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Homomorphism and Anti Homomorphism on Bipolar Fuzzy Sub HX Groups

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Abstract

In this paper, we introduce the concept of an image, pre image of a bipolar fuzzy subsets and discuss in detail a series of homomorphic and anti homomorphic properties of bipolar fuzzy sub groups.

Keywords: *Fuzzy set, HX group, Fuzzy HX group, Bipolar-valued fuzzy set, Bipolar fuzzy HX group, Homomorphism and Anti homomorphism of fuzzy HX group, Image and pre-image of bipolar fuzzy sets are discussed.*

1 Introduction

The concept of fuzzy sets was initiated by Zadeh [16]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [12] gave the idea of fuzzy subgroup. In fuzzy sets the membership degree of elements range over the interval [0, 1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0,1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Li Hongxing [3] introduced the concept of HX group and the authors Luo Chengzhong, Mi Honghai, Li Hongxing [4] introduced the concept of fuzzy HX group. The author W.R. Zhang [14], [15] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [-1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [-1, 0) indicates that elements somewhat satisfy the implicit counter-property. M. Marudai, V. Rajendran [5] introduced the pre image of bipolar Q fuzzy subgroup. In this paper we define the image and pre image of a bipolar fuzzy subgroup and bipolar fuzzy sub HX group and discuss some of its properties.

2 Preliminaries

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G, *) be a finite group, e is the identity element of G, and xy we mean x * y.

Definition 2.1[9]: *Let X be any non-empty set. A fuzzy subset μ of X is a function $\mu : X \rightarrow [0,1]$.*

Definition 2.2[3]: *In $2^G - \{\phi\}$, a non-empty set $\vartheta \subset 2^G - \{\phi\}$ is called a HX group on G, if ϑ is a group with respect to the algebraic operation defined by $AB = \{ab \,/\, a \in A \text{ and } b \in B\}$, which its unit element is denoted by E.*

Definition 2.3[10]: *Let μ be a fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. A fuzzy set λ_μ defined on ϑ is said to be a fuzzy subgroup induced by μ on ϑ or a fuzzy HX subgroup on ϑ if for any A, B $\in \vartheta$,*

i. $\lambda_\mu(AB) \geq \min\{\lambda_\mu(A), \lambda_\mu(B)\}$
ii. $\lambda_\mu(A^{-1}) = \lambda_\mu(A)$.

where, $\lambda_\mu(A) = \max\{\mu(x) \,/\, \text{for all } x \in A \subset G\}$.

Definition 2.4[10]: *Let G be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set* μ *in G is an object having the form* $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$*,where* $\mu^+ : G \to [0,1]$ *and* $\mu^- : G \to [-1,0]$ *are mappings. The positive membership degree* $\mu^+(x)$ *denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set* $\mu =\{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$ *and the negative membership degree* $\mu^-(x)$ *denotes the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar-valued fuzzy set* $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$*. If* $\mu^+(x) \neq 0$ *and* $\mu^-(x) = 0$*, it is the situation that x is regarded as having only positive satisfaction for* $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$*. If* $\mu^+(x) = 0$ *and* $\mu^-(x) \neq 0$*, it is the situation that x does not satisfy the property of* $\mu =\{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$*, but somewhat satisfies the counter property of* $\mu =\{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$*. It is possible for an element x to be such that* $\mu^+(x) \neq 0$ *and* $\mu^-(x) \neq 0$ *when the membership function of property overlaps that its counter property over some portion of G. For the sake of simplicity, we shall use the symbol* $\mu = (\mu^+,\mu^-)$ *for the bipolar-valued fuzzy set* $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle / \text{ for all } x\in G\}$*.*

Definition 2.5[10]: *A bipolar-valued fuzzy set or bipolar fuzzy set* μ *in G is a bipolar fuzzy subgroup of G. if for all x, y* $\in G$*,*

i. $\mu^+(xy) \geq \min\{\mu^+(x), \mu^+(y)$,
ii. $\mu^-(xy) \leq \max\{\mu^-(x), \mu^-(y)\}$,
iii. $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$.

Definition 2.6[10]: *Let* ϑ *be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set [BFS]* λ_μ *in* ϑ *is an object having the form* $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x\in A\subset G\}$*, where* $\lambda_\mu^+ : \vartheta \to [0,1]$ *and* $\lambda_\mu^- : \vartheta \to [-1,0]$ *are mappings. The positive membership degree* $\lambda_\mu^+(A)$ *denotes the satisfaction degree of an element A to the property corresponding to a bipolar-valued fuzzy set* $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x \in A \subset G\}$ *and the negative membership degree* $\lambda_\mu^-(A)$ *denotes the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar-valued fuzzy set* $\lambda_\mu =\{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x \in A \subset G\}$*. If* $\lambda_\mu^+(A) \neq 0$ *and* $\lambda_\mu^-(A) = 0$*, it is the situation that A is regarded as having only positive satisfaction for* $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x \in A \subset G\}$*. If* $\lambda_\mu^+(A) = 0$ *and* $\lambda_\mu^-(A) \neq 0$*, it is the situation that A does not satisfy the property of* $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x \in A \subset G\}$*, but somewhat satisfies the counter property of* $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x \in A \subset G\}$*. It is possible for an element A to be such that* $\lambda_\mu^+(A) \neq 0$ *and* $\lambda_\mu^-(A) \neq 0$ *when the membership function of property overlaps that its counter property over some portion of* ϑ*. For the sake of simplicity, we shall use the symbol* $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$*. For the bipolar-valued fuzzy set* $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle / \text{ for all } x\in A \subset G\}$*.*

Definition 2.7[10]: *Let* μ *be a bipolar fuzzy subset defined on G. Let* $\vartheta \subset 2^G - \{\phi\}$ *be a HX group on G. A bipolar fuzzy set* λ_μ *defined on* ϑ *is said to be a bipolar* *fuzzy subgroup induced by μ on ϑ or a bipolar fuzzy HX subgroup on ϑ, if for any $A, B \in \vartheta$,*

i. $\lambda_\mu^+(AB) \geq \min\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$,
ii. $\lambda_\mu^-(AB) \leq \max\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$,
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

Where, $\lambda_\mu^+(A) = \max\{\mu^+(x) / \text{for all } x \in A \subset G\}$ and
$\lambda_\mu^-(A) = \min\{\mu^-(x) / \text{for all } x \in A \subset G\}$.

Definition 2.8[13]: *A mapping f from a group G_1 to a group G_2 is said to be a homomorphism if $f(xy) = f(x) f(y)$ for all $x, y \in G_1$.*

Definition 2.9[13]: *A mapping f from a group G_1 to a group G_2 (G_1 and G_2 are not necessarily commutative) is said to be an anti homomorphism if $f(xy) = f(y) f(x)$ for all $x,y \in G_1$.*

Definition 2.10[9]: *A mapping f from a HX group ϑ_1 to a HX group ϑ_2 is said to be a homomorphism if $f(AB) = f(A) f(B)$ for all $A,B \in \vartheta_1$.*

Definition 2.11[9]: *A mapping f from a HX group ϑ_1 to a HX group ϑ_2 (ϑ_1 and ϑ_2 are not necessarily commutative) is said to be an anti homomorphism if $f(AB) = f(B) f(A)$ for all $A,B \in \vartheta_1$.*

3 Image and Pre-Image of a Bipolar Fuzzy Sub Group of a Group under Homomorphism and Anti Homomorphism

In this section, we introduce the notion of image and pre-image of the bipolar fuzzy subgroup of a group, and discuss some of its properties. Throughout this section, We mean that G1 and G2 are finite groups and e1 and e2 are the identity elements of G1 and G2 respectively, and xy we mean x * y.

Definition 3.1: *Let f be a mapping from a group G_1 to a group G_2 and let μ, φ be fuzzy subsets in G_1 and G_2 respectively. Then the image $f(\mu)$ of μ is the fuzzy subset of G_2 defined by for $u \in G_2$*

$$(f(\mu))(u) = \begin{cases} \max\{\mu(x) : x \in f^{-1}(u)\}, & \text{if } f^{-1}(u) \neq \phi \\ 0 & \text{, Otherwise} \end{cases}$$

and the pre-image $f^{-1}(\varphi)$ of φ under f is the fuzzy subset of G_1 defined by for $x \in G_1$, $(f^{-1}(\varphi))(x) = \varphi(f(x))$.

Definition 3.2: *Let f be a mapping a group G_1 to a group G_2 and let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets in G_1 and G_2 respectively. Then the image $f(\mu)$ of μ is the bipolar fuzzy subset $f(\mu) = ((f(\mu))^+, (f(\mu))^-)$ of G_2 defined by for $u \in G_2$,*

$$(f(\mu))^+(u) = \begin{cases} \max\{\mu^+(x) : x \in f^{-1}(u)\}, & \text{if } f^{-1}(u) \neq \phi \\ 0 & \text{, Otherwise} \end{cases}$$

and

$$(f(\mu))^-(u) = \begin{cases} \max\{\mu^-(x) : x \in f^{-1}(u)\}, & \text{if } f^{-1}(u) \neq \phi \\ 0 & \text{, Otherwise} \end{cases}$$

and the pre-image $f^{-1}(\varphi)$ of φ under f is the bipolar fuzzy subset of G_1 defined by for $x \in G_1$, $(f^{-1}(\varphi))^+(x) = \varphi^+(f(x))$, $(f^{-1}(\varphi))^-(x) = \varphi^-(f(x))$.

Theorem 3.1: *Let f be a homomorphism from a group G_1 into a group G_2. If $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of G_1 then $f(\mu)$, the image of μ under f, is a bipolar fuzzy subgroup of G_2.*

Proof: Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of G_1, $\mu^+ : G_1 \to [0,1]$ and $\mu^- : G_1 \to [-1,0]$ are mappings.

Let $u,v \in G_2$, since f is homomorphism and so there exist $x,y \in G_1$ such that $f(x) = u$ and $f(y) = v$ it follows that $xy \in f^{-1}(uv)$.

Now,
$$\begin{aligned} (f(\mu))^+(uv) &= \max\{\mu^+(z) : z = xy \in f^{-1}(uv)\} \\ &\geq \max\{\mu^+(xy) : x \in f^{-1}(u), y \in f^{-1}(v)\} \\ &\geq \max\{\min\{\mu^+(x), \mu^+(y)\} : x \in f^{-1}(u), y \in f^{-1}(v)\} \\ &= \min\{\max\{\mu^+(x) : x \in f^{-1}(u)\}, \max\{\mu^+(y) : y \in f^{-1}(v)\}\} \\ &= \min\{(f(\mu))^+(u), (f(\mu))^+(v)\} \end{aligned}$$

Therefore, $(f(\mu))^+(uv) \geq \min\{(f(\mu))^+(u), (f(\mu))^+(v)\}$

And
$$\begin{aligned} (f(\mu))^-(uv) &= \max\{\mu^-(z) : z = xy \in f^{-1}(uv)\} \\ &\leq \max\{\mu^-(xy) : x \in f^{-1}(u), y \in f^{-1}(v)\} \\ &\leq \max\{\max\{\mu^-(x), \mu^-(y)\} : x \in f^{-1}(u), y \in f^{-1}(v)\} \\ &= \max\{\max\{\mu^-(x) : x \in f^{-1}(u)\}, \max\{\mu^-(y) : y \in f^{-1}(v)\}\} \\ &= \max\{(f(\mu))^-(u), (f(\mu))^-(v)\} \end{aligned}$$

Therefore, $(f(\mu))^-(uv) \leq \max\{(f(\mu))^-(u), (f(\mu))^-(v)\}$

Now,
$$\begin{aligned} (f(\mu))^+(u^{-1}) &= \max\{\mu^+(x) : x \in f^{-1}(u^{-1})\} \\ &= \max\{\mu^+(x^{-1}) : x^{-1} \in f^{-1}(u)\} \\ &= (f(\mu))^+(u) \end{aligned}$$

And
$$(f(\mu))^-(u^{-1}) = \max\{\mu^-(x) : x \in f^{-1}(u^{-1})\}$$

$= \max\{\mu^-(x^{-1}) : x^{-1} \in f^{-1}(u)\}$
$= (f(\mu))^-(u)$

Therefore $f(\mu)$ is a bipolar fuzzy subgroup of G_2.
Hence, if μ be a bipolar fuzzy subgroup of G_1 then $f(\mu)$ is a bipolar fuzzy subgroup of G_2.

Theorem 3.2: *The homomorphic pre-image of a bipolar fuzzy subgroup $\varphi = (\varphi^+, \varphi^-)$ of a group of G_2 is a bipolar fuzzy subgroup of a group G_1.*

Proof: Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy subgroup of G_2, $\varphi^+ : G_2 \rightarrow [0,1]$ and $\varphi^- : G_2 \rightarrow [-1,0]$ are mappings.

Now, $(f^{-1}(\varphi))^+(xy) = \varphi^+(f(xy))$
$= \varphi^+(f(x)f(y))$
$\geq \min\{\varphi^+(f(x)), \varphi^+(f(y))\}$
$= \min\{(f^{-1}(\varphi))^+(x), (f^{-1}(\varphi))^+(y))\}$

Therefore, $(f^{-1}(\varphi))^+(xy) \geq \min\{(f^{-1}(\varphi))^+(x), (f^{-1}(\varphi))^+(y))\}$

And $(f^{-1}(\varphi))^-(xy) = \varphi^-(f(xy))$
$= \varphi^-(f(x)f(y))$
$\leq \max\{\varphi^-(f(x)), \varphi^-(f(y))\}$
$= \max\{(f^{-1}(\varphi))^-(x), (f^{-1}(\varphi))^-(y))\}$

Therefore, $(f^{-1}(\varphi))^-(xy) \leq \max\{(f^{-1}(\varphi))^-(x), (f^{-1}(\varphi))^-(y))\}$

Now, $(f^{-1}(\varphi))^+(x^{-1}) = \varphi^+(f(x^{-1}))$
$= \varphi^+(f(x)^{-1})$
$= \varphi^+(f(x))$
$= (f^{-1}(\varphi))^+(x)$

And $(f^{-1}(\varphi))^-(x^{-1}) = \varphi^-(f(x^{-1}))$
$= \varphi^-(f(x)^{-1})$
$= \varphi^-(f(x))$
$= (f^{-1}(\varphi))^-(x)$

Therefore, $f^{-1}(\varphi)$ is a bipolar fuzzy subgroup of G1.
Hence, if φ be a bipolar fuzzy subgroup of G_2 then $f^{-1}(\varphi)$ is a bipolar fuzzy subgroup of G1.

Theorem 3.3: *Let f be an anti homomorphism from a group G_1 into a group G_2. If $\mu = (\mu^+, \mu^-)$ is a bipolar fuzzy subgroup of G_1 then $f(\mu)$, the image of μ under f , is a bipolar fuzzy subgroup of a group G_2.*

Proof: Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of G_1, $\mu^+ : G_1 \to [0,1]$ and $\mu^- : G_1 \to [-1,0]$ are mappings.

Let $u, v \in G_2$, since f is an anti homomorphism and so there exist $x, y \in G_1$ such that $f(x) = u$ and $f(y) = v$ it follows that $xy \in f^{-1}(vu)$.

Now,
$$\begin{aligned}(f(\mu))^+(uv) &= \max\{\mu^+(z) : z = xy \in f^{-1}(vu)\}\\ &\geq \max\{\mu^+(xy) : x \in f^{-1}(u), y \in f^{-1}(v)\}\\ &\geq \max\{\min\{\mu^+(x), \mu^+(y)\} : x \in f^{-1}(u), y \in f^{-1}(v)\}\\ &= \min\{\max\{\mu^+(x) : x \in f^{-1}(u)\}, \max\{\mu^+(y) : y \in f^{-1}(v)\}\}\\ &= \min\{(f(\mu))^+(u), (f(\mu))^+(v)\}\end{aligned}$$

Therefore, $(f(\mu))^+(uv) \geq \min\{(f(\mu))^+(u), (f(\mu))^+(v)\}$

And
$$\begin{aligned}(f(\mu))^-(uv) &= \max\{\mu^-(z) : z = xy \in f^{-1}(vu)\}\\ &\leq \max\{\mu^-(xy) : x \in f^{-1}(u), y \in f^{-1}(v)\}\\ &\leq \max\{\max\{\mu^-(x), \mu^-(y)\} : x \in f^{-1}(u), y \in f^{-1}(v)\}\\ &= \max\{\max\{\mu^-(x) : x \in f^{-1}(u)\}, \max\{\mu^-(y) : y \in f^{-1}(v)\}\}\\ &= \max\{(f(\mu))^-(u), (f(\mu))^-(v)\}\end{aligned}$$

Therefore, $(f(\mu))^-(uv) \leq \max\{(f(\mu))^-(u), (f(\mu))^-(v)\}$

Now,
$$\begin{aligned}(f(\mu))^+(u^{-1}) &= \max\{\mu^+(x) : x \in f^{-1}(u^{-1})\}\\ &= \max\{\mu^+(x^{-1}) : x^{-1} \in f^{-1}(u)\}\\ &= (f(\mu))^+(u)\end{aligned}$$

And
$$\begin{aligned}(f(\mu))^-(u^{-1}) &= \max\{\mu^-(x) : x \in f^{-1}(u^{-1})\}\\ &= \max\{\mu^-(x^{-1}) : x^{-1} \in f^{-1}(u)\}\\ &= (f(\mu))^-(u)\end{aligned}$$

Therefore, $f(\mu)$ is a bipolar fuzzy subgroup of G_2.
Hence, if μ be a bipolar fuzzy subgroup of G_1 then $f(\mu)$ is a bipolar fuzzy subgroup of G_2.

Theorem 3.4: *The anti homomorphic pre-image of a bipolar fuzzy subgroup $\varphi = (\varphi^+, \varphi^-)$ of a group G_2 is a bipolar fuzzy subgroup of a group G_1.*

Proof: Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy subgroup of G_2, $\varphi^+ : G_2 \to [0,1]$ and $\varphi^- : G_2 \to [-1,0]$ are mappings.

Now,
$$\begin{aligned}(f^{-1}(\varphi))^+(xy) &= \varphi^+(f(xy))\\ &= \varphi^+(f(y)f(x))\\ &\geq \min\{\varphi^+(f(y)), \varphi^+(f(x))\}\\ &= \min\{(f^{-1}(\varphi))^+(y), (f^{-1}(\varphi))^+(x)\}\\ &= \min\{(f^{-1}(\varphi))^+(x), (f^{-1}(\varphi))^+(y)\}\end{aligned}$$

Therefore, $(f^{-1}(\varphi))^{+}(xy) \geq \min\{(f^{-1}(\varphi))^{+}(x), (f^{-1}(\varphi))^{+}(y))\}$

And $$\begin{aligned}(f^{-1}(\varphi))^{-}(xy) &= \varphi^{-}(f(xy)) \\ &= \varphi^{-}(f(y)f(x)) \\ &\leq \max\{\varphi^{-}(f(y)), \varphi^{-}(f(x))\} \\ &= \max\{(f^{-1}(\varphi))^{-}(y), (f^{-1}(\varphi))^{-}(x))\} \\ &= \max\{(f^{-1}(\varphi))^{-}(x), (f^{-1}(\varphi))^{-}(y))\}\end{aligned}$$

Therefore, $(f^{-1}(\varphi))^{-}(xy) \leq \max\{(f^{-1}(\varphi))^{-}(x), (f^{-1}(\varphi))^{-}(y))\}$

Now, $$\begin{aligned}(f^{-1}(\varphi))^{+}(x^{-1}) &= \varphi^{+}(f(x^{-1})) \\ &= \varphi^{+}(f(x)^{-1}) \\ &= \varphi^{+}(f(x)) \\ &= (f^{-1}(\varphi))^{+}(x)\end{aligned}$$

And $$\begin{aligned}(f^{-1}(\varphi))^{-}(x^{-1}) &= \varphi^{-}(f(x^{-1})) \\ &= \varphi^{-}(f(x)^{-1}) \\ &= \varphi^{-}(f(x)) \\ &= (f^{-1}(\varphi))^{-}(x)\end{aligned}$$

Therefore, $f^{-1}(\varphi)$ is a bipolar fuzzy subgroup of G_1.

Hence, if φ be a bipolar fuzzy subgroup of G_2 then $f^{-1}(\varphi)$ is a bipolar fuzzy subgroup of G1.

4 Image and Pre-Image of a Bipolar Fuzzy HX Group of a HX Group under Homomorphism and Anti Homomorphism

In this section, we introduce the notion of image and pre-image of the bipolar fuzzy sub HX group of a HX group, and discuss some of its properties. Throughout this section, We mean that ϑ_1 and ϑ_2 are HX groups and E_1 and E_2 are the identity elements of ϑ_1 and ϑ_2 respectively, and XY we mean X *Y.

Definition 4.1: *Let G_1 and G_2 be any two groups. Let $\vartheta_1 \subset 2^{G_1} - \{\phi\}$ and $\vartheta_2 \subset 2^{G_2} - \{\phi\}$ are HX groups defined on G_1 and G_2 respectively. Let $\mu = (\mu^{+}, \mu^{-})$ and $\varphi = (\varphi^{+}, \varphi^{-})$ are bipolar fuzzy subsets in G_1 and G_2 respectively, let $\lambda_\mu = (\lambda_\mu^{+}, \lambda_\mu^{-})$, and $\lambda_\varphi = (\lambda_\varphi^{+}, \lambda_\varphi^{-})$ are bipolar fuzzy subsets defined on ϑ_1 and ϑ_2 respectively induced by μ and φ. Let $f : \vartheta_1 \to \vartheta_2$ be a mapping then the image $f(\lambda_\mu)$ of λ_μ is the bipolar fuzzy subset $(f(\lambda_\mu) = ((f(\lambda_\mu))^{+}, (f(\lambda_\mu))^{-})$ of ϑ_2 defined by for $U \in \vartheta_2$,*

$$(f(\lambda_\mu))^{+}(U) = \begin{cases} \max\{(\lambda_\mu)^{+}(X) : X \in f^{-1}(U)\}, & \text{if } f^{-1}(U) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

And

$$(f(\lambda_\mu))^-(U)) = \begin{cases} \max\{(\lambda_\mu)^-(X) : X\in f^{-1}(U)\}, & \text{if } f^{-1}(U) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

also the pre-image $f^{-1}(\lambda_\varphi)$ of λ_φ under f is bipolar fuzzy subset of ϑ_1 defined by $(f^{-1}(\lambda_\varphi))^+(X) = \lambda_\varphi^+(f(X))$, $(f^{-1}(\lambda_\varphi))^-(X) = \lambda_\varphi^-(f(X))$.

Theorem 4.1: *Let f be a homomorphism from a HX group ϑ_1 into a HX group ϑ_2. If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a bipolar fuzzy sub HX group on ϑ_1 then $f(\lambda_\mu)$, the image of λ_μ under f, is a bipolar fuzzy sub HX group of ϑ_2.*

Proof: Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G_1, $\mu^+ : G_1 \to [0,1]$ and $\mu^- : G_1 \to [-1,0]$ are mapping, and λ_μ is a bipolar fuzzy sub HX group on ϑ_1.

Let U, V $\in \vartheta_2$, since f is homomorphism and so there exist X,Y $\in \vartheta_1$ such that f(X) = U and f(Y) = V it follows that $XY \in f^{-1}(UV)$.

Now,

$$\begin{aligned} (f(\lambda_\mu))^+(UV) &= \max\{\lambda_\mu^+(Z) : Z = XY \in f^{-1}(UV)\} \\ &\geq \max\{\lambda_\mu^+(XY) : X\in f^{-1}(U), Y\in f^{-1}(V)\} \\ &\geq \max\{\min\{\lambda_\mu^+(X), \lambda_\mu^+(Y)\} : X\in f^{-1}(U), Y\in f^{-1}(V)\} \\ &= \min\{\max\{\lambda_\mu^+(X) : X\in f^{-1}(U)\}, \max\{\lambda_\mu^+(Y) : Y\in f^{-1}(V)\}\} \\ &= \min\{(f(\lambda_\mu))^+(U), (f(\lambda_\mu))^+(V)\} \end{aligned}$$

Therefore, $(f(\lambda_\mu))^+(UV) \geq \min\{(f(\lambda_\mu))^+(U), (f(\lambda_\mu))^+(V)\}$

And,

$$\begin{aligned} (f(\lambda_\mu))^-(UV) &= \max\{\lambda_\mu^-(Z) : Z = XY \in f^{-1}(UV)\} \\ &\leq \max\{\lambda_\mu^-(XY) : X\in f^{-1}(U), Y\in f^{-1}(V)\} \\ &\leq \max\{\max\{\lambda_\mu^-(X), \lambda_\mu^-(Y)\} : X\in f^{-1}(U), Y\in f^{-1}(V)\} \\ &= \max\{\max\{\lambda_\mu^-(X) : X\in f^{-1}(U)\}, \max\{\lambda_\mu^-(Y) : Y\in f^{-1}(V)\}\} \\ &= \max\{(f(\lambda_\mu))^-(U), (f(\lambda_\mu))^-(V)\} \end{aligned}$$

Therefore, $(f(\lambda_\mu))^-(UV) \leq \max\{(f(\lambda_\mu))^-(U), (f(\lambda_\mu))^-(V)\}$

Now,
$$\begin{aligned} (f(\lambda_\mu))^+(U^{-1}) &= \max\{\lambda_\mu^+(X) : X\in f^{-1}(U^{-1})\} \\ &= \max\{\lambda_\mu^+(X^{-1}) : X^{-1}\in f^{-1}(U)\} \\ &= (f(\lambda_\mu))^+(U) \end{aligned}$$

And
$$\begin{aligned} (f(\lambda_\mu))^-(U^{-1}) &= \max\{\lambda_\mu^-(X) : X\in f^{-1}(U^{-1})\} \\ &= \max\{\lambda_\mu^-(X^{-1}) : X^{-1}\in f^{-1}(U)\} \\ &= (f(\lambda_\mu))^-(U) \end{aligned}$$

Therefore, $f(\lambda_\mu)$ is a bipolar fuzzy sub HX group of ϑ_2.

Hence, if λ_μ be a bipolar fuzzy sub HX group on ϑ_1 then f (λ_μ) is a bipolar fuzzy sub HX group of ϑ_2.

Theorem 4.2: *The homomorphic pre-image of a bipolar fuzzy sub HX group $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ of a HX group ϑ_2 is a bipolar fuzzy sub HX group of a HX group ϑ_1.*

Proof: Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy subset of G_2, $\varphi^+ : G_2 \to [0,1]$ and $\varphi^- : G_2 \to [-1,0]$ are mappings, and λ_φ be a bipolar fuzzy sub HX group on ϑ_2.

$$\begin{aligned}
\text{Now, } (f^{-1}(\lambda_\varphi))^+(XY) &= \lambda_\varphi^+(f(XY)) \\
&= \lambda_\varphi^+(f(X)\,f(Y)) \\
&\geq \min\{\lambda_\varphi^+(f(X)),\ \lambda_\varphi^+(f(Y))\} \\
&= \min\{(f^{-1}(\lambda_\varphi))^+(X),\ (f^{-1}(\lambda_\varphi))^+(Y)\}
\end{aligned}$$

Therefore, $(f^{-1}(\lambda_\varphi))^+(XY) \geq \min\{(f^{-1}(\lambda_\varphi))^+(X), (f^{-1}(\lambda_\varphi))^+(Y)\}$

$$\begin{aligned}
\text{And } (f^{-1}(\lambda_\varphi))^-(XY) &= \lambda_\varphi^-(f(XY)) \\
&= \lambda_\varphi^-(f(X)f(Y)) \\
&\leq \max\{\lambda_\varphi^-(f(X)),\ \lambda_\varphi^-(f(Y))\} \\
&= \max\{(f^{-1}(\lambda_\varphi))^-(X),\ (f^{-1}(\lambda_\varphi))^-(Y)\}
\end{aligned}$$

Therefore, $(f^{-1}(\lambda_\varphi))^-(XY) \leq \max\{(f^{-1}(\lambda_\varphi))^-(X), (f^{-1}(\lambda_\varphi))^-(Y)\}$

$$\begin{aligned}
\text{Now, } (f^{-1}(\lambda_\varphi))^+(X^{-1}) &= \lambda_\varphi^+(f(X^{-1})) \\
&= \lambda_\varphi^+(f(X)^{-1}) \\
&= \lambda_\varphi^+(f(X)) \\
&= (f^{-1}(\lambda_\varphi))^+(X) \\
\text{And } (f^{-1}(\lambda_\varphi))^-(X^{-1}) &= \lambda_\varphi^-(f(X^{-1})) \\
&= \lambda_\varphi^-(f(X)^{-1}) \\
&= \lambda_\varphi^-(f(X)) \\
&= (f^{-1}(\lambda_\varphi))^-(X)
\end{aligned}$$

Therefore, $f^{-1}(\lambda_\varphi)$ is a bipolar fuzzy sub HX group of ϑ_1.
Hence, if λ_φ be a bipolar fuzzy sub HX group on ϑ_2 then $f^{-1}(\lambda_\varphi)$ is a bipolar fuzzy sub HX group of ϑ_1.

Theorem 4.3: *Let f be an anti homomorphism from a HX group ϑ_1 into a HX group ϑ_2. If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a bipolar fuzzy sub HX group of ϑ_1 then f (λ_μ), the image of λ_μ under f, is a bipolar fuzzy sub HX group of ϑ_2.*

Proof: Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subgroup of G_1, $\mu^+ : G_1 \to [0,1]$ and $\mu^- : G_1 \to [-1,0]$ are mappings, and λ_μ is a bipolar fuzzy sub HX group of ϑ_1.

Let U, V $\in \vartheta_2$, since f is an anti homomorphism and so there exist X,Y $\in \vartheta_1$ such that f(X) = U and f(Y) = V it follows that XY $\in f^{-1}$ (VU).

Now,

$$\begin{aligned}(f(\lambda_\mu))^+(UV) &= \max\{\lambda_\mu^+(Z): Z = XY \in f^{-1}(VU)\}\\ &\geq \max\{\lambda_\mu^+(XY): X \in f^{-1}(U), Y \in f^{-1}(V)\}\\ &\geq \max\{\min\{\lambda_\mu^+(X), \lambda_\mu^+(Y)\}: X \in f^{-1}(U), Y \in f^{-1}(V)\}\\ &= \min\{\max\{\lambda_\mu^+(X): X \in f^{-1}(U)\}, \max\{\lambda_\mu^+(Y): Y \in f^{-1}(V)\}\}\\ &= \min\{(f(\lambda_\mu))^+(U), (f(\lambda_\mu))^+(V)\}\end{aligned}$$

Therefore, $(f(\lambda_\mu))^+(UV) \geq \min\{(f(\lambda_\mu))^+(U), (f(\lambda_\mu))^+(V)\}$

And

$$\begin{aligned}(f(\lambda_\mu))^-(UV) &= \max\{\lambda_\mu^-(Z): Z = XY \in f^{-1}(VU)\}\\ &\leq \max\{\lambda_\mu^-(XY): X \in f^{-1}(U), Y \in f^{-1}(V)\}\\ &\leq \max\{\max\{\lambda_\mu^-(X), \lambda_\mu^-(Y)\}: X \in f^{-1}(U), Y \in f^{-1}(V)\}\\ &= \max\{\max\{\lambda_\mu^-(X): X \in f^{-1}(U)\}, \max\{\lambda_\mu^-(Y): Y \in f^{-1}(V)\}\}\\ &= \max\{(f(\lambda_\mu))^-(U), (f(\lambda_\mu))^-(V)\}\end{aligned}$$

Therefore, $(f(\lambda_\mu))^-(UV) \leq \max\{(f(\lambda_\mu))^-(U), (f(\lambda_\mu))^-(V)\}$

$$\begin{aligned}\text{Now, } (f(\lambda_\mu))^+(U^{-1}) &= \max\{\lambda_\mu^+(X): X \in f^{-1}(U^{-1})\}\\ &= \max\{\lambda_\mu^+(X^{-1}): X^{-1} \in f^{-1}(U)\}\\ &= (f(\lambda_\mu))^+(U)\end{aligned}$$

$$\begin{aligned}\text{And } (f(\lambda_\mu))^-(U^{-1}) &= \max\{\lambda_\mu^-(X): X \in f^{-1}(U^{-1})\}\\ &= \max\{\lambda_\mu^-(X^{-1}): X^{-1} \in f^{-1}(U)\}\\ &= (f(\lambda_\mu))^-(U)\end{aligned}$$

Therefore, f (λ_μ) is a bipolar fuzzy sub HX group of ϑ_2.
Hence, if λ_μ be a bipolar fuzzy sub HX group on ϑ_1 then f (λ_μ) is a bipolar fuzzy sub HX group of ϑ_2.

Theorem 4.4: *The anti homomorphic pre-image of a bipolar fuzzy sub HX group $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ of a HX group ϑ_2 is a bipolar fuzzy sub HX group of a HX group ϑ_1.*

Proof: Let $\varphi = (\varphi^+, \varphi^-)$ be a bipolar fuzzy subgroup of G_2, $\varphi^+: G_2 \to [0,1]$ and $\varphi^-: G_2 \to [-1,0]$ are mappings and λ_φ be a bipolar fuzzy sub HX group on ϑ_2.

$$\begin{aligned}\text{Now, } (f^{-1}(\lambda_\varphi))^+(XY) &= \lambda_\varphi^+(f(XY))\\ &= \lambda_\varphi^+(f(Y)f(X))\\ &\geq \min\{\lambda_\varphi^+(f(Y)), \lambda_\varphi^+(f(X))\}\\ &= \min\{(f^{-1}(\lambda_\varphi))^+(Y), (f^{-1}(\lambda_\varphi))^+(X))\}\end{aligned}$$

$= \min\{(f^{-1}(\lambda_\varphi))^+(X), (f^{-1}(\lambda_\varphi))^+(Y))\}$

Therefore, $(f^{-1}(\lambda_\varphi))^+(XY) \geq \min\{(f^{-1}(\lambda_\varphi))^+(X), (f^{-1}(\lambda_\varphi))^+(Y))\}$

And $(f^{-1}(\lambda_\varphi))^-(XY) = \lambda_\varphi^-(f(XY))$

$= \lambda_\varphi^-(f(Y)f(X))$

$\leq \max\{\lambda_\varphi^-(f(Y)), \lambda_\varphi^-(f(X))\}$

$= \max\{(f^{-1}(\lambda_\varphi))^-(Y), (f^{-1}(\lambda_\varphi))^-(X))\}$

$= \max\{(f^{-1}(\lambda_\varphi))^-(X), (f^{-1}(\lambda_\varphi))^-(Y))\}$

Therefore, $(f^{-1}(\lambda_\varphi))^-(XY) \leq \max\{(f^{-1}(\lambda_\varphi))^-(X), (f^{-1}(\lambda_\varphi))^-(Y))\}$

Now, $(f^{-1}(\lambda_\varphi))^+(X^{-1}) = \lambda_\varphi^+(f(X^{-1}))$

$= \lambda_\varphi^+(f(X)^{-1})$

$= \lambda_\varphi^+(f(X))$

$= (f^{-1}(\lambda_\varphi))^+(X)$

And $(f^{-1}(\lambda_\varphi))^-(X^{-1}) = \lambda_\varphi^-(f(X^{-1}))$

$= \lambda_\varphi^-(f(X)^{-1})$

$= \lambda_\varphi^-(f(X))$

$= (f^{-1}(\lambda_\varphi))^-(X)$

Therefore, $f^{-1}(\lambda_\varphi)$ is a bipolar fuzzy sub HX group of ϑ_1.
Hence, if λ_φ be a bipolar fuzzy sub HX group on ϑ_2 then $f^{-1}(\lambda_\varphi)$ is a bipolar fuzzy sub HX group of ϑ_1.

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