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ON EXPONENTIAL CONVEXITY FOR POWER SUMS AND RELATED RESULTS

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Abstract. In this paper, we use parameterized class of increasing functions to give exponential convexity of the non-negative difference of certain inequality as a function of parameter in connection with power sums. We define new means of Cauchy type and give its relation to the means defined in [5] and [6]. Also we give related mean value theorems of Cauchy type.

1. Introduction and Preliminaries

Bernstein [3] introduced the important sub-class of convex functions in a given interval (a,b). Akhiezer [1, page 209] denoted this sub-class by $\mathrm{W}_{a,b}$. Independently of Bernstein, but somewhat later, Widder [7] also introduced the class $W_{a,b}$ and studied it. Bernstein called functions $f \in \mathrm{W}_{a,b}$ exponentially convex.

DEFINITION 1. A function $f:(a,b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} v_i v_j f(x_i + x_j) \geqslant 0$$

for all $n \in \mathbb{N}$ and all choices $v_i \in \mathbb{R}$, $i = 1,\ldots,n$ such that $x_i + x_j \in (a,b)$, $1 \leqslant i,j \leqslant n$.

PROPOSITION 1.1. *Let $f:(a,b) \to \mathbb{R}$. The following propositions are equivalent*

(i) f is exponentially convex

(ii) f is continuous and

$$\sum_{i,j=1}^{n} v_i v_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

for every $v_i \in \mathbb{R}$ and for every $x_i \in (a,b)$, $1 \leqslant i \leqslant n$.

COROLLARY 1.2. *If $f:(a,b) \to \mathbb{R}^+$ is exponentially convex function then f is a log-convex function.*

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In [5], we defined the following function:

$$\Delta_t = \Delta_t(\mathbf{x};\mathbf{p}) = \begin{cases} \frac{1}{t-1}\left(\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n p_i x_i^t\right), & t \neq 1; \\ \sum_{i=1}^n p_i x_i \log \sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i \log x_i, & t = 1, \end{cases}$$

where $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ are positive n-tuples such that, $\sum_{i=1}^n p_i x_i \geqslant x_j$ for $j = 1,\ldots,n$.

In [2], we proved that $t \mapsto \Delta_t$ is an exponentially convex function on $\mathbb{R}$. Also in [5], we introduced the Cauchy means by considering an increasing function of the type $f(x)/x$ related to power sums, that is, the following means were defined.

DEFINITION 2. Let $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ be two positive n-tuples $(n \geqslant 2)$ such that $p_i \geqslant 1$ $(i = 1,\ldots,n)$. Then for $t,r,s \in \mathbb{R}^+$,

$$A_{t,r}^s(\mathbf{x};\mathbf{p}) = \left\{\frac{r-s}{t-s}\,\frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r}\right\}^{\frac{1}{t-r}}, \quad t \neq r, r \neq s, t \neq s.$$

$$A_{s,r}^s(\mathbf{x};\mathbf{p}) = A_{r,s}^s(\mathbf{x};\mathbf{p}) = \left\{\frac{r-s}{s}\,\frac{\left(\sum_{i=1}^n p_i x_i^s\right)\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r}\right\}^{\frac{1}{s-r}}, \ s \neq r.$$

$$A_{r,r}^s(\mathbf{x};\mathbf{p}) = \exp\left(\frac{1}{s-r} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}}\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r\right\}}\right), \ s \neq r.$$

$$A_{s,s}^s(\mathbf{x};\mathbf{p}) = \exp\left(\frac{\left(\sum_{i=1}^n p_i x_i^s\right)\left(\log\sum_{i=1}^n p_i x_i^s\right)^2 - s^2\sum_{i=1}^n p_i x_i^s\left(\log x_i\right)^2}{2s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)\log\left(\sum_{i=1}^n p_i x_i^s\right) - s\sum_{i=1}^n p_i x_i^s \log x_i\right\}}\right).$$

In [6] we introduced the Cauchy means by considering convex function, that is, the following means were defined.

DEFINITION 3. Let $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ be two positive n-tuples such that $p_i \geqslant 1$ $(i = 1,\ldots,n)$. Then for $t,r,s \in \mathbb{R}^+$,

$$B_{t,r}^s(\mathbf{x};\mathbf{p}) = \left\{\frac{r(r-s)}{t(t-s)}\,\frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{t}{s}} - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r}\right\}^{\frac{1}{t-r}}, \quad t \neq r, r \neq s, t \neq s,$$

$$B_{s,r}^s(\mathbf{x};\mathbf{p}) = B_{r,s}^s(\mathbf{x};\mathbf{p}) = \left\{\frac{r(r-s)}{s^2}\,\frac{\left(\sum_{i=1}^n p_i x_i^s\right)\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i}{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r}\right\}^{\frac{1}{s-r}}, \ s \neq r,$$

$$B_{r,r}^s(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{2r-s}{r(r-s)} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}}\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)^{\frac{r}{s}} - \sum_{i=1}^n p_i x_i^r\right\}}\right), \ s \neq r,$$

$$B_{s,s}^s(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{1}{s} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)\left(\log\sum_{i=1}^n p_i x_i^s\right)^2 - s^2\sum_{i=1}^n p_i x_i^s\left(\log x_i\right)^2}{2s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)\log\sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i\right\}}\right).$$

One can found the following relation between $A^s_{t,r}(\mathbf{x},\mathbf{p})$ and $B^s_{t,r}(\mathbf{x},\mathbf{p})$ [6].

$$B^s_{t,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{t}\right)^{\frac{1}{t-r}} A^s_{t,r}(\mathbf{x};\mathbf{p}),$$
$$B^s_{r,s}(\mathbf{x};\mathbf{p}) = B^s_{s,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{s}\right)^{\frac{1}{s-r}} A^s_{s,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{s}\right)^{\frac{1}{s-r}} A^s_{r,s}(\mathbf{x};\mathbf{p}),$$
$$B^s_{r,r}(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{1}{r}\right) A^s_{r,r}(\mathbf{x};\mathbf{p}),$$
$$B^s_{s,s}(\mathbf{x};\mathbf{p}) = \exp\left(-\frac{1}{s}\right) A^s_{s,s}(\mathbf{x};\mathbf{p}).$$

In this paper, we use the class of increasing functions to give some results related to power sums as shown in [5] and [6]; we use the following theorem [4, page 151].

THEOREM 1.3. *Let* $(x_1,...,x_n) \in I^n$, *where* I *is an interval,* $(p_1,...,p_n)$ *and* $(q_1,...,q_n)$ *be non-negative n-tuples such that*

$$\textstyle\sum_{i=1}^n p_i x_i \geqslant x_j, \; for \; j = 1,...,n \; and \; \sum_{i=1}^n p_i x_i \in I. \tag{1}$$

If $f : I \to \mathbb{R}$ *is an increasing function, then*

$$\sum_{i=1}^n q_i f\left(\sum_{i=1}^n p_i x_i\right) \geqslant \sum_{i=1}^n q_i f(x_i). \tag{2}$$

REMARK 1.4. If f is strictly increasing on I and all x_i's are not equal, then

$$\textstyle\sum_{i=1}^n p_i x_i > x_j,$$

implies

$$f\left(\textstyle\sum_{i=1}^n p_i x_i\right) > f(x_j).$$

Thus we obtain strict inequality in (2).

In this paper we use parameterized class of an increasing functions to give exponential convexity of non-negative difference of (2) as a function of parameter. We introduce means of Cauchy type and use logarithmic convexity of the difference to prove a monotonicity property of newly defined means. We also prove related mean value theorem of Cauchy type.

2. Main results

Let $t \in \mathbb{R}$ and $h_t : (0,\infty) \to \mathbb{R}$ be the function defined as

$$h_t(x) = \begin{cases} \frac{x^t}{t}, & t \neq 0; \\ \log x, & t = 0. \end{cases} \tag{3}$$

It is easy to check that h_t is strictly increasing on $(0,\infty)$ for each $t \in \mathbb{R}$.

THEOREM 2.1. *Let* $\mathbf{x} = (x_1,...,x_n)$, $\mathbf{p} = (p_1,...,p_n)$ *and* $\mathbf{q} = (q_1,...,q_n)$ *be positive n-tuples* $(n \geqslant 2)$ *such that* $\sum_{i=1}^{n} p_i x_i \geqslant x_j$ *for* $j = 1,...,n$. *Also let* $\{h_t : t \in \mathbb{R}\}$ *be the family of functions define in* (3) *and*

$$\mho_t := \mho_t(\mathbf{x};\mathbf{p};\mathbf{q}) = \sum_{i=1}^{n} q_i h_t\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i h_t(x_i). \tag{4}$$

(a) For $m \in \mathbb{N}$, *let* $r_1,...,r_m$ *be arbitrary real numbers. Then the matrix*

$$\left[\mho_{\frac{r_i+r_j}{2}}\right], \quad \textit{where} \quad 1 \leqslant i,j \leqslant m,$$

is a positive semi-definite matrix. Particularly

$$\det\left[\mho_{\frac{r_i+r_j}{2}}\right]_{i,j=1}^{k} \geqslant 0 \textit{ for all } k = 1,...,m.$$

(b) The function $t \mapsto \mho_t$, *where* $t \in \mathbb{R}$, *is an exponentially convex.*

(c) If all x_i*'s are not equal, then* $t \mapsto \mho_t$ *is log-convex function.*

Proof. (a) Define a $m \times m$ matrix $M = \left[h_{\frac{r_i+r_j}{2}}\right]$, where $i,j = 1,...,m$, and let $\mathbf{v} = (v_1,...,v_m)$ be a nonzero arbitrary vector from $\mathbb{R}^m$.

Consider the function

$$\zeta(x) = \mathbf{v}M\mathbf{v}^{\tau} = \sum_{i,j=1}^{m} v_i v_j h_{\frac{r_i+r_j}{2}}(x).$$

Now we have

$$\zeta'(x) = \sum_{i,j=1}^{m} v_i v_j x^{\frac{r_i+r_j}{2}-1} = \left(\sum_{i=1}^{m} v_i x^{\frac{r_i-1}{2}}\right)^2 \geqslant 0 \text{ for all } x \in \mathbb{R}^+,$$

concluding ζ is an increasing on $\mathbb{R}^+$. Now by Theorem 1.3 with $f = \zeta$, we have

$$\sum_{k=1}^{n} q_k \zeta\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} q_k \zeta(x_k) \geqslant 0,$$

this implies

$$\sum_{i,j=1}^{m} v_i v_j \left(\sum_{k=1}^{n} q_k h_{\frac{r_i+r_j}{2}}\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} q_k h_{\frac{r_i+r_j}{2}}(x_k)\right) \geqslant 0,$$

and finally we have

$$\sum_{i,j=1}^{m} v_i v_j \mho_{\frac{r_i+r_j}{2}} \geqslant 0.$$

Therefore the given matrix is positive semi-definite.

Specially, we get

$$\begin{vmatrix} \mho_{r_1} & \cdots & \mho_{\frac{r_1+r_k}{2}} \\ \vdots & \ddots & \vdots \\ \mho_{\frac{r_k+r_1}{2}} & \cdots & \mho_{r_k} \end{vmatrix} \geqslant 0 \tag{5}$$

for all $k = 1, ..., m$.

(b) Since $\lim_{t\to 0} \mho_t = \mho_0$, it follows that $t \mapsto \mho_t$ is continuous on $\mathbb{R}$. Now using Proposition 1.1 we have exponential convexity of the function $t \mapsto \mho_t$.

(c) Since all x_i's are not equal and $x \mapsto h_t(x)$ is strictly increasing for any $t \in \mathbb{R}$ therefore from Remark 1.4 we have $\mho_t > 0$. Now logarithmic convexity of $t \mapsto \mho_t$ is follows from the Corollary 1.2.

Let us introduce the following:

DEFINITION 4. Let $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{p} = (p_1, ..., p_n)$ and $\mathbf{q} = (q_1, ..., q_n)$ be positive n-tuples $(n \geqslant 2)$ such that $\sum_{i=1}^n p_i x_i \geqslant x_j$ for $j = 1, ..., n$. Then for $t, r, \in \mathbb{R}$, we define

$$H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \left(\frac{r}{t} \frac{\sum_{i=1}^n q_i \left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n q_i x_i^t}{\sum_{i=1}^n q_i \left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r} \right)^{\frac{1}{t-r}}, \ r \neq t, \ r,t \neq 0.$$

$$H_{r,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \exp\left(-\frac{1}{r} + \frac{\sum_{i=1}^n q_i \left(\sum_{i=1}^n p_i x_i\right)^r \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i x_i^r \log x_i}{\sum_{i=1}^n q_i \left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r} \right), \ r \neq 0.$$

$$H_{r,0}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{0,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \left(\frac{\sum_{i=1}^n q_i \left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r}{r\left\{\sum_{i=1}^n q_i \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i \log x_i\right\}} \right)^{\frac{1}{r}}, \ r \neq 0.$$

$$H_{0,0}(\mathbf{x};\mathbf{p};\mathbf{q}) = \exp\left(\frac{\sum_{i=1}^n q_i \left\{\log\left(\sum_{i=1}^n p_i x_i\right)\right\}^2 - \sum_{i=1}^n q_i \left(\log x_i\right)^2}{2\left\{\sum_{i=1}^n q_i \log\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i \log x_i\right\}} \right).$$

REMARK 2.2. Note that $\lim_{t\to r} H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{r,r}(\mathbf{x};\mathbf{p};\mathbf{q})$, $\lim_{t\to 0} H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = \lim_{t\to 0} H_{r,t}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{0,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{r,0}(\mathbf{x};\mathbf{p};\mathbf{q})$ and $\lim_{r\to 0} H_{r,r}(\mathbf{x};\mathbf{p};\mathbf{q}) = H_{0,0}(\mathbf{x};\mathbf{p};\mathbf{q})$.

We shall use a following lemma [5] to prove the monotonicity of the means defined above.

LEMMA 2.3. *Let f be a log-convex function and assume that if $x_1 \leqslant y_1, x_2 \leqslant y_2, x_1 \neq x_2, y_1 \neq y_2$. Then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \leqslant \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{6}$$

THEOREM 2.4. *Let $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{p} = (p_1, ..., p_n)$ and $\mathbf{q} = (q_1, ..., q_n)$ be positive n-tuples $(n \geqslant 2)$ such that $\sum_{i=1}^n p_i x_i \geqslant x_j$ for $j = 1, ..., n$. Also let $r, t, u, v \in \mathbb{R}$ such that $r \leqslant u$, $t \leqslant v$. Then we have*

$$H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q}) \leqslant H_{v,u}(\mathbf{x};\mathbf{p};\mathbf{q}). \tag{7}$$

Proof. Let $\mho_t$ be defined by (4). Taking $x_1 = r,\ x_2 = t,\ y_1 = u,\ y_2 = v$, where $r \neq t,\ u \neq v$, and $f(t) = \mho_t$ in Lemma 2.3, we have

$$\left(\frac{r}{t}\frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n q_i x_i^t}{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n q_i x_i^r}\right)^{\frac{1}{t-r}} \leqslant \left(\frac{u}{v}\frac{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^v - \sum_{i=1}^n q_i x_i^v}{\sum_{i=1}^n q_i\left(\sum_{i=1}^n p_i x_i\right)^u - \sum_{i=1}^n q_i x_i^u}\right)^{\frac{1}{v-u}}.$$

This is equivalent to (7) for $t \neq r,\ u \neq v$. From Remark 2.2, we get (7) is also valid for $t = r,\ u = v$.

REMARK 2.5. If we put $r \to r-1,\ t \to t-1$ and $q_i \to p_i x_i$ in $H_{t,r}(\mathbf{x};\mathbf{p};\mathbf{q})$, we have

$$\widetilde{H}_{t,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r-1}{t-1}\frac{\left(\sum_{i=1}^n p_i x_i\right)^t - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n p_i x_i^r}\right)^{\frac{1}{t-r}},\ r \neq t,\ r,t \neq 1.$$

$$\widetilde{H}_{r,r}(\mathbf{x};\mathbf{p}) = \exp\left(\frac{1}{1-r} + \frac{\left(\sum_{i=1}^n p_i x_i\right)^r \log \sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i^r \log x_i}{\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n p_i x_i^r}\right),\ r \neq 1.$$

$$\begin{aligned}\widetilde{H}_{r,0}(\mathbf{x};\mathbf{p}) &= \widetilde{H}_{0,r}(\mathbf{x};\mathbf{p}) \\ &= \left(\frac{\left(\sum_{i=1}^n p_i x_i\right)^r - \sum_{i=1}^n p_i x_i^r}{(r-1)\left\{\left(\sum_{i=1}^n p_i x_i\right)\log \sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i \log x_i\right\}}\right)^{\frac{1}{r-1}},\ r \neq 1.\end{aligned}$$

$$\widetilde{H}_{0,0}(\mathbf{x};\mathbf{p}) = \exp\left(\frac{\sum_{i=1}^n p_i x_i\left(\log \sum_{i=1}^n p_i x_i\right)^2 - \sum_{i=1}^n p_i x_i (\log x_i)^2}{2\left\{\left(\sum_{i=1}^n p_i x_i\right)\log \sum_{i=1}^n p_i x_i - \sum_{i=1}^n p_i x_i \log x_i\right\}}\right).$$

Now if $x_i \to x_i^s,\ r \to \frac{r}{s}$ and $t \to \frac{t}{s}$ where $r,t \neq s$ and $s \neq 0$, we have

$$\begin{aligned}\widetilde{H}_{\frac{t}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p}) &= \left(A^s_{t,r}(\mathbf{x};\mathbf{p})\right)^s, \\ \widetilde{H}_{\frac{r}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p}) &= \left(A^s_{r,r}(\mathbf{x};\mathbf{p})\right)^s, \\ \widetilde{H}_{\frac{r}{s},0}(\mathbf{x}^s;\mathbf{p}) &= \widetilde{H}_{0,\frac{r}{s}}(\mathbf{x}^s;\mathbf{p}) = \left(A^s_{r,s}(\mathbf{x};\mathbf{p})\right)^s = \left(A^s_{s,r}(\mathbf{x};\mathbf{p})\right)^s, \\ \widetilde{H}_{0,0}(\mathbf{x}^s;\mathbf{p}) &= \left(A^s_{s,s}(\mathbf{x};\mathbf{p})\right)^s.\end{aligned}$$

Also note that

$$\begin{aligned}B^s_{t,r}(\mathbf{x};\mathbf{p}) &= \left(\frac{r}{t}\right)^{\frac{1}{t-r}}\left(\widetilde{H}_{\frac{t}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}, \\ B^s_{r,s}(\mathbf{x};\mathbf{p}) &= B^s_{s,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{s}\right)^{\frac{1}{s-r}}\left(\widetilde{H}_{0,\frac{r}{s}}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}} = \left(\frac{r}{s}\right)^{\frac{1}{s-r}}\left(\widetilde{H}_{\frac{r}{s},0}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}, \\ B^s_{r,r}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{r}\right)\left(\widetilde{H}_{\frac{r}{s},\frac{r}{s}}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}, \\ B^s_{s,s}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{s}\right)\left(\widetilde{H}_{0,0}(\mathbf{x}^s;\mathbf{p})\right)^{\frac{1}{s}}.\end{aligned}$$

The following result has been proved in [5].

COROLLARY 2.6. *Let* $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{p} = (p_1, ..., p_n)$ *be positive n-tuples* $(n \geqslant 2)$ *such that* $\sum_{i=1}^{n} p_i x_i \geqslant x_j$ *for* $j = 1, ..., n$*. Also let* $t, r, u, v \in \mathbb{R}^+$ *such that* $r \leqslant u$*,* $t \leqslant v$*. Then we have*

$$A^s_{t,r}(\mathbf{x};\mathbf{p}) \leqslant A^s_{v,u}(\mathbf{x};\mathbf{p}). \tag{8}$$

Proof. Taking $r \to r-1$, $t \to t-1$, $u \to u-1$, $v \to v-1$ and $q_i \to p_i x_i$ in (7), we have

$$\widetilde{H}_{t,r}(\mathbf{x};\mathbf{p}) \leqslant \widetilde{H}_{v,u}(\mathbf{x};\mathbf{p}).$$

Now taking $x_i \to x_i^s$, $r \to \frac{r}{s}$, $t \to \frac{t}{s}$, $u \to \frac{u}{s}$, $v \to \frac{v}{s}$ where $r,t,u,v \neq s$ and $s \neq 0$, we have

$$\left(A^s_{t,r}(\mathbf{x};\mathbf{p})\right)^s \leqslant \left(A^s_{v,u}(\mathbf{x};\mathbf{p})\right)^s.$$

This follows (8).

REMARK 2.7. Similarly, we can prove the monotonicity of $B^s_{t,r}(\mathbf{x};\mathbf{p})$ which we have given in [6], that is, for $t, r, u, v \in \mathbb{R}^+$ such that $r \leqslant u$, $t \leqslant v$, we have

$$B^s_{t,r}(\mathbf{x};\mathbf{p}) \leqslant B^s_{v,u}(\mathbf{x};\mathbf{p}). \tag{9}$$

In fact we have shown in [6] that such results can be obtained from the results given in [5].

3. Mean value theorems

In this section, we prove mean value theorems of Cauchy type by using Theorem 1.3 with the help of functions defined in a following lemma.

LEMMA 3.1. *Let* $f \in C^1(I)$*, such that*

$$m \leqslant f'(x) \leqslant M, \; x \in I. \tag{10}$$

Consider the functions ϕ_1*,* ϕ_2 *defined as,*

$$\phi_1(x) = Mx - f(x)$$

$$\phi_2(x) = f(x) - mx.$$

Then ϕ_i *for* $i = 1, 2$ *are monotonically increasing.*

Proof. We have that

$$\phi_1'(x) = M - f'(x) \geqslant 0,$$

$$\phi_2'(x) = f'(x) - m \geqslant 0.$$

i.e. ϕ_i for $i = 1, 2$ are monotonically increasing.

THEOREM 3.2. *Let $(x_1,...,x_n) \in I^n$, where I is a compact interval, $(p_1,...,p_n)$ and $(q_1,...,q_n)$ be non-negative n-tuples such that all x_i's are not equal and condition* (1) *is satisfied. If $f \in C^1(I)$, then there exists $\xi \in I$ such that*

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) = f'(\xi) \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right). \tag{11}$$

Proof. Since I is compact and $f \in C^1(I)$, therefore let $m = \min f'$ and $M = \max f'$.
In Theorem 1.3, setting $f = \phi_1$ and $f = \phi_2$ respectively as defined in Lemma 3.1, we get the following inequalities

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) \leqslant M \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right), \tag{12}$$

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i) \geqslant m \sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right). \tag{13}$$

Taking $f(x) = x$ in Theorem 1.3 with all x_i's are not equal, we get

$$\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right) > 0,$$

therefore combining (12) and (13), we have

$$m \leqslant \frac{\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i)}{\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right)} \leqslant M. \tag{14}$$

Hence, there exists $\xi \in I$ such that

$$\frac{\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i)}{\sum_{j=1}^{n} q_j \left(\sum_{i=1}^{n} p_i x_i - x_j\right)} = f'(\xi).$$

Which implies (11).
From above Theorem we can deduce the results which we have proved in [5].

COROLLARY 3.3. *Let $(x_1,...,x_n) \in I^n$, where $I \subseteq (0,\infty)$ is a compact interval, $(p_1,...,p_n)$ be non-negative n-tuple such that all x_i's are not equal and condition* (1) *is satisfied. If $f \in C^1(I)$, then there exists $\xi \in I$ such that*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) = \frac{\xi f'(\xi) - f(\xi)}{\xi^2} \left\{ \left(\sum_{i=1}^{n} p_i x_i\right)^2 - \sum_{i=1}^{n} p_i x_i^2 \right\}. \tag{15}$$

Proof. Taking $q_i \to p_i x_i$, $f(x) \to f(x)/x$ in (11), we get (15).

THEOREM 3.4. *Let* $(x_1,...,x_n) \in I^n$*, where* I *is a compact interval,* $(p_1,...,p_n)$ *and* $(q_1,...,q_n)$ *be non-negative n-tuples such that all* x_i*'s are not equal and condition* (1) *is satisfied. If* $f_1, f_2 \in C^1(I)$*, then there exists* $\xi \in I$ *such that*

$$\frac{\sum_{i=1}^n q_i f_1\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i f_1(x_i)}{\sum_{i=1}^n q_i f_2\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i f_2(x_i)} = \frac{f_1'(\xi)}{f_2'(\xi)}. \tag{16}$$

Provided that the denominators are non-zero.

Proof. Let a function $k \in C^1(I)$ be defined as

$$k = c_1 f_1 - c_2 f_2,$$

where c_1 and c_2 are defined as

$$c_1 = \sum_{i=1}^n q_i f_2\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i f_2(x_i),$$

$$c_2 = \sum_{i=1}^n q_i f_1\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n q_i f_1(x_i).$$

Then, using Theorem 3.2 with $f = k$, we have

$$0 = \left(c_1 f_1'(\xi) - c_2 f_2'(\xi)\right) \sum_{j=1}^n q_j \left(\sum_{i=1}^n p_i x_i - x_j\right). \tag{17}$$

$\sum_{j=1}^n q_j\left(\sum_{i=1}^n p_i x_i - x_j\right)$ is non-zero, so we have

$$\frac{c_2}{c_1} = \frac{f_1'(\xi)}{f_2'(\xi)}.$$

After putting the values of c_1 and c_2, we get (16).

COROLLARY 3.5. [5] *Let* $(x_1,...,x_n) \in I^n$*, where* $I \subseteq (0,\infty)$ *is a compact interval,* $(p_1,...,p_n)$ *be non-negative n-tuple such that all* x_i*'s are not equal and condition* (1) *is satisfied. If* $f_1, f_2 \in C^1(I)$*, then there exists* $\xi \in I$ *such that*

$$\frac{f_1\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i f_1(x_i)}{f_2\left(\sum_{i=1}^n p_i x_i\right) - \sum_{i=1}^n p_i f_2(x_i)} = \frac{\xi f_1'(\xi) - f_1(\xi)}{\xi f_2'(\xi) - f_2(\xi)}. \tag{18}$$

Provided that the denominators are non-zero.

Proof. Taking $q_i \to p_i x_i$, $f(x) \to f(x)/x$ in (16), we get (18).

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