures 1b–1d, viscosity shifts the jet latitudes poleward. With increasing γ, the summer jet is shifted farther poleward than the winter jet, thus setting the two subtropical jets at their observed relative latitudes. The dependence of the jet latitudes on the latitude of maximal heating observed in Figure 1c for γ = .005 and in Figure 3 of FT are notably similar. In addition, in contrast to the predictions of inviscid theory, increasing h_1 while keeping R_T constant moves the latitudes of the subtropical jets equatorward. Like the subtropical jets, but to a much lesser extent, the tropical easterly jet latitude (shown on Figures 1a and 1c) is shifted poleward with increasing viscosity and equatorward with increasing h_1.

[12] Figure 2 displays the dependence of the steady Circulation Strength (CS), defined as the global absolute maximum of V, scaled by the Equinoctial Circulation Strength (ECS) (i.e., μ_0 = 0 case for each respective γ), on the latitude of maximal heating for several values of γ. Figure 2 also compares the scaled CS for steady heating

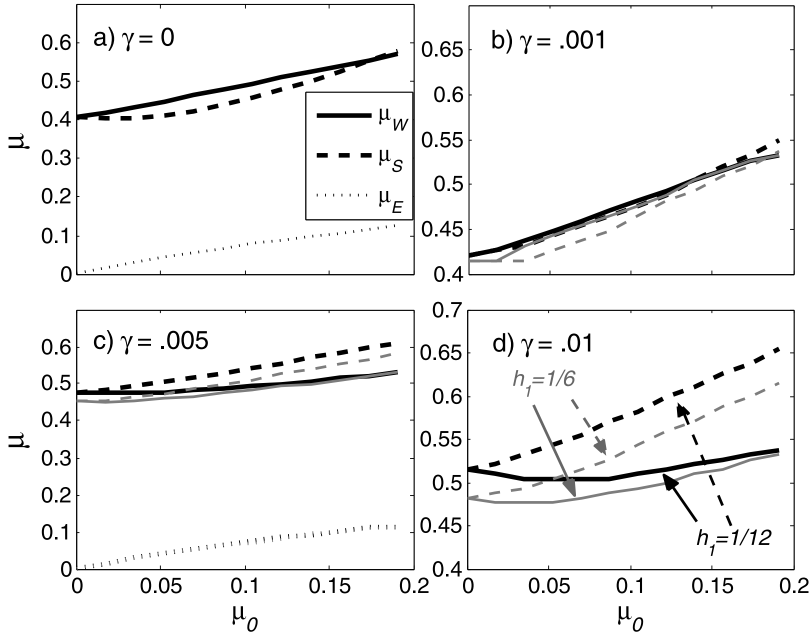

Figure 1. The steady latitudes of the winter (μ_w, solid), summer (μ_s, dashed) and easterly (μ_E, dotted) jets, for R_T = 0.133. (a) Analytical solutions for γ = 0 (AP10). Numerical solutions for h_1 = 1/12 (bold black) and h_1 = 1/6 (grey) and (b) γ = .001 (t~80 years); (c) γ = .005 (t~40 years); (d) γ = .01 (t~10 years). For simplicity, the absolute value of the jet latitudes is presented. The vertical ordinates of Figures 1b and 1d are truncated for clarity of presentation and the solutions for μ_E (that are nearly identical for the two values of h_1 as in Figure 1c) are not shown.

with h_1 = 1/12 (black) and h_1 = 1/6 (grey) while keeping R_T = .113. As in PS, the scaled CS increases with μ_0, while the amplification decreases with γ due to the strengthening of the ECS with increasing γ (as discussed in the next section). The scaled CS amplification with increasing μ_0 in the inviscid case is 50% stronger than the amplification observed in the nearly-inviscid numerical results of LH and FT. An amplification and jet latitudes similar to those presented by LH and FT are obtained for $\gamma \approx 0.003$ (not shown). In addition, as mentioned by WS, the effect of baroclinic eddies on the momentum fluxes at the subtropics is similar to the effect of viscosity in axially symmetric models. Indeed, the scaled CS and jet latitudes in the viscous SWM are similar to those obtained in the eddy permitting simulations of WS (R_T = .105, $h_1 \approx 1/10$) for $\gamma \approx 0.1$, though jet latitudes in the SWM are slightly shifted poleward.

[13] Figure 3 displays the Annually Varying Circulation Strength (AVCS) for the steady (solid) and time dependent forcing (τ = 20days – dashed, τ = 40days – dotted), scaled by the ECS for each respective γ. The annual average is obtained by weighting the relative contribution of each steady state corresponding to a particular latitude of maximal heating according to the duration spent in the vicinity of that latitude, where $\mu_0(t) = 0.2 \sin(2\pi\tau t/365)$, as in FT.

[14] A "nearly-inviscid" regime can be identified for $\gamma < 10^{-4}$ in which the scaled AVCS for steady and for time dependent forcing are nearly identical, with values of approximately 2.2 and 2.4 for time dependent and steady forcing, respectively. This result is somewhat larger than value obtained by FT who also found that the AVCS is slightly weaker in the time dependent case and that the AVCS is about twice the ECS. The scaled AVCS for the annually varying case and τ = 40 days is found to be stronger than the τ = 20 days case. The larger scaled AVCS in the SWM may be attributed to the longer adjustment times discussed above.

[15] A 'viscous' regime can be identified for $\gamma > 10^{-4}$, in which the scaled AVCS for the steady and time dependent

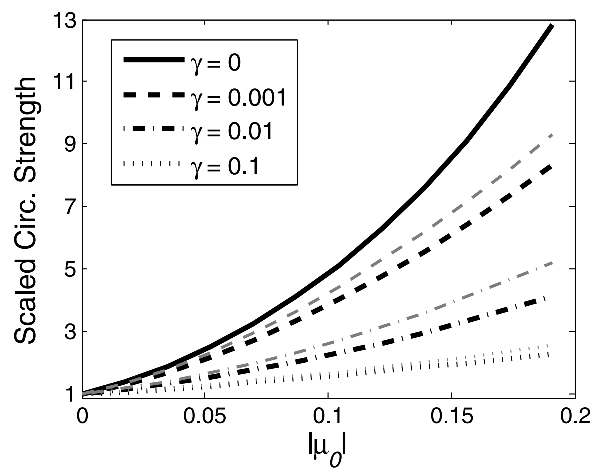

Figure 2. The dependence of the circulation strength, scaled by the circulation strength in the equinoctial heating case, on the latitude of maximal heating μ_0 for γ = 0, γ = .001, γ = .01 and γ = 0.1. The plot also compares integrations with a relative pole-to-equator temperature difference of 1/6 (h_1 = 1/12, bold black) and 1/3 (h_1 = 1/6, grey) for a fixed thermal Rossby number, R_T = 0.113. The circulation strength is defined as the global absolute maximum of V.

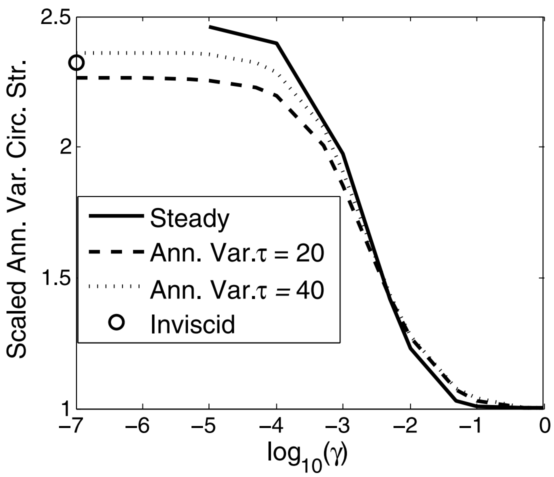

Figure 3. The dependence of the annually averaged circulation strength, scaled by the circulation strength in the equinoctial case, on $\log_{10}(\gamma)$. The circulation strength is defined as the global maximum of absolute V. Steady viscous solutions (solid) have been calculated for $\gamma \geq 10^{-5}$. The steady inviscid (γ = 0, AP10) case is marked by a circle. In the time dependent annually varying case, the latitude of maximal heating, μ_0, is given by $\mu_0(t) = 0.2 \sin(2\pi\tau t/365)$ for τ = 20 (dashed) and 40 (dotted) days. In all integrations, R_T = 0.113 and h_1 = 1/12.

forcing cases strongly decrease with γ and converge to 1 for $\gamma \to 1$.

4. Discussion and Conclusions

[16] The SWM is a relatively simple platform that enables a qualitative study of the contributions of vertical advection of momentum, viscosity and pole–to-equator temperature difference (PETD) to global Hadley circulations. In particular, a study of the dependence of this circulation on viscosity added to the SWM is afforded without requiring unphysical boundary conditions or limitations due to instabilities that occur in 2D models at low viscosities. In addition, unlike other 2D models that can only be solved by numerical integration of the time dependent equations, analytical insight emerges in inviscid 1D circulations model based on the SWM (AP09; AP10).

[17] The scenario emerging from the study of the SWM with and without Vertical Advection of Momentum (VAM) (AP09; AP10) and viscosity is the following:

[18] 1. A comparison between the inviscid SWM with and without VAM reveals that in the absence of VAM, the circulation follows a 2-region paradigm: a) an angular momentum conserving region at the Tropics and b) radiative-equilibrium regions at the extratropics, as predicted by the Angular Momentum Conserving (AMC) solutions of HH and LH. The addition of VAM to the inviscid SWM introduces an additional tropical, mass acquiring, strong mixing region in which the temperature gradient is weak and angular momentum is not uniform.

[19] 2. A comparison of the inviscid and viscous circulation (including VAM) reveals that the effect of viscosity is dominant at the subtropics where it prevents angular momentum conservation, strengthens the circulation and extends the latitudes of the subtropical jets poleward. At the Tropics, the VAM solutions capture well the latitude and strength of the easterly jet even at relatively high viscosities.

[20] 3. A study of the circulation in the limit of $\gamma \to 0$ reveals that while the annually averaged circulation strength is insensitive to the value of γ in this limit, the Circulation Strength (CS) and the relative latitudes of the subtropical jets for steady off-equatorial heating states strongly depend on γ in this limit. In particular, a non-vanishing viscosity is required in order to set the subtropical jets at their observed relative latitudes.

[21] 4. Nearly-inviscid ($\gamma \to 0$) circulations do not have the same parameter dependence as inviscid (γ = 0) solutions, which depend only on the latitude of maximal heating and the thermal Rossby number, R_T. In particular, significant differences in the latitudes and strengths of the subtropical jets and in the CS of nearly-inviscid solutions are observed when the PETD is varied while keeping R_T constant.

[22] The increase in the circulation width and strength with increasing viscosity has been observed in many works (e.g., PS; FT). In addition, as mentioned by WS, and in agreement with the results shown in Figures 1 and 2, viscosity can be used to model the effect of baroclinic eddies on the momentum fluxes at the subtropics.

[23] The effect of viscosity on the mean features of the circulation can also be inferred from an examination of the effect of viscosity on the actual PETD, i.e., the difference between the uniform value of h at the Tropics and the value of h at the poles (in contrast to $2h_1$ that quantifies the "prescribed" PETD). For a given radiative-equilibrium height, increased viscosity results in a more efficient dissipation of energy, which accounts for the decrease in the actual available potential energy (proportional to the actual PETD). The reduced height at the Tropics (where $Q > 0$ and h is uniform) causes the Hadley cell to widen due to the

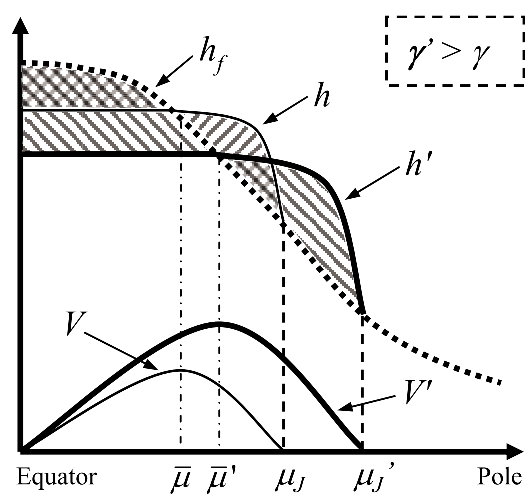

Figure 4. A depiction of the dependence of the circulation strength and width on γ according to the "equal-area" construction. For equinoctial heating, the width of the tropical uniform-h region ($\bar{\mu}$) and the subtropical jet latitude (μ_J, coinciding with the Hadley circulation width) are shifted poleward as γ increases and the height at the equator decreases ($\bar{\mu}' > \bar{\mu}$, and $\mu_J' > \mu_J$, respectively). Bold lines depict the functional forms of V' and h', corresponding to the γ' case.

increased outward flux (cooling) required to balance the increased in-ward flux (heating) by the forcing at the Tropics (this is the so called "equal area" argument). This effect is depicted in Figure 4 where it is shown that for equinoctial heating, the boundaries of the tropical uniform-h region and the latitudes of the subtropical jets are shifted poleward as γ increases. For off-equatorial heating, this effect is more pronounced at the summer hemisphere, causing the summer jet to be shifted farther poleward than the winter jet in the viscous atmosphere. Similarly, the global maximum of V is proportional to the total amount of heating, causing the CS to increase with viscosity. We also note that the actual PETD is set by the integral of $h_f(\mu)$ and not only by the value of the prescribed PETD.

[24] In contradiction to the assumption of the nearly-inviscid limit, inviscid solutions do not predict well nearly-inviscid circulations outside the Tropics. The SWM is perhaps the only framework that allows a simple bridge between inviscid and viscous global circulations. Further analysis of the solutions of the viscous SWM can therefore be instrumental in obtaining a revised nearly inviscid theory, more in accordance with simulated nearly-inviscid circulations and the observed atmosphere.

[25] **Acknowledgment.** This study is part of OA's Ph.D. dissertation at HU.

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