Classes of general H-matrices *

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Abstract

Let $\mathcal{M}(A)$ denote the comparison matrix of a square H-matrix A, that is, $\mathcal{M}(A)$ is an M-matrix. H-matrices such that their comparison matrices are nonsingular are well studied in the literature. In this paper, we study characterizations of H-matrices with either singular or nonsingular comparison matrices. The spectral radius of the Jacobi matrix of $\mathcal{M}(A)$ and the generalized diagonal dominance property are used in the characterizations. Finally, a classification of the set of general H-matrices is obtained.

1 Introduction

In the literature on iterative methods of linear systems, H-matrices are widely used because they appear in many applications when discretizing certain nonlinear parabolic equations and when solving the linear complementarity problem. Furthermore, H-matrices are closely related to M-matrices [2, 19].

Matrices of this kind are currently the subject of much interest as noted in [6] and [5] and the references therein. For instance, to study the convergence of block iterative methods, the concepts of Z-matrix, M-matrix and H-matrix have been generalized to block matrices in [7, 17] while generalized H-matrices are defined in [14]. In [11] additional properties of generalized H-matrices are described. Another subject of recent attention is the determination of H-matrices. Most of the equivalent conditions given in [2] are not of practical use to know if a given matrix is an H-matrix. To test

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this condition, several iterative algorithms based on the generalized diagonal dominance of the matrix have been proposed (see [1, 4, 12, 13]), for a good discussion of this kind of algorithms we refer to the recent paper [1]. Further, direct criteria for H-matrices can be found in [9, 10].

Given the H-matrix A, if the comparison matrix $\mathcal{M}(A)$ is nonsingular, then A is nonsingular, and this fact has led many authors who consider only nonsingular M-matrices to conclude that H-matrices are always nonsingular. In fact, it is known that if $\mathcal{M}(A)$ is a nonsingular M-matrix then all equimodular matrices are nonsingular (see [16]). We show here that H-matrices can be singular. Furthermore, the converse of the above statement is not true, i.e., an H-matrix A can be invertible, while $\mathcal{M}(A)$ may be singular, as we show in Example 1 below. Moreover, if the M-matrix $\mathcal{M}(A)$ is singular, the invertible H-matrix may not satisfy all properties corresponding to the case of nonsingular $\mathcal{M}(A)$.

Different characterizations of singular and nonsingular M-matrices are given in [2]. In the case of H-matrices we shall see that the nonsingularity of the matrix A and of its comparison matrix $\mathcal{M}(A)$ may yield different types of H-matrices. Other classifications of Z-matrices, including M-matrices and inverses of Z-matrices, were given in [8] and [15].

The case of nonsingular H-matrices with nonsingular comparison matrices has been widely studied and characterized (see for instance [18, 2]). Some characterizations of H-matrices of this case are given in Table 1. However, it seems that the remaining cases have not been studied. In addition, some conclusions may be uncertain as explained in the next section.

We define three classes of H-matrices one of them with nonsingular comparison matrix and the other two with singular comparison matrix. In one class with singular comparison matrix all equimodular matrices are singular while in the other class there are both singular and nonsingular matrices. To define the classes we determine the properties that identify these three types of H-matrices. Facts related with the nullity of diagonal elements, irreducibility or generalized diagonal dominance are used to obtain necessary or sufficient conditions to conclude that a given matrix is an H-matrix, and if so, to which of three types it belongs.

The structure of the paper is as follows. In section 2, we recall concepts, results, notations we may need in the sequel. In particular, two examples will illustrate the singularity of the matrices. In section 3, we discuss the results characterizing H-matrices and study irreducible H-matrices in more detail. Finally, in section 4 we obtain three types or classes of H-matrices providing the last characterization of the singular type of H-matrices; in addition, we summarize all properties of any type of H-matrices that we have obtained.

2 Preliminaries and motivation

We recall that a square real matrix A is said to be Z-matrix if $a_{ij} \leq 0$ for all $i \neq j,\ \ i,j = 1,2,\ldots,n$.

The comparison matrix of the (complex) matrix $A \in \mathbb{C}^{n\times n}$ is defined as the Z-matrix

$$\mathcal{M}(A) = 2\,|D_A| - |A| = \begin{cases} -\left|a_{ij}\right|, & \text{if } i \neq j \\ \left|a_{ii}\right|, & \text{if } i = j \end{cases}, \quad i,j = 1,2,\ldots,n, \tag{1}$$

where D_A denotes the diagonal matrix $D_A = \mathrm{diag}(a_{ii})$. The set of equimodular matrices associated with A, denoted by $\Omega(A)$, is

$$\Omega(A) \equiv \{B \in \mathbb{C}^{n\times n} \ :\ \mathcal{M}(B) = \mathcal{M}(A)\}. \tag{2}$$

Note that both A and $\mathcal{M}(A)$ are in $\Omega(A)$.

Let us recall that a splitting $A = M - N$, where M is invertible, is called regular if $M^{-1} \geq O$ and $N \geq 0$. Associated with the splitting $A = D_A - (-E - F)$, we consider the Jacobi iteration matrix

$$J_A = -D_A^{-1}(E+F), \tag{3}$$

where E and F are the strictly lower and upper triangular parts of A, respectively.

With these notations, if $\tau = \max_i\{a_{ii}\}$, a Z-matrix A can be written as $A = \tau I - C$ where the matrix C is nonnegative. In particular, the matrix A is an M-matrix if

$$A = sI - B \quad \text{with} \quad B \geq 0 \ \text{ and } \ s \geq \rho(B), \tag{4}$$

where $\rho(B)$ denotes the spectral radius of matrix B. Recall that an M-matrix A has $a_{ii} \geq 0,\ i = 1,2,\ldots,n,\ \ s \geq \tau$ and A is invertible if and only if $s > \rho(B)$; in this case, $a_{ii} > 0,\ i = 1,2,\ldots,n$. Finally, A is said to be an H-matrix if its comparison matrix $\mathcal{M}(A)$ is an M-matrix.

Properties and characterizations of H-matrices such that their comparison matrices are nonsingular M-matrices are obtained by characterizations of nonsingular M-matrices. It is well-known that $\mathcal{M}(A)$ is an invertible M-matrix if and only if A is generalized strictly diagonally dominant (GSDD), that is, if there exists a positive diagonal matrix $D = \mathrm{diag}(d_i)$ such that AD is strictly diagonally dominant (SDD), i.e.,

$$\sum_{j\neq i} |a_{ij}|\, d_j < |a_{ii}|\, d_i, \quad i = 1,2,\ldots,n,$$

or, there exists a positive vector $d = (d_i)$ such that the above inequalities hold.

Another characterization considered is that $\mathcal{M}(A)$ is an invertible M-matrix if and only if $\rho(J_{\mathcal{M}(A)}) < 1$.

In this study we consider general H-matrices, i.e., when the comparison matrix may be singular. From points (X) and (XI) of Theorem 1 in Varga [18] (and from the original paper by Ostrowski [16]), one may deduce that if any $B \in \Omega(A)$ is singular then $\mathcal{M}(A)$ is singular, i.e., if $\mathcal{M}(A)$ is invertible then all matrices in $\Omega(A)$, including A, are invertible. However, when the matrix A is nonsingular, the nonsingularity of $\mathcal{M}(A)$ is not guaranteed as the following example shows.

Example 1. *The matrix*

$$A = \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix}$$

is nonsingular, while its comparison matrix $\mathcal{M}(A)$ is a singular M-matrix. Therefore, A is an H-matrix and it is nonsingular.

Thus, the singularity of $\mathcal{M}(A)$ does not imply the singularity of all matrices in $\Omega(A)$.

Example 1 proves somewhat confusing when working with general H-matrices without taking into account the invertibility of $\mathcal{M}(A)$. One example of this confusion appears in Theorem 7.5.14, page 185, of [2], where statement (1) assures that a nonsingular H-matrix A is such that $\mathcal{M}(A)$ satisfies any one of the 50 equivalent conditions of a nonsingular M-matrix given in Theorem 6.2.3 of [2]. However, $\mathcal{M}(A)$ in Example 1 does not satisfy conditions of said Theorem 6.2.3. In fact, in this statement the nonsingularity should be imposed on the matrix $\mathcal{M}(A)$ instead of the H-matrix A.

Moreover, if A is an H-matrix such that $\mathcal{M}(A)$ is nonsingular, then it is clearly understood that all diagonal entries of A are nonzero. However, there are H-matrices with some zero diagonal element:

Example 2. *Matrices* $A = \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ -a & b \end{bmatrix}$ *and* $C = \begin{bmatrix} 0 & -1 & 0 \\ 0 & a & 0 \\ -1 & -1 & b \end{bmatrix}$ *with $a, b > 0$, are singular H-matrices and some diagonal entries are null.*

As the three matrices of Example 2 have singular comparison matrices, it is not possible to compute their Jacobi matrices and the GSDD property is not satisfied. Additionally, these matrices are reducible and only the matrix B satisfies the GDD property, i.e., B is a generalized diagonally dominant matrix (but not strictly): there exists a positive diagonal matrix D such that BD is diagonally dominant, That is,

$$\sum_{j\neq i} |a_{ij}|\, d_j \leq |a_{ii}|\, d_i, \quad i = 1, 2, \ldots, n.$$

Specifically, $D = \mathrm{diag}(b, a)$ proves that matrix B of Example 2 is GDD.

These examples illustrate the complexity of the set of general H-matrices, which we shall study in the following section.

3 Characterization and properties of general H-matrices

Let us start by characterizing general H-matrices.

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

1. *A is an H-matrix*
2. *for each $B \in \mathbb{C}^{n\times n}$, $\mathcal{M}(B) \geq \mathcal{M}(A) \Rightarrow B$ is an H-matrix.*

Proof. $(2 \Rightarrow 1)$ It is clear taking $B = A$.
$(1 \Rightarrow 2)$ Since $\mathcal{M}(B) \geq \mathcal{M}(A)$, $|b_{ii}| \geq |a_{ii}|$ and $|b_{ij}| \leq |a_{ij}|$ for $j \neq i$ and $i, j = 1, 2, \ldots, n$. Let us write $\mathcal{M}(B) = mI - C$ with $m = \max_i |b_{ii}|$ and $C \geq 0$. Then $\mathcal{M}(A) = mI - P$ where $m \geq \max_i |a_{ii}|$ and then $P \geq 0$. Note that, $\rho(P) \leq m$ since $\mathcal{M}(A)$ is M-matrix. Since $0 \leq C \leq P$ we have $\rho(C) \leq \rho(P) \leq m$ and hence B is an H-matrix. □

The property of generalized diagonally dominance, without any strict inequality, does not characterize general H-matrices. Note that the H-matrix of Example 1 is GDD, but the H-matrices A and C of Example 2 are not. So, we first restrict our analysis to the case of nonzero diagonal elements.

Now, we recall the following well-known result.

Lemma 1. *Let A be a Z-matrix. Then A is an M-matrix, if and only if DA is an M-matrix for each positive diagonal matrix D.*

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$ be such that $a_{ii} \neq 0$, for $i = 1, 2, \ldots, n$. The following statements are equivalent:*

1. *A is an H-matrix*
2. $\rho\left(J_{\mathcal{M}(A)}\right) \leq 1$

3. for any $B \in \Omega(A)$, $\rho(J_B) \leq 1$.

Proof. Let $D_{\mathcal{M}(A)} = \operatorname{diag}(\mathcal{M}(A))$ and consider the regular splitting of $\mathcal{M}(A)$ yielding the Jacobi matrix $J_{\mathcal{M}(A)} = I - D^{-1}_{\mathcal{M}(A)}\mathcal{M}(A) \geq 0$. Then, by Lemma 1, A is an H-matrix, or equivalently $\mathcal{M}(A)$ is an M-matrix, if and only if the matrix $D^{-1}_{\mathcal{M}(A)}\mathcal{M}(A) = I - J_{\mathcal{M}(A)}$ is an M-matrix. Note that this is equivalent to $\rho\left(J_{\mathcal{M}(A)}\right) \leq 1$ by (4). This proves the equivalence of the first two statements.

Statement 2 implies 3 since for any $B \in \Omega(A)$, we have

$$\rho(J_B) \leq \rho(|J_B|) = \rho(J_{\mathcal{M}(A)}).$$

The converse (3 implies 2) follows taking $B = \mathcal{M}(A) \in \Omega(A)$. □

Now, we shall characterize the irreducible H-matrices using the GDD property. First, we prove the following result.

Theorem 3. *Let $A \in \mathbb{C}^{n\times n}$ be an irreducible H-matrix. Then $a_{ii} \neq 0$, for $i = 1, 2, \ldots, n$.*

Proof. Consider the splitting of the comparison matrix $\mathcal{M}(A) = mI - C$, $C \geq 0$, $\rho(C) = \rho \leq m$. Since C is an irreducible nonnegative matrix, there exists a positive vector u such that $Cu = \rho u$. Then, $\mathcal{M}(A)u = mu - Cu = (m - \rho)u$. Thus, for each row we have

$$|a_{ii}|\, u_i - \sum_{j\neq i} |a_{ij}|\, u_j = (m - \rho)u_i \geq 0 \tag{5}$$

and, if $a_{ii} = 0$, the corresponding row will be zero, and so, A becomes reducible. The proof follows. □

With this result, we obtain the following characterization.

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ be an irreducible matrix. Then, A is an H-matrix if and only if A is GDD, that is, there exists a positive vector d such that*

$$|a_{ii}|\, d_i \geq \sum_{j\neq i} |a_{ij}|\, d_j, \qquad i = 1, 2, \ldots, n. \tag{6}$$

Proof. Let us suppose that A is an H-matrix. Since A is irreducible, the inequality (5) holds for $i = 1, 2, \ldots, n$, and thus A is generalized diagonally dominant, for $d = u$.

Conversely, since A is GDD and irreducible then $a_{ii} \neq 0$, for all $i = 1, 2, \ldots, n$ and we can construct the Jacobi matrix $J_{\mathcal{M}(A)}$. Then the inequalities (6) may be written as

$$\sum_{j \neq i} \frac{|a_{ij}|}{|a_{ii}|} \frac{d_j}{d_i} \leq 1, \qquad i = 1, 2, \ldots, n$$

which means that the spectral radius of the nonnegative irreducible matrix $D^{-1} J_{\mathcal{M}(A)} D$ is bounded by 1, where $D = \mathrm{diag}(d_i)$. Therefore, $\rho(J_{\mathcal{M}(A)}) \leq 1$ and using Theorem 2 we deduce that A is an H-matrix. $\square$

Note that in the proof of the converse, the irreducibility is needed only to assure the nonnullity of the diagonal elements of the matrix. Thus, we can conclude the following more general statement: *if A is a GDD matrix with $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$, then A is an H-matrix.*

The case when H-matrices are reducible is studied in the following result. First, we shall recall that the normal form of a reducible matrix A is given by a block triangular matrix $PAP^T = (R_{ij})$, $i, j = 1, \ldots, p$, in which each square diagonal block R_{ii} is either irreducible or a 1×1 null matrix and P is a permutation matrix.

Theorem 5. *Let $A \in \mathbb{C}^{n \times n}$ be a reducible matrix. Then, A is an H-matrix if and only if in the normal form of A, $PAP^T = (R_{ij})$, each square diagonal block is an H-matrix.*

Proof. The only part follows from the fact that PAP^T is an H-matrix and hence all its principal submatrices.

To prove the if part, we construct the normal form of $\mathcal{M}(A)$, $P\mathcal{M}(A)P^T = (S_{ij})$ from the normal form of A, $PAP^T = (R_{ij})$. Now consider as usual the splitting $P\mathcal{M}(A)P^T = mI - C$, where $C \geq 0$. Then, we obtain the splittings of the M-matrices $S_{kk} = mI - C_{kk}$, where the identity matrix I has an adequate order, satisfying $\rho(C_{kk}) \leq m$ for $k = 1, \ldots, p$. Since $\rho(C) = \max_k \rho(C_{kk})$, then, PAP^T is an H-matrix and so is A. $\square$

Note that by Theorem 4 the irreducible diagonal blocks of the normal form of an H-matrix are GDD. We obtain the following converse result.

Theorem 6. *Let $A \in \mathbb{C}^{n \times n}$. If A is GDD, then A is an H-matrix.*

Proof. The case that A is irreducible follows from Theorem 4. The reducible case is studied for each diagonal block of the normal form, which is either GDD and irreducible and then, by the same theorem, an H-matrix, or a 1×1 null matrix which is also an H-matrix. Finally, by Theorem 5, we conclude that A is an H-matrix. $\square$

Remark: Symmetric H-matrices are characterized as GDD matrices in [3] (Theorem 8). This characterization is also deduced from our results. In the irreducible case, H-matrices are characterized as GDD in Theorem 4. In the reducible case, a symmetric H-matrix is a block diagonal matrix; then by Theorem 5, the diagonal blocks are irreducible H-matrices and GDD matrices and thus, the whole matrix. In addition, our Theorem 6 leads to the converse in this case.

4 Classification of H-matrices

With the results given in the previous section, we can observe an initial classification of the family of H-matrices. The first set contains all H-matrices such that its comparison matrix is nonsingular. These H-matrices are invertible and are characterized as GSDD matrices or as those matrices such that the spectral radius of the corresponding Jacobi matrix is less than 1, in addition to all characterizations of nonsingular M-matrices on the comparison matrix (see [2, 18]). This class will be called the "invertible class".

The second set contains H-matrices with a singular comparison matrix. In this set we observe from the aforestated two examples that in $\Omega(A)$ there are singular and maybe nonsingular matrices. Thus, we shall study this second set in order to differentiate it in two classes.

Theorem 7. *Let A be an H-matrix. Then, A has some null diagonal element if and only if B is singular for all $B \in \Omega(A)$.*

Proof. Assume null the diagonal element a_{ii} of A and let $B \in \Omega(A)$. Then, B is an H-matrix and $b_{ii} = 0$, and, by Theorem 3, B is reducible. If $Q = PBP^T$ is the normal form of B, the irreducible diagonal blocks of Q are H-matrices, by Theorem 5, and so its diagonal elements are different from zero by Theorem 3. Then, the 1×1 submatrix $(b_{ii}) = (0)$ is a diagonal block of Q. Thus, Q is singular and so is B.

Conversely, consider now that $\mathcal{M}(A)$ is singular and suppose that $a_{ii} \neq 0$ for all i. We shall construct a nonsingular matrix $B \in \Omega(A)$, in fact we shall construct by induction a matrix B with all leading principal submatrices B_k nonsingular.

Obviously, the 1×1 leading principal submatrix is nonsingular. Suppose now that a $k \times k$ nonsingular matrix B_k equimodular with the principal $k \times k$ submatrix of A has been constructed. Then consider the $(k+1) \times (k+1)$ matrix,

$$\tilde{B}_{k+1} = \begin{bmatrix} B_k & a_{k+1} \\ a^{k+1} & a_{k+1,k+1} \end{bmatrix}$$

where a_{k+1} (a^{k+1}) is the column (row) formed by the first k components of the $(k+1)$-th column (row) of A. If $\tilde{B}_{k+1}$ is nonsingular then $B_{k+1} = \tilde{B}_{k+1}$. Otherwise, construct the matrix

$$B_{k+1} = \begin{bmatrix} B_k & a_{k+1} \\ a^{k+1} & a_{k+1,k+1} - 2a_{k+1,k+1} \end{bmatrix}$$

whose determinant is

$$\det B_{k+1} = \det \tilde{B}_{k+1} + 2\det \begin{bmatrix} B_k & 0 \\ a^{k+1} & -a_{k+1,k+1} \end{bmatrix} = -2a_{k+1,k+1} \det B_k \neq 0,$$

since $\tilde{B}_{k+1}$ is singular and B_k is nonsingular joint with $a_{k+1,k+1}$ is not null. □

This result leads us to consider two separate classes of the second set: the class of H-matrices in which all matrices B in $\Omega(A)$ are singular, which we will call the "singular class", and the class of H-matrices such that $\Omega(A)$ contains singular and nonsingular matrices, which we will call the "mixed class".

The singular class of H-matrices is characterized by the existence of null diagonal entries (Theorem 7). In addition, H-matrices of this class have the following properties.

Corollary 1. *Let A be an H-matrix of the singular class. Then*
(i) A is reducible.
(ii) If A is GDD then the ith row is null whenever $a_{ii} = 0$.

Proof. (i) The proof follows from Theorem 3.
(ii) Obvious. □

Note that the H-matrices in the third class, the mixed class, all have diagonal elements different from zero, but their comparison matrices are singular. These matrices may be singular or not, as well as reducible or not. In the irreducible case, they are GDD. In the reducible case, all irreducible diagonal blocks of their normal form are GDD and the nonsingular diagonal blocks are GSDD; thus, the values of the elements in the nonzero off-diagonal blocks do not play any role in the fact that the matrix is of this class, i.e., if

$$PAP^T = (R_{ij})$$

is the normal form of the reducible H-matrix A of the mixed class, the block diagonal matrix

$$D = \operatorname{diag}(R_{ii})$$

is a GDD H-matrix, and all block triangular matrices with the same block diagonal D are H-matrices of the mixed class.

As a result, we obtain a complete classification of the set of H-matrices, denoted by $\mathcal{H}$, in the three following classes:

- *Invertible class:* $\{A \in \mathcal{H} \ : \ B \in \Omega(A) \Rightarrow B$ is nonsingular$\} = \{A \in \mathcal{H} \ : \ \mathcal{M}(A)$ is nonsingular$\}$.
- *Singular class:* $\{A \in \mathcal{H} \ : \ B \in \Omega(A) \Rightarrow B$ is singular$\}$.
- *Mixed class:* $\{A \in \mathcal{H} \ : \ \mathcal{M}(A)$ is singular and $\exists B \in \Omega(A)$ nonsingular$\}$.

We note that the matrix of Example 1 belongs to the mixed class and matrices of Example 2 are in the singular class. Finally, the properties of these three classes are summarized in Table 1.

	Invertible class	Singular class	Mixed class
$\mathcal{M}(A) = sI - C$	$\boldsymbol{\rho(C) < s}$	$\rho(C) = s$	$\rho(C) = s$
$\mathcal{M}(A)$	**Invertible**	Singular	Singular
Diag. elements	Nonzero	$\boldsymbol{\exists a_{ii} = 0}$	Nonzero
$B \in \Omega(A)$	**Invertible**	**Singular**	**Inv. or sing.**
Reducibility		Yes	
Diag. dominance	**GSDD**		Irred. $\Rightarrow$ GDD
$J = J_{\mathcal{M}(A)}$	$\boldsymbol{\rho(J) < 1}$	$\boldsymbol{J}$ **does not exist**	$\boldsymbol{\rho(J) = 1}$

Table 1: Main properties of each class of H-matrices. The bold properties determine the classes.

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