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# Brane Solutions in Supergravity


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## Summary

This project report covers an MSci investigation into the basics of supergravity. We start by looking at the original Kaluza-Klein theory. Supergravity is then introduced and a truncated toy system is constructed. Equations of motion for this toy system are derived and the simplifying assumptions required to obtain a solution are presented.

This basic framework of supergravity is applied to the sample problem of the braneprobe. The braneprobe is a system of two super-membranes (branes) where a light test brane (the braneprobe) orbits a stationary heavy brane. The action integral for this system is varied to find equations of motion, which are then solved to determine the nature of the braneprobe's orbit.

Supergravity is introduced without the more complex aspects of the mathematics of membranes and super-membranes. Although some results from supersymmetry are used, a thorough understanding of this topic is not required for the development in this project.

# 1. Introduction

The problem of unification in physics is to find a single framework to describe all phenomena observed in the universe. The first significant progress towards this goal was Maxwell's unification of the electric and magnetic forces into electromagnetism. This laid the foundations for Einstein's development of special relativity in 1905 [1] which redefined electricity and magnetism to be two aspects of the same force.

In 1914, the Finnish physicist Gunnar Nordstrøm attempted to take this one step further by unifying electromagnetism with his own theory of gravity. He postulated a fifth dimension of spacetime, in which the two forces were coupled. This was also the first time that the new mathematical tool of differential geometry was applied to physics. The apparent success of this theory made it extremely popular at the time. In 1915, Einstein applied differential geometry to derive a new theory of spacetime and gravity, General Relativity [2].

For some time, Nordstrøm's theory was considered a viable competitor to Einstein's General Relativity (GR). However, it was displaced after GR's accurate predictions concerning the orbit of Mercury. With the success of GR, Nordstrøm's unification of electromagnetism with gravity was largely forgotten, but the idea of postulating higher dimensions of spacetime to encompass extra fields was revived in 1921 when Theodor Kaluza published [3]. This later became Kaluza-Klein theory.

Kaluza showed that by adding an extra spatial dimension to GR a theory could be constructed with a four dimensional metric tensor, a vector gauge potential and an extra scalar. Restricting these fields to depend only on the four normal dimensions of spacetime, he found that the metric would obey a standard Einstein action and the gauge potential would obey a Maxwell action. Therefore, by postulating this extra spatial dimension, electromagnetism was obtained as a consequence of gravity.

Kaluza-Klein theory will be explored in some detail in §2. Starting with the Einstein-Hilbert action in five dimensions, a new action integral will be formed in four dimensions. From this reduced action integral, the equations of motion will be derived.

These developments were taken further by Oskar Klein in 1926 [4] who added physical substance to the mathematical structure. Klein suggested that the extra dimensions should be physically real rather than just mathematical constructs, and postulated that Kaluza's extra spatial dimension should by curled up into a tiny circle. Hence the quantisation of electric charge was obtained as a by-product of the circle's topology. In order to get the correct value for the electric charge, the radius of the circle would have to be very small— of the order of the Plank length, which conveniently explained why this extra circular dimension had not been observed.

Kaluza-Klein theory does however have significant problems. Imposing a fixed topology on part of spacetime violates the principle of background independence enshrined in General Relativity. In GR, masses and energies change the curvature of spacetime, and as the quanta of electric charge are determined by the radius of the circle, gravity would be able to change this charge. This is not observed in the real world.

Another problem with Kaluza-Klein theory is the extra field produced by the dimensional reduction. Alongside the expected metric tensor and the gauge potential, there is an extra scalar that does not couple neatly to any observed phenomena. This scalar is investigated in §2 and its equations of motion are found. It is not possible to delete this scalar by setting it to zero as it interacts with the electromagnetic gauge potential.

Whilst Kaluza-Klein theory is ultimately a failure as a physical description of the universe, the idea of adding new dimensions of spacetime to accommodate extra fields is still being developed. Throughout the twentieth century there were many attempts to unify electromagnetism, gravity and later the strong and weak nuclear forces in this kind of geometric theory. In spite of some success, all these attempts encountered similar problems to the original Kaluza-Klein theory. They either included extra fields that could not be related to observed phenomena, or predicted unphysical interactions between the four forces.

The preferred solution of mainstream physics has been to ignore the gravitational force as it is indeed rather weak compared to the other three forces. This approach has led to the Standard Model, which unifies the strong, weak and electromagnetic forces. The Standard Model (SM) is one of the great achievements of twentieth century physics, but as it does not include gravity or GR it cannot be a complete solution. However, trying to incorporate GR into the SM leads to inconsistencies as the theory's predictions rapidly diverge towards infinity.

A possible solution to this problem was supersymmetry. Supersymmetry was independently proposed by several different physicists, but is most commonly attributed to Wess and Zumino in 1974 [5]. Supersymmetry is an abstract symmetry that relates the two different classes of elementary particles. For each known fermion a partner boson was proposed and for each known boson a partner fermion was postulated. None of these supersymmetric partners has ever been observed, so it must be assumed that these new particles must be much heavier than their known counterparts.

Despite the lack of evidence for supersymmetry, it was hoped that stringent requirements of the symmetry would force these diverging infinities to cancel. The first successful theory to incorporate supersymmetry into a theory of gravity was supergravity. Supergravity is a classical theory born in the late 1970s [6], initially as a four dimensional theory, but it was quickly generalised to a set of many different theories with different numbers of dimensions. In 1978, Cremmer, Julia and Scherk [7] found the explicit action of the eleven dimensional, maximally symmetric theory. This action will be explored in §3.2.

Supergravity was initially usurped in 1984 by what has become known as the first superstring revolution. The innovation of superstring theory was to describe particles as small one-dimensional strings rather than simple point particles [8].

The idea of modeling particles as extended objects had been first proposed by Dirac in 1962 [9]. Dirac was unsuccessful in his aim of modeling a muon as an excited state of an electron, but despite mathematical complexities, extended objects still have some advantages over point particles. As point particles have no size, they can get arbitrarily close to each other, which leads to infinite forces of interactions. Extended objects neatly avoid this problem.

String theory has now grown to include extended objects of any dimension. These extended objects are collectively known as membranes or more simply branes. In the terminology of branes, a zero-brane is identical to a point particle, a one-brane is the same as a string and so on up to the $p$ dimensional $p$-brane. Modern supergravity has developed from a theory only including point particles to include branes of any dimension. The requirements of the supersymmetric membranes or superbranes mean that the eleven dimensional, maximally symmetric supergravity theory is prefered over its lower dimensional variants.

Modern string theory is split into five distinct theories. The second superstring revolution in 1995 was the beginning of a new understanding that these five theories are just different facets of a new underlying theory known as M-Theory [10]. M-Theory is written in 10 dimensions, which would seem to be in conflict the 11 dimensional supergravity.

However, in 1987, a deep link was revealed between these two classes of theory, showing that by simultaneously reducing the dimensions of both the spacetime and the worldvolume, 10 dimensional superstrings can be derived from 11 dimensional superbranes [11]. In particular, this seems to suggest that maximally supersymmetric supergravity is the low energy limit to M-Theory generally and particularly to type-IIA string theory [12].

Supergravity in eleven dimensions is outlined in §3 and the dimensional reduction linking supergravity with string theory is explored in §3.4.1. With these elements, a toy supergravity system is constructed in §3.4, and by making appropriate simplifying assumptions in §3.5, the action integral can be varied to find equations of motion in §3.6.

Supergravity, as a classical theory, is not applicable to the high energies of modern particle physics. It is more suited to cosmological problems concerning the interactions of very large objects such as black holes. In §4, such a situation will be investigated. As supergravity is the only part of M-Theory with an action that can be written explicitly, this kind of theoretical setup is currently the only viable way to extract a testable prediction from the grand unified theory.

In §4, we investigate the sample problem of a braneprobe. This is a situation with two branes, where one static, stationary heavy brane provides a background in which a light test brane orbits. A simplistic classical analogy to this setup would be an electron orbiting a black hole.

In §4.4, a Lagrangian formalism is used to find an equation goverening the radial motion of the test brane. Then in §4.5, the properties of the test brane's orbit are revealed. We will see that the brane can form unbounded orbits for larger energies and bounded orbits for smaller energies.

Some of the longer calculations are presented in Appendix A. As some of the algebraic manipulation is rather tortuous it was reassuring to check the accuracy of the equations with the symbolic manipulation package Cadabra - see Appendix B.

# 2. The Five Dimensional World

## 2.1. Kaluza-Klein Theory

In 1921, Theodore Kaluza suggested a novel approach to the unification of electromagnetism and general relativity [3]. This idea, which was extended by Oskar Klein, was to write the equations of general relativity in five dimensions rather than four, and by compactifying the extra dimension, return to a realistic model of the universe. This idea was surprisingly successful, as Einstein's equations in five dimensions yield not only the correct equations for gravity in four dimensions, but also Maxwell's equations of electromagnetism [13].

There is a remarkable elegance to this theory, as the conservation of electric charge can be viewed as the conservation of momentum in the hidden fifth dimension. Also, by requiring the fifth dimension to be of the group $\mathcal{U}(1)$, i.e. having the same topology as a circle, the quantization of electric charge is an emergent product of the formulation. To retrieve the correct value for the electric charge, we must take the size of this compactified circular dimension to be extremely small (of the order of the Planck length $\sim 10^{-35}$m) which explains why the extra dimension is not observed.

## 2.2. The Action and the Metric

We can now demonstrate Kaluza-Klein theory following Duff [13], Pope [14] and Carroll [15]. A more in-depth discussion of Kaluza-Klein in the context of branes and supergravity is found in Stelle [16, §6]. Start by writing the standard action of general relativity (the Einstein Hilbert action [17, 18]) in $(D+1)$ dimensions:

$$\mathcal{I} = \int \mathrm{d}^{D+1}x \sqrt{-\hat{g}}\, \hat{R} \tag{2.1}$$

where this action contains $\hat{g}$, which denotes the determinant of the metric, along with $\hat{R}$, the Ricci or curvature scalar. The ansatz of $D$ normal dimensions and an extra curled up dimension is imposed by splitting the $(D+1)$ dimensional metric $\hat{g}_{MN}$ into:

$$\hat{g}_{MN} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{\mu z} & \hat{g}_{zz} \end{pmatrix} \qquad \mu, \nu = 0 \ldots (D-1), \quad z = D \tag{2.2}$$

where in the $D$ dimensional world, $\hat{g}_{\mu\nu}$ is the ordinary metric tensor, $\hat{g}_{\mu z}$ is a gauge potential and $\hat{g}_{zz}$ is an extra scalar. Furthermore, these new fields are independent of the curled up, extra dimension and only depend on the $D$ flat dimensions. The $(D+1)$ dimensional quantities are given hats and the $D$ dimensional quantities are written without hats. A note on the choice of indices is given in Box 2.1.

For later convenience, the $(D+1)$ dimensional line element is written as follows:

$$\mathrm{d}\hat{s}^2 = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}\left(\mathrm{d}z + \mathcal{A}_\mu \mathrm{d}x^\mu\right)^2 \tag{2.3}$$

where $\mathcal{A}_\mu$ is chosen as the gauge potential and $\phi$ the scalar field. Also $\alpha$ and $\beta$ are constants to be specified later. This writing of the line element results in the components of the metric:

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu \mathcal{A}_\nu, \quad \hat{g}_{\mu z} = e^{2\beta\phi}\mathcal{A}_\mu, \quad \hat{g}_{zz} = e^{2\beta\phi} \tag{2.4}$$

Throughout this project, we will be writing equations with indices that cover different ranges and occupy different spaces.

In this section the following conventions are observed:

- Capital Latin indices $M$ describe the $(D+1)$ dimensional world volume
- Greek indices $\mu$ describe the $D$ dimensional $\mathcal{M}^D$ subspace
- The single $z$ index denotes the $\mathcal{S}$ circular subspace
- Lower case Latin indices are used for the tangent space

This convention will be revisited in §3, Box 3.4 when the situation becomes even more complicated and requires further choice of indices.

**Box 2.1:** Indices

### 2.2.1. Dimensional Reduction

In order to carry out dimensional reduction, the (hatted) $(D+1)$ dimensional quantities need to be written in terms of $D$ dimensional quantities. The action integral in (2.1) is split into two terms. The root of the metric is easily dealt with by inserting the components given in (2.4):

$$\begin{aligned}\sqrt{-\hat{g}} &= \sqrt{-\det\left(\hat{g}_{\mu\nu}\hat{g}_{zz} - \hat{g}_{\mu z}\hat{g}_{z\nu}\right)} \\ &= \sqrt{-\det\left(\left[e^{2\alpha\phi}g_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu\right]e^{2\beta\phi} - e^{2\beta\phi}\mathcal{A}_\mu e^{2\beta\phi}\mathcal{A}_\nu\right)} \\ &= \sqrt{-\det\left(e^{2\alpha\phi+2\beta\phi}g_{\mu\nu}\right)} \\ &= e^{2\alpha\phi}\sqrt{-g} \qquad (2.5)\end{aligned}$$

where the determinants of the metrics are written simply without indices as $\hat{g}$ and $g$. A convenient choice linking the constants $\alpha$ and $\beta$ has been made:

$$\beta = -\left(D-2\right)\alpha \qquad (2.6)$$

The other term in the action integral (2.1), the curvature scalar $\hat{R}$ is more complicated. The method originally used by Kaluza in 1921 [3] was to expand the curvature in terms of Christoffel Symbols and expand these in terms of the metric. Then by inserting the metric ansatz, the desired result is retrieved.

$$\hat{g}_{\mu\nu} \quad \rightarrow \quad \Gamma^\lambda_{\mu\nu} \quad \rightarrow \quad \hat{R}$$

This is a difficult calculation to do efficiently, and here the calculations have been done with the help of a computer algebra system called Cadabra [19, 20]. The working is detailed in Appendix B.2.

A more recent approach to the problem is to use the language of differential forms. Forms are introduced in Box 2.2 and allow us to use the metric to define *vielbeins* (see Box 2.5). These vielbeins can be used to define the spin connection $\omega$, which can then be used to calculate $\hat{R}$.

$$\hat{g}_{\mu\nu} \quad \rightarrow \quad e^a_\mu \quad \rightarrow \quad \hat{\omega}^{ab} \quad \rightarrow \quad \hat{R}$$

This second method is detailed in §2.2.2, §2.2.3 and §2.2.4 and will also be useful preparation for applying the same method to a supergravity action in §3.

Throughout this report, we will be using the language of components alongside the more esoteric language of forms. A general $p$-form can be written:

$$F_{[p]} = \frac{1}{p!} F_{\mu_1\mu_2\ldots\mu_n} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}$$

Where the $\mathrm{d}x$ terms are elements of integration. These are explained further in Box 2.3. Due to the use of the wedge product, $\wedge$, forms are always antisymmetric. The wedge product is explained further in Box 2.4. To illustrate the use of forms, consider the following product:

$$C_{[3]} = A_{[1]} \wedge B_{[2]}$$

where,

$$A_{[1]} = A_{\mu_1} \mathrm{d}x^{\mu_1}$$
$$B_{[2]} = \frac{1}{2!} B_{\nu_1\nu_2} \mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\nu_2}$$

$\therefore$

$$C_{[3]} = A_{[1]} \wedge B_{[2]} = A_{mu_1} \frac{B_{\nu_1\nu_2}}{2!} \mathrm{d}x^{mu_1} \wedge \mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\nu_2}$$

as

$$C_{[3]} = \frac{1}{3!} C_{\mu_1\mu_2\mu_3} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^{\mu_3}$$

$\Rightarrow$

$$\begin{aligned} C_{\mu_1\mu_2\mu_3} &= \frac{3!}{2!} A_{[\mu_1} B_{\mu_2\mu_3]} \\ &= \frac{2}{2!} \left( A_{\mu_1} B_{\mu_2\mu_3} + A_{\mu_2} B_{\mu_3\mu_1} + A_{\mu_3} B_{\mu_1\mu_2} \right) \end{aligned}$$

This can lead to some complicated combinatorial factors when moving between form and component language.

**Box 2.2:** Differential Forms

The integration elements $\mathrm{d}^n x$ can be written in terms of one-forms as

$$\sqrt{|g|}\mathrm{d}^n x = \frac{1}{n!} \varepsilon_{\mu_1\ldots\mu_n} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}$$

Since a differential form includes the basic one-forms $\mathrm{d}x^\mu$ as integration elements, one can place them directly under an integral sign. In terms of components this becomes:

$$\int F_{[n]} = \frac{1}{n!} \int F_{\mu_1\ldots\mu_n} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_n}$$

**Box 2.3:** Forms and Integration Elements

The wedge product is defined as an extension of the tensor product to a higher dimensional analogy of the anti-symmetric cross product:

$$A \wedge B = (A \otimes B) - (B \otimes A)$$

It is linear, associative and anti–commutative, thus for general forms $A$, $B$ and $C$ and scalars $\alpha$ and $\gamma$:

$$\begin{aligned} A \wedge (\alpha B + \gamma C) &= \alpha\,(A \wedge B) + \gamma\,(A \wedge C) \\ A \wedge (B \wedge C) &= (A \wedge B) \wedge C \\ A \wedge B &= -\,(B \wedge A) \end{aligned}$$

**Box 2.4:** Wedge Products: $\wedge$

The vielbein or tetrad formalism allows us to write equations that do not refer explicitly to any particular coordinate system. To do this, we define an orthonormal set of vectors ${e_\mu}^a$ (the vielbeins) that span a tangent space. As discussed in Box 2.1, we refer to the components of the vielbeins with Latin indices in order to avoid confusion.

We can define the metric in terms of the inner product of vielbeins:

$$g_{\mu\nu} = {e_\mu}^a {e_\nu}^b\ \eta_{ab}, \qquad g_{\mu\nu}\ {e^\mu}_a {e^\nu}_b = \eta_{ab}$$

where the second equation introduces the inverse vielbeins, ${e^\mu}_a$. This relationship with the metric leads to the vielbeins being described as the 'square root' of the metric. The inverse vielbeins also satisfy:

$${e^\mu}_a {e_\nu}^a = \delta^\mu_\nu$$

We will also use the vielbein 1-forms:

$$e^a = \mathrm{d}x^\mu\ {e_\mu}^a$$

Vielbeins are sometimes referred to as non-coordinate bases, and are introduced more throughly by Carroll [15, Appendix J].

**Box 2.5:** The Vielbein Formalism

The exterior derivative produces a $(p+1)$–form $\mathrm{d}\sigma$ from a $p$–form $\sigma$. It obeys the generalised Leibniz product rule:

$$\mathrm{d}\,(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$$

where $\alpha$ is a $p$-form (and $\beta$ any $q$-form). We will also need to use:

$$\mathrm{dd} = 0$$

which embodies an elementary rule of topology: the boundary of a boundary is zero.

**Box 2.6:** Exterior Derivatives

### 2.2.2. Finding the Spin Connection

Start by defining the vielbein one-forms as:

$$\hat{e}^a = e^{\alpha\phi} e^a, \qquad \hat{e}^z = e^{\beta\phi} \left( \mathrm{d}z + \mathcal{A} \right) \tag{2.7}$$

where the gauge potential $\mathcal{A}_\mu$ is also written as a one-form $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$.

As before, the terms with hats belong to the $(D+1)$ dimensional Kaluza-Klein universe, and the terms without hats are the apparent terms in the $D$ dimensional visible universe. Now proceed towards finding the spin connection terms by applying the torsion-free condition:

$$\mathrm{d}\hat{e}^M = -\hat{\omega}^M{}_N \wedge \hat{e}^N \tag{2.8}$$

In (2.8), we see for the first time a wedge product ($\wedge$) and an exterior derivative (d). These are introduced in Box 2.4 and Box 2.6. We will also make use of the antisymmetry of $\hat{\omega}$:

$$\hat{\omega}^z{}_z = 0, \qquad \hat{\omega}^z{}_a = \hat{\omega}^a{}_z, \qquad \hat{\omega}^a{}_b = -\hat{\omega}^b{}_a \tag{2.9}$$

In the lower dimensions, the torsion free condition (2.8) gives:

$$\mathrm{d}e^a = -\omega^a{}_b \wedge e^b \tag{2.10}$$

Starting by choosing $M = z$ in (2.8), inserting our choice of vielbeins from (2.7) into the left hand side and splitting the indices (as described in Box 2.7) on the right hand side of torsion free condition (2.8) gives:

$$\begin{aligned}
\mathrm{d}\hat{e}^z = \mathrm{d}\left( e^{\beta\phi} \left( \mathrm{d}z + \mathcal{A} \right) \right) &= -\hat{\omega}^z{}_{\tilde{N}} \wedge \hat{e}^{\tilde{N}} \\
\mathrm{d}\left( e^{\beta\phi} \right) \left( \mathrm{d}z + \mathcal{A} \right) + e^{\beta\phi} \mathrm{d}\left( \mathrm{d}z + \mathcal{A} \right) &= -\hat{\omega}^z{}_b \wedge \hat{e}^b - \hat{\omega}^z{}_z \wedge \hat{e}^z \\
\beta\ \partial\phi\ e^{\beta\phi} \left( \mathrm{d}z + \mathcal{A} \right) + e^{\beta\phi} \mathrm{d}\mathrm{d}z + e^{\beta\phi} \mathrm{d}\mathcal{A} &= -\hat{\omega}^z{}_b \wedge \hat{e}^b \\
\beta\ \partial\phi\ \hat{e}^z + e^{\beta\phi} \mathcal{F} &= -\hat{\omega}^z{}_a \wedge \hat{e}^a
\end{aligned} \tag{2.11}$$

where the derivative of the one-form gauge potential is now written as the two-form field strength $\mathrm{d}\mathcal{A} = \mathcal{F}$. We can now pull out the $\hat{\omega}^{az}$ term:

$$\begin{aligned}
\hat{\omega}^z{}_a \wedge \hat{e}^a &= -\beta\ \partial\phi\ \hat{e}^z - e^{\beta\phi} \mathcal{F} \\
\hat{\omega}^z{}_a \wedge \hat{e}^a \wedge \hat{e}_b &= -\beta\ \partial\phi\ \hat{e}^z \wedge \hat{e}_b - e^{\beta\phi} \mathcal{F} \wedge \hat{e}_b \\
\hat{\omega}^z{}_b &= -\beta\ \partial\phi\ \hat{e}^z \wedge e^{-\alpha\phi} e_b - e^{\beta\phi} \mathcal{F} \wedge \hat{e}_b \wedge \frac{1}{2} \left( \hat{e}^a \wedge \hat{e}_a \right) \\
&= \beta e^{-\alpha\phi}\ \partial\phi\ e_b \wedge \hat{e}^z + \frac{1}{2} e^{\beta\phi} \mathcal{F} \wedge \hat{e}_b \wedge \hat{e}_a \wedge \hat{e}^a \\
&= \beta e^{-\alpha\phi}\ \partial^b\phi\ \hat{e}^z + \frac{1}{2} e^{\beta\phi} \mathcal{F} \wedge \hat{e}^b \wedge \hat{e}_a \wedge \hat{e}^a \\
&= \beta e^{-\alpha\phi}\ \partial^a\phi\ \hat{e}^z + \frac{1}{2} e^{\beta\phi} e^{-2\alpha\phi} \mathcal{F}^a{}_b\ \hat{e}^b \\
\hat{\omega}^{az} = -\hat{\omega}^{za} &= -\beta e^{-\alpha\phi}\ \partial^a\phi\ \hat{e}^z - \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_b\ \hat{e}^b
\end{aligned} \tag{2.12}$$

We will sometimes encounter equations in which it is necessary to split an implicit sum over indices into an explicit sum of terms with indices that operate over a smaller range.

For a short example, define an index $M$ which runs $1 \dots D$. Also define indices $\mu$ running $1 \dots N$ and $m$ running $(N+1) \dots D$. With a term such as:

$$A_M B^M$$

we can split the implicit sum over $M$ into the explicit sum:

$$A_M B^M = A_\mu B^\mu + A_m B^m$$

**Box 2.7:** Splitting Indices Into Subranges

where the derivative of $\phi$ and the field strength $\mathcal{F}$ have been written including vielbein components: $\partial_a \phi = \partial\phi \wedge e_a$ and $\mathcal{F}^a{}_b = \mathcal{F} \wedge e^a \wedge e_b$. (2.12) makes use of the fact given in Box 2.6 that $\forall\ T, \mathrm{dd}T = 0$ and the symmetries of $\omega$ given in (2.9).

The other component of the spin connection can be found be choosing $M = \mu$ in (2.8):

$$\begin{aligned}
\mathrm{d}\hat{e}^a = \mathrm{d}\left(e^{\alpha\phi} e^a\right) &= -\hat{\omega}^a{}_{\tilde{N}} \wedge \hat{e}^{\tilde{N}} \\
\mathrm{d}\left(e^{\alpha\phi}\right) e^a + e^{\alpha\phi} \mathrm{d}\left(e^a\right) &= -\hat{\omega}^a{}_b \wedge \hat{e}^b - \hat{\omega}^a{}_z \wedge \hat{e}^z \\
\alpha\ \partial\phi\ e^{\alpha\phi} e^a - e^{\alpha\phi} \omega^a{}_b \wedge e^b &= -\hat{\omega}^a{}_b \wedge \hat{e}^b - \hat{\omega}^a{}_z \wedge \hat{e}^z \\
\alpha\ \partial\phi\ \hat{e}^a - \omega^a{}_b \wedge \hat{e}^b &= -\hat{\omega}^a{}_b \wedge \hat{e}^b - \hat{\omega}^a{}_z \wedge \hat{e}^z
\end{aligned} \tag{2.13}$$

In (2.13) we have used the relationship (2.10) to give an equation containing both $\hat{\omega}$ and $\omega$, connecting the $(D+1)$ dimensional universe with the $D$ dimensional visible world. Hence, from (2.13),

$$\begin{aligned}
\hat{\omega}^a{}_b \wedge \hat{e}^b &= \omega^a{}_b \wedge \hat{e}^b + \hat{\omega}^a{}_z \wedge \hat{e}^z - \alpha\ \partial\phi\ \hat{e}^a \\
\hat{\omega}^a{}_b &= \omega^a{}_b + \hat{\omega}^a{}_z \wedge \hat{e}^z \wedge \hat{e}_b - \alpha\ \partial\phi \left(\hat{e}^a \wedge \hat{e}_b - \hat{e}_b \wedge \hat{e}^a\right) \\
&= \omega^a{}_b - \left(\beta e^{-\alpha\phi} \partial^a\phi\ \hat{e}^z + \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_b \hat{e}^b\right) \wedge \hat{e}^z \wedge \hat{e}_b - \alpha e^{-\alpha\phi} \left(\partial^a\phi\ \hat{e}_b - \partial_b\phi\ \hat{e}^a\right) \\
&= \omega^a{}_b - \beta e^{-\alpha\phi}\ \partial^a\phi\ \hat{e}^z \wedge \hat{e}^z \wedge \hat{e}_b - \alpha e^{-\alpha\phi} \left(\partial^a\phi\ \hat{e}_b - \partial_b\phi\ \hat{e}^a\right) - \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_b\ \hat{e}^z \\
&= \omega^a{}_b - 0 - \alpha e^{-\alpha\phi} \left(\partial^a\phi\ \hat{e}_b - \partial_b\phi\ \hat{e}^a\right) - \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^a{}_b\ \hat{e}^z \\
\hat{\omega}^{ab} &= \omega^{ab} - \alpha e^{-\alpha\phi} \left(\partial^a\phi\ \hat{e}^b - \partial^b\phi\ \hat{e}^a\right) - \frac{1}{2} e^{(\beta-2\alpha)\phi} \mathcal{F}^{ab}\ \hat{e}^z
\end{aligned} \tag{2.14}$$

The final component of the spin connection has already been defined by the antisymmetric nature of $\omega$ described in (2.9):

$$\omega^z{}_z = 0 \tag{2.15}$$

With these components of the spin connection found in (2.12), (2.14) and (2.15), the vielbein components of the curvature two-form can now be found.

### 2.2.3. Calculating the Curvature

With the components of the spin connection given above in §2.2.2, the curvature two-forms can now be calculated. As in (2.6), convenient values of the constants $\alpha$ and $\beta$ can be chosen to simplify (2.12) and (2.14):

$$
\begin{aligned}
\alpha &= [2(D-1)(D-2)]^{-1/2}\,, \qquad \beta = -(D-2)\alpha && (2.16)\\
\hat{\omega}^{ab} &= \omega^{ab} + \alpha e^{-\alpha\phi}\left(\partial^b\phi\ \hat{e}^a - \partial^a\phi\ \hat{e}^b\right) - \frac{1}{2}e^{-D\alpha\phi}\mathcal{F}^{ab}\ \hat{e}^z \\
\hat{\omega}^{az} &= -\hat{\omega}^{za} = (D-2)\alpha e^{-\alpha\phi}\partial^a\phi\ \hat{e}^z - \frac{1}{2}\mathcal{F}^a{}_b e^{-D\alpha\phi}\ \hat{e}^b && (2.17)
\end{aligned}
$$

The definition of the curvature two-form in terms of the spin connection $\omega$ gives:

$$
R^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.18}
$$

for the lower dimensional case and:

$$
\begin{aligned}
\hat{R}^M{}_N &= \mathrm{d}\hat{\omega}^M{}_N + \hat{\omega}^M{}_P \wedge \hat{\omega}^P{}_N \\
&= \mathrm{d}\hat{\omega}^M{}_N + \hat{\omega}^M{}_a \wedge \hat{\omega}^a{}_N + \hat{\omega}^M{}_z \wedge \hat{\omega}^z{}_N && (2.19)
\end{aligned}
$$

for the higher dimensional case.

Inserting the components in (2.17) into these relations (2.18) and (2.19) to find the components of the Ricci tensor is a rather tedious calculation. The final results are:

$$
\begin{aligned}
\hat{R}_{ab} &= e^{-2\alpha\phi}\left(R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi\right) - \frac{1}{2}e^{-2D\alpha\phi}\mathcal{F}_a{}^c\mathcal{F}_{bc} \\
\hat{R}_{az} &= \hat{R}_{za} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla^b\left(e^{-2(D-1)\alpha\phi}\mathcal{F}_{ab}\right) \\
\hat{R}_{zz} &= (D-2)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2 && (2.20)
\end{aligned}
$$

The $R_{az}$ term is actually not necessary for specifying the Ricci scalar as below, but is included here for completeness.

The Ricci scalar can then be calculated by contracting the Ricci tensor using:

$$
\hat{R} = \eta^{AB}\hat{R}_{AB} = \eta^{ab}\hat{R}_{ab} + \hat{R}_{zz} \tag{2.21}
$$

Contracting $\hat{R}_{ab}$ gives three terms. The first term, $\eta^{ab}R_{ab}$ contracts trivially to give $R$. The second term gives $\eta^{ab}\partial_a\phi\partial_b\phi = (\partial\phi)^2$. Contracting the third term is also simple with $\eta^{ab}\eta_{ab} = 1$. Therefore, inserting (2.20) into (2.21) gives:

$$
\begin{aligned}
\hat{R} &= \eta^{AB}\hat{R}_{AB} = \eta^{ab}\hat{R}_{ab} + \hat{R}_{zz} \\
&= e^{-2\alpha\phi}\left(R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi\right) - \frac{1}{4}e^{-2D\alpha\phi}\mathcal{F}^2 && (2.22)
\end{aligned}
$$

### 2.2.4. Retrieving the Dimensionally Reduced Action

Combining the contributions found in (2.5) and (2.22) allows the dimensionally reduced action to be written:

$$\begin{aligned}\mathcal{I} &= \int \mathrm{d}^{D+1}x\,\sqrt{-\hat{g}}\,\hat{R} \\ &= \int \mathrm{d}^{D}x\; e^{2\alpha\phi}\sqrt{-g}\; e^{-2\alpha\phi}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2\right) \\ &= \int \mathrm{d}^{D}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2\right) \qquad (2.23)\end{aligned}$$

where the $\Box\phi$ term in (2.22) is dropped as it is a total derivative and therefore does not contribute to the action integral.

Comparing this new action in (2.23) with the action of classical relativistic electrodynamics [17]:

$$\int \mathrm{d}^{n}x\sqrt{-g}\left(R - \mathcal{F}^2\right) \qquad (2.24)$$

we can see that the action (2.23) contains an Einstein action $\sqrt{-g}R$ for $g_{\mu\nu}$ and a Maxwell action $\mathcal{F}^2$ for $\mathcal{A}$. This means that the $(D+1)$ dimensional theory of pure gravity contains $D$ dimensional gravity alongside electromagnetism. As the electromagnetism term is just a product of the dimensional reduction, we could say that electromagnetism is a consequence of the higher dimensional gravity.

There is also a mysterious kinetic term for the scalar $\phi$. In order to investigate this more fully, the this action is varied and the equations of motion found in §2.3.

## 2.3. Equations of Motion

The variation of action integrals to find equations of motion is a technique that will be used repeatedly in this project. In §2.3.1 the simple Einstein-Hilbert action is varied as a template for the more difficult variation of the full Kaluza-Klein action (2.23) in §2.3.2 and for the other variations in §3.

### 2.3.1. Varying the Einstein-Hilbert Action

The Einstein-Hilbert action integral is given by:

$$\mathcal{I}_{\mathrm{EH}} = \int \mathrm{d}^{D}x\sqrt{-g}R \qquad (2.25)$$

In this equation, we see the determinant of the metric $g$ alongside the Ricci scalar $R$. Einstein's field equations are produced by varying this action with respect to $g_{\mu\nu}$. We will treat the curvature as a function of the connections $R = R(\Gamma)$ and independent of the metric, ignoring the fact that actually $\Gamma = \Gamma(g_{\mu\nu})$.

There are several relationships that will be useful when varying expressions that contain the metric. Firstly, to contract the metric with its inverse:

$$
\begin{aligned}
g^{\mu\nu} g_{\nu\rho} &= \delta^{\mu}_{\rho} \\
g^{\mu\nu} g_{\nu\mu} &= \delta^{\mu}_{\mu} = D - 1
\end{aligned}
$$

where D is the number of dimensions of the spacetime. This gives the formula for varying an inverse metric:

$$
\delta\left(g^{\mu\nu}\right) g_{\nu\rho} + g^{\mu\nu} \delta\left(g_{\nu\rho}\right) = 0 \qquad \Rightarrow \qquad \delta\left(g^{\mu\nu}\right) = -g^{\mu\rho} g^{\nu\sigma} \delta\left(g_{\rho\sigma}\right)
$$

It will also be useful to vary the determinant of the metric:

$$
\delta(g) = g g^{\mu\nu} \delta\left(g_{\mu\nu}\right)
$$

where the determinant has been written without indices as $g$.

**Box 2.8:** Varying the Metric

So, making use of the definitions in Box 2.8, varying (2.25) with respect to $g_{\mu\nu}$ gives:

$$
\begin{aligned}
\delta\left(\mathcal{I}_{\mathrm{EH}}\right) &= \int \mathrm{d}^D x \; \delta\left(\sqrt{-g} R\right) \\
&= \int \mathrm{d}^D x \left[\delta\left(\sqrt{-g}\right) R + \sqrt{-g} \delta(R)\right] \\
&= \int \mathrm{d}^D x \left[\frac{-1}{2\sqrt{-g}} \; \delta(g) R + \sqrt{-g} \; \delta\left(g^{\mu\nu}\right) R_{\mu\nu}\right] \\
&= \int \mathrm{d}^D x \left[\frac{-1}{2\sqrt{-g}} g g^{\rho\sigma} \delta\left(g_{\rho\sigma}\right) R - \sqrt{-g} \; g^{\mu\rho} g^{\sigma\nu} \delta\left(g_{\rho\sigma}\right) R_{\mu\nu}\right] \\
&= \int \mathrm{d}^D x \; \sqrt{-g} \left[\frac{1}{2} g^{\rho\sigma} R - R^{\rho\sigma}\right] \delta\left(g_{\rho\sigma}\right)
\end{aligned}
\tag{2.26}
$$

which gives the equation of motion:

$$
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 0 \tag{2.27}
$$

We can choose to rewrite the left-hand side using index free notation as the Einstein Tensor $\mathbf{G}$:

$$
\mathbf{G} = \mathbf{R} - \frac{1}{2} \mathbf{g} R \tag{2.28}
$$

using the convention that bold quantities $\mathbf{G}$, $\mathbf{R}$ are tensors.

The right hand side of (2.27) is zero for this vacuum case. When matter and energy are included, the right hand side would involve the stress energy tensor $T^{\mu\nu}$. This gives the familiar Einstein Field Equations (written with a cosmological constant $\Lambda$).

$$
\mathbf{G} + \Lambda \mathbf{g} = \frac{8\pi G}{c^4} \mathbf{T} \tag{2.29}
$$

### 2.3.2. Varying the Kaluza-Klein Action

We can now proceed to finding the equations of motion by varying (2.23) with respect to all the variables of interest. Here, only the results are presented, as an almost identical set of calculations are performed in §3.4.2.

The three equations of motions found by varying with respect to the metric $g$, the scalar $\phi$ and the gauge potential $\mathcal{A}$ are:

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{-2(D-1)\alpha\phi}\left(\mathcal{F}^2_{\mu\nu} - \frac{1}{2(D-2)}\mathcal{F}^2 g_{\mu\nu}\right) \tag{2.30}$$

$$0 = \nabla^\mu\left(e^{-2(D-2)\alpha\phi}\mathcal{F}_{\mu\nu}\right) \tag{2.31}$$

$$\Box\,\phi = \frac{-1}{2}\alpha(D-1)\ e^{-2(D-1)\alpha\phi}\ \mathcal{F}^2 \tag{2.32}$$

The final equation of motion for $\phi$ in (2.32) leads to the failure of the original Kaluza-Klein theory. The scalar $\phi$ is interacting with the electromagnetic fields and so cannot be simply truncated, set to zero and ignored.

## 2.4. Conclusions

In this section, it has been shown that by dimensionally reducing gravity from $(D+1)$ to $D$ dimensions we arrive at (2.23). In this new action, the familiar $R$ and $\mathcal{F}^2$ terms from the Einstein-Hilbert action are obtained together with an unfamiliar scalar term.

If this scalar $\phi$ could be simply set to zero, it could be claimed that a successful unification of gravity and electromagnetism has been achieved, formulated as pure five dimensional gravity. However, in the derivation of the equations of motion in §2.3.2, we see in (2.32) that setting $\phi = 0$ also sets $\mathcal{F} = 0$ thereby loosing all electromagnetic interaction. For further discussion see [14] and [16, §6]. In must be therefore be concluded that this theory is ultimately unphysical.

While this original Kaluza-Klein theory is unsuccessful, it has spawned many similar geometric theories seeking to unify gravity with electromagnetism and later with the nuclear forces. Throughout the twentieth century, there have been many attempts at unification involving a higher dimensional reality with one or more dimensions curled up into spheres or tori. In fact, almost all theories that are written in more than four dimensions rely on some version of Kaluza-Klein style reduction to connect them to the physical world.

In §3, the results found in §2.2 and §2.3 will be used to connect 11 dimensional supergravity to 10 dimensional M-Theory. The same process of using vielbeins to find spin connections and curvature components will also be used to investigate the supergravity action in §3.5.

# 3. The Eleven Dimensional World

## 3.1. Supergravity

The current favourite candidate for a unified theory to embrace all aspects of modern physics is M-Theory, which is a unified field theory of strings and higher dimensional extended objects known as branes. The five different superstring theories developed in the late 1980s were found to be connected by various dualities [10] which linked them together to form this M-Theory. Whilst most of the details of this grand unified theory are unknown, it can be shown that the low energy limit is supergravity [12].

Supergravity is a classical theory from the late 1970s [6] which was originally abandoned in favour of superstrings, but has continued to be relevant as a low energy approximation to M-Theory. It has many possible areas of study, as unlike most string theories and M-Theory, it has an action that can be written explicitly [7].

A key feature of both M-Theory, supergravity and almost all modern attempts at a unified theory is supersymmetry. Supersymmetry is an abstract set of symmetries that rotate the two classes of particles (bosons and fermions) into each other. None of these partner particles have ever been observed, so we do know that if supersymmetry exists, it must be a partially broken symmetry, as evidenced by the very different masses of the leptons and the quarks. Supersymmetry is also important as it limits the number of dimensions to eleven.

In this section, we will investigate the bosonic sector of supergravity. Starting by investigating the action found by Cremmer, Julia and Scherk [7], we will also apply some ansätze and simplifying assumptions in order to find and solve the equations of motion.

## 3.2. The Bosonic Action

The action of the bosonic sector of $D = 11$ supergravity was found in 1978 by Cremmer, Julia and Scherk [7]. Its action integral can be written as:

$$\mathcal{I} = \int \mathrm{d}^{11}x\sqrt{-g}\left(R - \frac{1}{48}F_{[4]}^2\right) - \frac{1}{6}\int F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \tag{3.1}$$

In some places, for example Stelle [16, eq. 1.1], the second integral in (3.1) is written with a positive sign. We choose here to use a negative sign for later consistency, but the two forms are identical through a redefinition of the potential $A_{[3]}$.

In the action in (3.1), the metric $g$ can be identified, written without indices to denote the determinant along with $R$, the Ricci Scalar. The three form antisymmetric gauge potential $A_{[3]}$ and the four form field strength $F_{[4]}$ can also be seen. These quantities are related as:

$$F_{[4]} = \mathrm{d}A_{[3]}. \tag{3.2}$$

The first integral in (3.1) is written in component notation and the second integral is written in the language of forms (introduced in Box 2.2, Box 2.6 and Box 2.4). The first integral can be identified as the action of classical relativistic electrodynamics as seen in (2.24). This first term will be rewritten in §3.3.1 to also be in form language.

## 3.3. Magnetic and Electric Charges

### 3.3.1. Rewriting the Action

To proceed, we would like to rewrite the action integral (3.1) totally in the language of forms. The first term is recognised as the Einstein-Hilbert action (2.1) and can be rewritten in form language using the Hodge dual, such that:

$$\begin{aligned}\int R\sqrt{-g}\ \mathrm{d}^n x &= \frac{1}{11!}\int R\sqrt{-g}\ \varepsilon_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_d}\\ &= \int\frac{1}{11!}R\ \epsilon_{\mu_1\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_d}\\ &= \int {}^\star R \end{aligned} \tag{3.3}$$

where ${}^\star R$ is the Hodge dual of $R$. The equality of lines 1 and 2 is due to the definitions of $\epsilon$ and $\varepsilon$ as outlined in Box 3.1. A precise definition of the Hodge dual is given in Box 3.2. Turning to the second, electric term, we need to show the equivalence of:

$$-\sqrt{-g}\frac{1}{48}\int \mathrm{d}^{11}xF_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4} = -\frac{1}{2}\int F_{[4]}\wedge{}^\star F_{[4]} \tag{3.4}$$

This is shown in Appendix A.1. As a result of these calculations, the action (3.1) can now be written in the form:

$$\mathcal{I} = \int\left[{}^\star R - \frac{1}{2}F_{[4]}\wedge{}^\star F_{[4]} - \frac{1}{6}F_{[4]}\wedge F_{[4]}\wedge A_{[3]}\right] \tag{3.5}$$

We can now proceed and vary this action and find the equations of motion.

This is defined in various ways across the literature, so in this report, we shall follow the following conventions:

- The *tensor density*, $\epsilon$ is defined as follows:

$$\epsilon^{\mu\nu\rho\sigma\ldots} = \begin{cases} +1 & \text{for even permutations}\\ -1 & \text{for odd permutations}\\ 0 & \text{otherwise}\end{cases}$$

  For consistency, we shall only use this $\epsilon$ with upper indices.
- The *proper tensor*, $\varepsilon$ is defined as:

$$\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma\ldots} = \epsilon^{\mu\nu\rho\sigma\ldots}$$

  where $\sqrt{-g}$ is a *density factor*, defined by $g$: the determinant of the metric:

$$g = g^{\mu\nu}g_{\mu\nu}$$

The advantage of using the proper tensor $\varepsilon$ is that its indices can be raised and lowered by the metric as usual:

$$\varepsilon_{\mu\nu\rho\sigma\ldots} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\eta}\ldots\epsilon^{\alpha\beta\gamma\eta\ldots}$$

**Box 3.1:** The Levi-Civita Symbol: $\epsilon$ and $\varepsilon$

On an $n$–dimensional manifold the Hodge dual produces a $(n-p)$-form $^\star\sigma$ from a $p$-form $\sigma$. Dualizing is defined for one–forms as

$$^\star(\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}) = \frac{1}{q!}\ \varepsilon_{\nu_1 \ldots \nu_q}{}^{\mu_1 \ldots \mu_p}\ \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_p}$$

where $p + q = n$. This definition allows us to dualize any $p$-form without affecting the tensor components. Dualizing twice gives the identity on the algebra (up to a sign) such that:

$$^{\star\star}\eta = (-1)^{p(n-p)}\,\eta$$

with $\eta$ a general $p$-form.

**Box 3.2:** The Hodge Dual: $\star$

### 3.3.2. Variation With Respect to $A_{[3]}$

Now that the action is in the form of (3.5), it is now possible to proceed to vary this action to find the equations of motion. Applying the variational derivative gives:

$$\begin{aligned}
\delta\mathcal{I} &= \int \left[\delta\left(^\star R\right) - \delta\left(\frac{1}{2}F_{[4]} \wedge {}^\star F_{[4]}\right) - \delta\left(\frac{1}{6}F_{[4]} \wedge F_{[4]} \wedge A_{[3]}\right)\right] \\
&= \int \left[-\delta\left(\frac{1}{2}F_{[4]} \wedge {}^\star F_{[4]}\right) - \delta\left(\frac{1}{6}F_{[4]} \wedge F_{[4]} \wedge A_{[3]}\right)\right] \\
&= \frac{-1}{2}\int \delta\left(F_{[4]} \wedge {}^\star F_{[4]}\right) + \frac{-1}{6}\int \delta\left(F_{[4]} \wedge F_{[4]} \wedge A_{[3]}\right)
\end{aligned} \tag{3.6}$$

where in the second line, the $R$ term has been set to zero as it has no $A_{[3]}$ dependence, and then the constants have been taken outside the variational derivatives. Now the relationship between $A_{[3]}$ and $F_{[4]}$ given in (3.2) can be used to find:

$$\begin{aligned}
F_{[4]} &= \mathrm{d}A_{[3]} \\
\mathrm{d}F_{[4]} &= \mathrm{dd}A_{[3]} \\
\delta\left(F_{[4]}\right) &= \delta\left(\mathrm{d}A_{[3]}\right) = \mathrm{d}\delta\left(F_{[4]}\right)
\end{aligned} \tag{3.7}$$

Taking the first term of (3.6) gives:

$$\begin{aligned}
\frac{1}{2}\int \delta\left(F_{[4]} \wedge {}^\star F_{[4]}\right) &= \frac{1}{2}\int \delta\left(F_{[4]}\right) \wedge {}^\star F_{[4]} + F_{[4]} \wedge \delta\left({}^\star F_{[4]}\right) \\
&= \frac{1}{2}\int \delta\left(F_{[4]}\right) \wedge {}^\star F_{[4]} + F_{[4]} \wedge {}^\star\delta\left(F_{[4]}\right) \\
&= \frac{1}{2}\int 2\delta\left(F_{[4]}\right) \wedge {}^\star F_{[4]} \\
&= \int \delta\left(\mathrm{d}A_{[3]}\right) \wedge {}^\star F_{[4]} \\
&= \int \delta\left(A_{[3]}\right) \wedge \mathrm{d}^\star F_{[4]}
\end{aligned} \tag{3.8}$$

where in the first line the variational derivative has been applied using the product rule. To arrive at the last line, a technique of partial integration is used which is explained in Box 3.3.

Now turning to the second term in (3.6), the variational derivative is again applied using the product rule:

$$\begin{aligned}&\frac{-1}{6}\int\delta\left(F_{[4]}\wedge F_{[4]}\wedge A_{[3]}\right) \qquad (3.9)\\ &\quad=\frac{-1}{6}\int\left[\delta\left(F_{[4]}\right)\wedge F_{[4]}\wedge A_{[3]}+F_{[4]}\wedge\delta\left(F_{[4]}\right)\wedge A_{[3]}+F_{[4]}\wedge F_{[4]}\wedge\delta\left(A_{[3]}\right)\right]\end{aligned}$$

The first two of these terms are identical as $F_{[4]}$ commutes with $\delta\left(F_{[4]}\right)$. For the final term we see that:

$$\begin{aligned}\int F_{[4]}\wedge F_{[4]}\wedge\delta\left(A_{[3]}\right)&=\int \mathrm{d}A_{[3]}\wedge F_{[4]}\wedge\delta\left(A_{[3]}\right)\\ &=\int A_{[3]}\wedge F_{[4]}\wedge \mathrm{d}\delta\left(A_{[3]}\right)\\ &=\int A_{[3]}\wedge F_{[4]}\wedge\delta\left(\mathrm{d}A_{[3]}\right)\\ &=\int A_{[3]}\wedge F_{[4]}\wedge\delta\left(F_{[4]}\right) \qquad (3.10)\end{aligned}$$

which is then identical to the first two terms. Therefore, (3.9) becomes:

$$\frac{-1}{6}\int\delta\left(F_{[4]}\wedge F_{[4]}\wedge A_{[3]}\right)=\frac{-1}{2}\int\delta\left(A_{[3]}\right)\wedge F_{[4]}\wedge F_{[4]} \qquad (3.11)$$

Combining the contributions from (3.8) and (3.11) gives:

$$\delta\left(\mathcal{I}\right)=\int-\delta\left(A_{[3]}\right)\wedge \mathrm{d}^{\star}F_{[4]}+\frac{-1}{2}\delta\left(A_{[3]}\right)\wedge F_{[4]}\wedge F_{[4]} \qquad (3.12)$$

and now, as seen in [16, eq. 1.2], $\delta\left(\mathcal{I}\right)$ can be set to zero in order to retrieve the equations of motion. (3.13) can can be rewritten using the relations in (3.7) to retrieve (3.14).

$$\mathrm{d}^{\star}F_{[4]}+\frac{1}{2}F_{[4]}\wedge F_{[4]}=0 \qquad (3.13)$$

$$\mathrm{d}\left({}^{\star}F_{[4]}+\frac{1}{2}A_{[3]}\wedge F_{[4]}\right)=0 \qquad (3.14)$$

### 3.3.3. Conserved Quantities

The equation of motion found in (3.14) allows us to identify two different conserved charges. There is an electric type charge $U$:

$$U=\int_{\partial\tilde{\mathcal{M}}_8}\left({}^{\star}F_{[4]}+\frac{1}{2}A_{[3]}\wedge F_{[4]}\right) \qquad (3.15)$$

and an magnetic type charge $V$:

$$V=\int_{\partial\tilde{\mathcal{M}}_5}F_{[4]} \qquad (3.16)$$

as seen in Stelle [16, eqs. 1.3 & 1.4].

## 3.4. Building a Toy System

### 3.4.1. Dimensional Reduction of the Supergravity Action

Returning to the bosonic action in (3.1), a Kaluza-Klein style dimensional reduction can be applied using the techniques introduced in §2 to give another action integral. By compressing the $11^{\text{th}}$ dimension into a circle, using the line element:

$$\mathrm{d}s_{11}^2 = e^{-\phi/6}\mathrm{d}s_{10}^2 + e^{4\phi/3}\left(\mathrm{d}z + \mathcal{A}_M \mathrm{d}x^M\right)^2 \tag{3.17}$$

where $M = 0 \ldots 9$, we retrieve:

$$\begin{aligned}\mathcal{I}_{\text{IIA}} = \int \mathrm{d}^{10}x\,\sqrt{-g}\Big[&\left(R - \frac{1}{2}\nabla_M\phi\nabla^M\phi - \frac{1}{12}e^{-\phi}F_{MNP}F^{MNP}\right)\\ &- \frac{1}{48}e^{\phi/2}F_{MNPQ}F^{MNPQ} - \frac{1}{4}e^{3\phi/2}\mathcal{F}_{MN}\mathcal{F}^{MN} + \ldots\Big]\end{aligned} \tag{3.18}$$

The gauge fields in the original action (3.1) are written as $A$ and $F$, whereas the gauge fields introduced by the dimensional reduction are written with curly $\mathcal{A}$ and $\mathcal{F}$. This action can be identified as the Einstein-frame type action of a type IIA bosonic string, written in a supergravity form.

Whilst this action is clearly intractable, we can still use it to identify the main features needed to build a working model of a supergravity system. As in Stelle [16, §2], three terms can be chosen to build an effective theory containing gravity, gauge fields and scalars:

$$\mathcal{I} = \int \mathrm{d}^D x\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2n!}e^{a\phi}F_{[n]}^2\right] \tag{3.19}$$

giving an action simple enough to be varied in order to find the equations of motion. Note that in this toy action, we are now considering a general $n$-form gauge potential. Unsurprisingly, as it is also formed by a Kaluza-Klein style dimensional reduction, this action is very similar to the one found in §2.2.1.

Choosing these three terms to form the action integral may seem somewhat arbitrary, but by this choice, we have formed a system that is in fact a consistent truncation of the full supergravity theory. This means that the solutions to this truncated system are solutions of the full untruncated theory. Further discussion can be found in Stelle [16].

This action can be varied in order to find the equations of motion. It will be useful to refer back to §2.3.1, where the variation of the Einstein-Hilbert action is presented. In the following section, this toy action will be varied with respect to the metric $g$, the scalar $\phi$ and the gauge potential $A_{[n-1]}$.

### 3.4.2. Varying the Toy Action

We can now proceed to finding the equations of motion by varying (3.19) with respect to all the variables of interest. This process is rather similar to that in §2.3.2.

**Variation With Respect to $\phi$** Starting with the most simple case, varying with respect to $\phi$, we remove terms with no $\phi$ dependence to find:

$$
\begin{aligned}
\delta(\mathcal{I}) &= \int \mathrm{d}^D x\sqrt{-g}\;\delta\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2n!}e^{a\phi}F^2_{[n]}\right)\\
&= \int \mathrm{d}^D x\sqrt{-g}\left[-\frac{1}{2}\delta\left(\nabla_\mu\phi\nabla^\mu\phi\right) - \frac{1}{2n!}\delta\left(e^{a\phi}\right)F^2_{[n]}\right]\\
&= \int \mathrm{d}^D x\sqrt{-g}\left[-\delta\left(\nabla_\mu\phi\right)\nabla^\mu\phi - \frac{a}{2n!}e^{a\phi}\delta\left(\phi\right)F^2_{[n]}\right]\\
&= \int \mathrm{d}^D x\sqrt{-g}\left[\nabla_\mu\nabla^\mu\phi - \frac{a}{2n!}e^{a\phi}F^2_{[n]}\right]\delta\left(\phi\right)
\end{aligned}
\tag{3.20}
$$

where the technique of partial integration from Box 3.3 has been used. Setting $\delta(\mathcal{I})$ to zero gives:

$$
\nabla_\mu\nabla^\mu\phi = \frac{a}{2n!}e^{a\phi}F^2_{[n]} \tag{3.21}
$$

In order to get this equation in the same form as Stelle [16, eq. 2.2d], we rewrite (3.21) using the $\Box$ operator:

$$
\Box\phi = \nabla^2\phi = \nabla_\mu\nabla^\mu\phi = \frac{a}{2n!}e^{a\phi}F^2_{[n]} \tag{3.22}
$$

**Variation With Respect to $\mathcal{A}$** The next equation of motion comes from varying the toy action integral (3.19) with respect to the gauge potential $A_{[n-1]}$. Again starting by removing terms with no $A_{[n-1]}$ dependence, we also rewrite the $F^2_{[n]}$ term in form language using the Hodge Dual (see Box 3.2):

$$
\begin{aligned}
\delta(\mathcal{I}) &= \int \mathrm{d}^D x\sqrt{-g}\;\delta\left(-\frac{1}{2n!}e^{a\phi}F^2_{[n]}\right)\\
&= K\int \mathrm{d}^D x\; e^{a\phi}\;\delta\left(F_{[n]}\wedge{}^\star F_{[n]}\right)\\
&= K\int \mathrm{d}^D x\; e^{a\phi}\left[\delta\left(F_{[n]}\right)\wedge{}^\star F_{[n]} + F_{[n]}\wedge\delta\left({}^\star F_{[n]}\right)\right]\\
&= K\int \mathrm{d}^D x\; e^{a\phi}\;\delta\left(F_{[n]}\right)\wedge{}^\star F_{[n]}\\
&= K\int \mathrm{d}^D x\; e^{a\phi}\;\delta\left(\mathrm{d}A_{[n-1]}\right)\wedge{}^\star F_{[n]}\\
&= K\int \mathrm{d}^D x\; e^{a\phi}\;\delta\left(A_{[n-1]}\right)\wedge\mathrm{d}\left({}^\star F_{[n]}\right)
\end{aligned}
\tag{3.23}
$$

where all numerical constants have been included into $K$ and again a partial integration has been used. It is now possible to extract the equation of motion from (3.23):

$$
e^{a\phi}\;\mathrm{d}\left({}^\star F_{[n]}\right) = 0 \tag{3.24}
$$

To convert this equation to the form of Stelle [16, eq.2.2c], it needs to be rewritten in coordinate form:

$$
{}^\star\mathrm{d}\left(e^{a\phi}\;{}^\star F_{[n]}\right) = \nabla_{M_1}\left(e^{a\phi}\;\mathcal{F}^{M_1\ldots M_2}\right) = 0 \tag{3.25}
$$

There is a common technique in manipulating these expressions which uses partial integration. With an expression like:

$$\int \delta\left(A\right) B$$

the identity of the product rule gives:

$$\int \delta\left(AB\right) = \int \delta\left(A\right) B + \int A\ \delta\left(B\right)$$

or equivalently, perform a partial integration to get:

$$\begin{aligned}\int \delta\left(A\right) B &= \int \delta\left(AB\right) - \int \delta\left(B\right) A \\ &= -\int \delta\left(B\right) A\end{aligned}$$

where we have dropped the first term $\int \delta\left(AB\right)$ as it is a total derivative and so vanishes when integrated over a closed surface.

**Box 3.3:** Partial Integration

**Variation With Respect to the Metric** The last equation of motion is found by again varying the toy action integral (3.19) now with respect to the metric.

$$\delta\left(\mathcal{I}\right) = \int \mathrm{d}^D x\ \delta\left(\sqrt{-g}\left[R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2n!}e^{a\phi}F_{[n]}^2\right]\right) \tag{3.26}$$

This can be split into three terms. The first term with $\delta\left(R\right)$ is a simple Einstein-Hilbert action and has already been addressed in §2.3.1. Therefore, this term simply contributes the result from (2.26):

$$\delta\left(\mathcal{I}_{\mathrm{EH}}\right) = \int \mathrm{d}^D x\ \sqrt{-g}\left[\frac{1}{2}g^{\rho\sigma}R - R^{\rho\sigma}\right]\delta\left(g_{\rho\sigma}\right) \tag{3.27}$$

Again using the definitions in Box 2.8, the second term gives:

$$\begin{aligned}\delta\left(\mathcal{I}_\phi\right) &= \int \mathrm{d}^D x\ \delta\left(\sqrt{-g}g^{\mu\nu}\right)\left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right) \\ &= \int \mathrm{d}^D x\left[\delta\left(\sqrt{-g}\right)g^{\mu\nu} + \sqrt{-g}\ \delta\left(g^{\mu\nu}\right)\right]\left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right) \\ &= \int \mathrm{d}^D x\left[\frac{-1}{2\sqrt{-g}}\delta\left(g\right)g^{\mu\nu} - \sqrt{-g}\ g^{\mu\rho}g^{\nu\sigma}\delta\left(g_{\rho\sigma}\right)\right]\left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right) \\ &= \int \mathrm{d}^D x\left[\frac{-1}{2\sqrt{-g}}gg^{\rho\sigma}g^{\mu\nu} - \sqrt{-g}\ g^{\mu\rho}g^{\nu\sigma}\right]\left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right)\delta\left(g_{\rho\sigma}\right) \\ &= \int \mathrm{d}^D x\ \sqrt{-g}\left[\frac{-1}{4}g^{\rho\sigma}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}g^{\mu\rho}g^{\nu\sigma}\nabla_\mu\phi\nabla_\nu\phi\right]\delta\left(g_{\rho\sigma}\right) \\ &= \int \mathrm{d}^D x\ \sqrt{-g}\left[\frac{1}{2}\nabla^\rho\phi\nabla^\sigma\phi - \frac{1}{4}g^{\rho\sigma}\nabla_\mu\phi\nabla^\mu\phi\right]\delta\left(g_{\rho\sigma}\right)\end{aligned} \tag{3.28}$$

Now turning to the final term, the $\mathcal{F}^2$ term is rewritten in components as:

$$\mathcal{F}^2 = g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\mathcal{F}_{\mu_1\ldots\mu_n}\mathcal{F}_{\nu_1\ldots\nu_n} \tag{3.29}$$

so, again using the definitions in Box 2.8, the variation with respect to the metric gives:

$$
\begin{aligned}
&\delta\left(\mathcal{I}_{\mathcal{F}}\right) \\
&= \int \mathrm{d}^D x\ \delta\left(\sqrt{-g}\left(\frac{-e^{a\phi}}{2n!} g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\mathcal{F}_{\mu_1\ldots\mu_n}\mathcal{F}_{\nu_1\ldots\nu_n}\right)\right) \\
&= \int \mathrm{d}^D x\left[\delta\left(\sqrt{-g}\right) g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n} + \sqrt{-g}\ \delta\left(g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\right)\right]\left(\frac{-e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\ldots\mu_n}\mathcal{F}_{\nu_1\ldots\nu_n}\right) \\
&= \int \mathrm{d}^D x\left[\frac{-1}{2\sqrt{-g}} g g^{\rho\sigma} g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n} - \sqrt{-g}\ g^{\rho\mu_1} g^{\sigma\nu_1} n\left(g^{\mu_2\nu_2}\cdots g^{\mu_n\nu_n}\right)\right]\times \\
&\qquad\qquad \times\left(\frac{-e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\ldots\mu_n}\mathcal{F}_{\nu_1\ldots\nu_n}\right)\delta\left(g_{\rho\sigma}\right) \\
&= \int \mathrm{d}^D x\ \sqrt{-g}\left[\frac{1}{2} g^{\rho\sigma} g^{\mu_1\nu_1} - n\ g^{\rho\mu_1} g^{\sigma\nu_1}\right]\left(\frac{-e^{a\phi}}{2n!} g^{\mu_2\nu_2}\cdots g^{\mu_n\nu_n}\mathcal{F}_{\mu_1\ldots\mu_n}\mathcal{F}_{\nu_1\ldots\nu_n}\right)\delta\left(g_{\rho\sigma}\right) \\
&= \int \mathrm{d}^D x\ \sqrt{-g}\left[\frac{1}{2} g^{\rho\sigma} g^{\mu_1\nu_1} - n\ g^{\rho\mu_1} g^{\sigma\nu_1}\right]\left(\frac{-e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\mu_2\ldots\mu_n}\right)\delta\left(g_{\rho\sigma}\right) \qquad (3.30)
\end{aligned}
$$

The contributions from the three terms in (3.26) found in (3.27), (3.28) and (3.30) can now be combined. Setting $\delta\left(\mathcal{I}\right)$ to zero gives:

$$
\begin{aligned}
0 = \frac{1}{2} g_{\rho\sigma} R - R_{\rho\sigma} + \frac{1}{2}\nabla_\rho\phi\nabla_\sigma\phi - \frac{1}{4} g_{\rho\sigma}\nabla_\mu\phi\nabla^\mu\phi + \qquad (3.31) \\
+\left(\frac{1}{2} g_{\rho\sigma} g^{\mu_1\nu_1} - n\ g_\rho{}^{\mu_1} g_\sigma{}^{\nu_1}\right)\left(\frac{-1}{2n!} e^{a\phi}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\mu_2\ldots\mu_n}\right)
\end{aligned}
$$

It is now necessary to eliminate the $R$ term, which can be achieved by contracting our equation (3.31) with $g^{\rho\sigma}$:

$$
\begin{aligned}
0 &= \frac{D}{2} R - R + \frac{1}{2}\left(\nabla\phi\right)^2 - \frac{D}{4}\left(\nabla\phi\right)^2 + \\
&\quad -\left(\frac{D}{2} g^{\mu_1\nu_1} - n\ g^{\mu_1\nu_1}\right)\left(\frac{-e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\mu_2\ldots\mu_n}\right) \\
R\,\frac{2-D}{2} &= \frac{2-D}{4}\left(\nabla\phi\right)^2 + \frac{D-2n}{2}\left(\frac{e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\nu_2\ldots\mu_n}\right) \\
R &= \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{D-2n}{2-D}\left(\frac{e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\mu_2\ldots\mu_n}\right) \qquad (3.32)
\end{aligned}
$$

This term is now reinserted into (3.31) to give the equation of motion as seen in Stelle [16, eqs. 2.2a, 2.2b]:

$$
\begin{aligned}
R_{\rho\sigma} &= \frac{1}{2}\nabla_\rho\phi\nabla_\sigma\phi + S_{\rho\sigma} \\
S_{\rho\sigma} &= \left(\frac{1}{2} g_{\rho\sigma} g^{\mu_1\nu_1} - n\ g_\rho{}^{\mu_1} g_\sigma{}^{\nu_1}\right)\left(\frac{-1}{2n!} e^{a\phi}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\mu_2\ldots\mu_n}\right) + \\
&\qquad + \frac{1}{2}\frac{D-2n}{2-D}\left(\frac{e^{a\phi}}{2n!}\mathcal{F}_{\mu_1\mu_2\ldots\mu_n}\mathcal{F}_{\nu_1}{}^{\mu_2\ldots\mu_n}\right) \\
&= \frac{e^{a\phi}}{2(n-1)!}\left(\mathcal{F}_{\rho\,\mu_2\ldots\mu_n}\mathcal{F}_\sigma{}^{\mu_2\ldots\mu_n} - \frac{n-1}{n(D-2)}\mathcal{F}^2 g_{\rho\sigma}\right) \qquad (3.33)
\end{aligned}
$$

**The Equations of Motion** The equations found in this section can now be collected together from (3.22), (3.25) and (3.33):

$$\begin{aligned} R_{MN} &= \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN} \\ S_{MN} &= \frac{1}{2(n-1)!}e^{a\phi}\left(\mathcal{F}_{M\dots}\mathcal{F}_N{}^{\dots} - \frac{n-1}{n(D-2)}\mathcal{F}^2 g_{MN}\right) \\ 0 &= \nabla_{M_1}\left(e^{a\phi}\mathcal{F}^{M_1M_2\dots M_n}\right) \\ \Box\phi &= \frac{a}{2n!}e^{a\phi}\mathcal{F}^2 \end{aligned} \tag{3.34}$$

as seen in Stelle [16, eq. 2.2].

These equations are too difficult to be solved directly in this form. In §3.5, a simplifying ansatz will be applied and these equations of motion solved for a simple, free moving, flat brane.

## 3.5. The *p*-Brane Ansatz

The equations (3.34) found in §3.4.2 are generally intractable. To proceed, some simplifying assumptions must be made. For a $p$-brane moving in a $D$ dimensional spacetime we shall require a translational symmetry in directions on the brane and also an isotropic symmetry for the directions transverse to the brane. This can be generally written as $(\text{Poincaré})_d \times \mathrm{SO}\,(D-d)$ symmetry where $d = p + 1$.

Put more simply, the $d$ dimensions with translational symmetry form the worldvolume of the brane and the remaining $(D - d)$ dimensions are isotropic. Therefore, the history of the brane will be formed of $p$-dimensional, flat spatial surfaces. To impose this ansatz, the spatial coordinates are split into two ranges:

$$x^M = (x^\mu, y^m) \tag{3.35}$$

where the $x$ coordinates are on the worldvolume directions and the $y$ coordinates are transverse to the worldvolume. A note on the choice of indices is given in Box 3.4. The line element is written:

$$\mathrm{d}s^2 = e^{2A(r)}\mathrm{d}x^\mu\mathrm{d}x^\nu\eta_{\mu\nu} + e^{2B(r)}\mathrm{d}y^m\mathrm{d}y^n\delta_{mn} \tag{3.36}$$

where the distance from the brane is given by the radial coordinate $r$:

$$r = \sqrt{y^m y^m} \tag{3.37}$$

Note that the suspicious contraction over two raised indices is appropriate in this case, as indices in the $y$ direction are raised and lowered with a simple $\delta$ metric. The effect of this ansatz on the scalar is such that $\phi = \phi(r)$. A full discussion is found in Stelle [16, §2.2].

In order to apply this ansatz, the same procedure as §2.2 is followed. In §3.5.1, the spin connection terms are found from the vielbein one-forms. In §3.5.2, these are used to find the components of the Ricci Tensor which can then be inserted into the equations of motion found above in (3.34), to finally give the $p$-brane equations in §3.5.4.

As before in §2, where the spacetime was split into the flat and curled up dimensions, here the spacetime will have to be split into directions *on* the brane's worldvolume and directions *transverse* to it.

For a $p$-brane in an ambient $D$ dimensional spacetime, the world indices will be split:

$$M = (\mu, m)$$

where the Greek indices cover the range $\mu = 0 \dots p$ and the Latin $m = (p+1) \dots D$. The tangent space in which the vielbeins live will be split in a similar way:

$$\underline{M} = (\underline{\mu}, \underline{m})$$

**Box 3.4:** Indices Revisited

### 3.5.1. Calculating Spin Connections

Noting the convention on choosing indices in Box 3.4, we start by choosing to define the vielbein one-forms as:

$$e^{\underline{\mu}} = e^{A(r)}\mathrm{d}x^{\mu}, \qquad e^{\underline{m}} = e^{B(r)}\mathrm{d}y^{m} \tag{3.38}$$

where care should be taken to distinguish between viebeins and exponentials. As noted above, the $x$ coordinate has been used for directions on the brane's worldvolume and $y$ for directions transverse to the worldvolume. To use these vielbeins to find the spin connection components, the torsion free condition is imposed:

$$\mathrm{d}e^{\underline{E}} + \omega^{\underline{E}}{}_{\underline{F}} \wedge e^{\underline{F}} = 0 \tag{3.39}$$

By splitting the space into two subspaces, we have given the antisymmetric spin connection one-form four different components:

$$\omega^{\underline{MN}} = \begin{pmatrix} \omega^{\underline{\mu\nu}} & \omega^{\underline{\mu n}} \\ \omega^{\underline{m\nu}} & \omega^{\underline{mn}} \end{pmatrix} \tag{3.40}$$

In order to find these different components, the vielbeins in (3.38) are inserted into the torsion free condition in (3.39). Starting by choosing $\underline{E} = \underline{\mu}$, we get:

$$\begin{aligned} 0 &= \mathrm{d}e^{\underline{\mu}} + \omega^{\underline{\mu}}{}_{\underline{F}} \wedge e^{\underline{F}} \\ &= \mathrm{d}e^{\underline{\mu}} + \omega^{\underline{\mu}}{}_{\underline{\nu}} \wedge e^{\underline{\nu}} + \omega^{\underline{\mu}}{}_{\underline{n}} \wedge e^{\underline{n}} \\ &= \mathrm{d}\left(e^{A(r)}\mathrm{d}x^{\mu}\right) + \omega^{\underline{\mu}}{}_{\underline{\nu}} \wedge e^{A(r)}\mathrm{d}x^{\nu} + \omega^{\underline{\mu}}{}_{\underline{n}} \wedge e^{B(r)}\mathrm{d}y^{n} \end{aligned} \tag{3.41}$$

where the implicit sum over $\underline{F}$ in the first line was split into a explicit sum over the two ranges $\underline{\nu}$ and $\underline{n}$ as encountered before in Box 2.7. Now the vielbeins have been inserted, the product rule is applied. The first term of (3.41) gives:

$$\begin{aligned} \mathrm{d}\left(e^{A(r)}\mathrm{d}x^{\mu}\right) &= \mathrm{d}\left(e^{A(r)}\right)\mathrm{d}x^{\mu} + e^{A(r)}\mathrm{d}\left(\mathrm{d}x^{\mu}\right) \\ &= e^{A(r)}\mathrm{d}\left(A(r)\right)\mathrm{d}x^{\mu} \\ &= e^{A(r)}A'(r)\mathrm{d}r \wedge \mathrm{d}x^{\mu} \end{aligned} \tag{3.42}$$

where $A'(r)$ has been written as a shorthand for $\mathrm{d}A(r)/\mathrm{d}r$. $\mathrm{d}r$ can be expressed in terms of $\mathrm{d}y$ using (3.37):

$$\begin{aligned} r &= \sqrt{y^m y^m} \\ 2\,r\,\mathrm{d}r &= 2\,y^m\,\mathrm{d}y^m \\ \mathrm{d}r &= \frac{y^m\,\mathrm{d}y^m}{r} \end{aligned} \tag{3.43}$$

inserting this into (3.42) and then (3.41) gives:

$$\begin{aligned} 0 &= e^{A(r)}A'(r)\mathrm{d}r \wedge \mathrm{d}x^\mu + \omega^{\mu}{}_{\underline{\nu}} \wedge e^{A(r)}\mathrm{d}x^\nu + \omega^{\mu}{}_{\underline{n}} \wedge e^{B(r)}\mathrm{d}y^n \\ &= e^{A(r)}A'(r)\frac{y^m}{r}\mathrm{d}y^m \wedge \mathrm{d}x^\mu + \omega^{\mu}{}_{\underline{\nu}} \wedge e^{A(r)}\mathrm{d}x^\nu + \omega^{\mu}{}_{\underline{n}} \wedge e^{B(r)}\mathrm{d}y^n \end{aligned} \tag{3.44}$$

Now, due to the linear independence of the orthogonal basis vectors $\mathrm{d}x^M$, it can be seen that the middle term of (3.44) is independently zero. This gives one of the components of the spin connection, as seen in Stelle [16, eq. 2.9]:

$$\begin{aligned} \omega^{\mu}{}_{\underline{\nu}} \wedge e^{A(r)}\mathrm{d}x^\nu &= 0 \\ \omega^{\mu}{}_{\underline{\nu}} &= 0 \\ \Rightarrow \qquad \omega^{\underline{\mu\nu}} &= 0 \end{aligned} \tag{3.45}$$

The remaining part of (3.44) is:

$$\begin{aligned} e^{A(r)}A'(r)\frac{y^m}{r}\mathrm{d}y^m \wedge \mathrm{d}x^\mu + \omega^{\mu}{}_{\underline{n}} \wedge e^{B(r)}\mathrm{d}y^n &= 0 \\ \left(e^{B(r)}\omega^{\mu}{}_{\underline{n}} - e^{A(r)}A'(r)\frac{y^n}{r}\mathrm{d}x^\mu\right) \wedge \mathrm{d}y^n &= 0 \\ e^{B(r)}\omega^{\mu}{}_{\underline{n}} - e^{A(r)}A'(r)\frac{y^n}{r}\mathrm{d}x^\mu &= 0 \end{aligned} \tag{3.46}$$

$A'(r)$ can now be rewritten:

$$A'(r) = \frac{r}{y^n}\frac{\partial}{\partial y^n}A(r) \tag{3.47}$$

and inserted into (3.46) to give:

$$\begin{aligned} \omega^{\mu}{}_{\underline{n}} &= e^{-B(r)}\partial_n A(r)\ e^{A(r)}\,\mathrm{d}x^\mu \\ &= e^{-B(r)}\partial_n A(r)\ e^{\underline{\mu}} \end{aligned} \tag{3.48}$$

where $\partial_n$ has been used as a shorthand for $\frac{\partial}{\partial y^n}$. Raising the index on $\omega$ to retrieve the desired form gives:

$$\begin{aligned} \omega^{\underline{\mu n}} &= \delta^{\underline{np}}\ \omega^{\mu}{}_{\underline{p}} \\ &= e^{-B(r)}\partial_n A(r)\,e^{\underline{\mu}} \end{aligned} \tag{3.49}$$

as seen in Stelle [16, eq. 2.9].

To find the final term of the spin connection, we now return to the torsion free condition in (3.39) and pick $\underline{E} = \underline{m}$:

$$
\begin{aligned}
0 &= \mathrm{d}e^{\underline{m}} + \omega^{\underline{m}}{}_{\underline{F}} \wedge e^{\underline{F}} \\
&= \mathrm{d}e^{\underline{m}} + \omega^{\underline{m}}{}_{\underline{\nu}} \wedge e^{\underline{\nu}} + \omega^{\underline{m}}{}_{\underline{n}} \wedge e^{\underline{n}} \\
&= \mathrm{d}\left(e^{B(r)}\mathrm{d}y^m\right) + \omega^{\underline{m}}{}_{\underline{\nu}} \wedge e^{A(r)}\mathrm{d}x^\nu + \omega^{\underline{m}}{}_{\underline{n}} \wedge e^{B(r)}\mathrm{d}y^n
\end{aligned} \tag{3.50}
$$

In the same way as above, the middle term of (3.50) is zero independently of the others due to the orthogonality of the basis vectors. Setting this term to zero gives an equation for $\omega^{\underline{m}}{}_{\underline{\nu}}$, but due to the symmetry of $\omega$, this term has already been found in (3.49).

So, removing the middle term from (3.50) and calculating the derivative in the first term using the same process as in (3.42) gives:

$$
\begin{aligned}
e^{B(r)}\,\partial_n B(r)\ \mathrm{d}y^n \wedge \mathrm{d}y^m + \omega^{\underline{m}}{}_{\underline{n}} \wedge e^{B(r)}\mathrm{d}y^n &= 0 \\
\left(\omega^{\underline{m}}{}_{\underline{n}} - \partial_n B(r)\ \mathrm{d}y^m\right) \wedge e^{B(r)}\mathrm{d}y^n &= 0 \\
\omega^{\underline{m}}{}_{\underline{n}} - \partial_n B(r)\ \mathrm{d}y^m &= 0
\end{aligned} \tag{3.51}
$$

Now, by reinserting the vielbeins and raising the indices, we retrieve the final component of $\omega$ as written in Stelle [16, eq. 2.9].

$$
\begin{aligned}
\omega^{\underline{m}}{}_{\underline{n}} &= \partial_n B(r)\ \mathrm{d}y^m \\
&= e^{-B(r)}\,\partial_n B(r)\ e^{\underline{m}} \\
\omega^{\underline{mn}} &= \delta^{\underline{np}}\ \omega^{\underline{m}}{}_{\underline{p}} - \delta^{\underline{pm}}\ \omega^{\underline{n}}{}_{\underline{p}} \\
&= e^{-B(r)}\,\partial_n B(r)\ e^{\underline{m}} - e^{-B(r)}\,\partial_m B(r)\ e^{\underline{n}}
\end{aligned} \tag{3.52}
$$

The three independent components of the spin connection can now be collected from (3.45), (3.49) and (3.52):

$$
\begin{aligned}
\omega^{\underline{\mu\nu}} &= 0 \qquad\qquad \omega^{\underline{\mu n}} = e^{-B(r)}\,\partial_n A(r)\ e^{\underline{\mu}} \\
\omega^{\underline{mn}} &= e^{-B(r)}\,\partial_n B(r)\ e^{\underline{m}} - e^{-B(r)}\,\partial_m B(r)\ e^{\underline{n}}
\end{aligned} \tag{3.53}
$$

### 3.5.2. Finding Curvature Components

The components found in §3.5.1 above can now be used to find the components of the curvature two-form. We use the definition:

$$
R^{\underline{EF}} = \mathrm{d}\omega^{\underline{EF}} + \omega^{\underline{ED}} \wedge \omega_{\underline{D}}{}^{\underline{F}} \tag{3.54}
$$

Inserting the components of the spin connection found in (3.53) in the definition (3.54) is a long but fairly mechanical calculation, so it is not presented here. From the components of the curvature two-form, the components of the Ricci tensor can be found:

$$\begin{aligned} R_{\mu\nu} &= -\eta_{\mu\nu} e^{2(A-B)} \left( A'' + d(A')^2 + \tilde{d} A' B' + \frac{\tilde{d}+1}{r} A' \right) \\ R_{mn} &= -\delta_{mn} \left( B'' + dA'B' + \tilde{d}(B')^2 + \frac{2\tilde{d}+1}{r} B' + \frac{d}{r} A' \right) \\ &\quad - \frac{y^m y^n}{r^2} \left( \tilde{d} B'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' \right) \end{aligned} \tag{3.55}$$

where as before, the primes denote derivatives with respect to $r$. The new constant $\tilde{d} = D - d - 2$ is described in §3.5.3 below.

### 3.5.3. Elementary and Solitonic Cases

In applying the $p$-brane ansatz to the field strength $F_{[n]}$, there is a choice in how to relate the rank $n$ of $F_{[n]}$ to the dimension $d$ of the $p$-brane's worldvolume. Following Stelle [16, §2.2], we can choose to write an elementary case where the gauge potential $A_{[n-1]}$ supports a $d = n - 1$ dimensional worldvolume. Written with non-zero values of the gauge potential lying only on the worldvolume:

$$A_{\mu_1 \ldots \mu_{n-1}} = \epsilon_{\mu_1 \ldots \mu_{n-1}} e^{C(r)} \tag{3.56}$$

where the isotropicity and the required transverse symmetry are automatically supported as the function $C(r)$ is purely radial.

It is also possible to write a solotonic case where $F_{[n]}$ couples to a $\tilde{d} = D - n - 1$ dimensional worldvolume:

$$F_{m_1 \ldots m_n} = \lambda \epsilon_{m_1 \ldots m_n p} \frac{y^p}{r^{n+1}} \tag{3.57}$$

where we see that the field strength is only non-zero in the transverse directions, again guaranteeing the required $(\text{Poincaré})_d \times \mathrm{SO}\,(D - d)$ symmetry.

### 3.5.4. The $p$-Brane Equations

The Ricci tensor found above in §3.5.2, (3.55) can be contracted to the Ricci scalar using the same method as before in §2.2.3, (2.21). This definition of the curvature scalar can be inserted into the equations of motion from (3.34) to give:

$$A'' + d(A')^2 + \tilde{d} A' B' + \frac{\tilde{d}+1}{r} A' = \frac{\tilde{d}}{2(D-2)} S^2 \tag{3.58}$$

$$B'' + dA'B' + \tilde{d}(B')^2 + \frac{2\tilde{d}+1}{r} B' + \frac{d}{r} A' = -\frac{d}{2(D-2)} S^2 \tag{3.59}$$

$$\tilde{d} B'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' + \frac{1}{2}(\phi')^2 = \frac{1}{2} S^2 \tag{3.60}$$

$$\phi'' + dA'\phi' + \tilde{d} B' \phi' + \frac{\tilde{d}+1}{r} \phi' = -\frac{1}{2} \varsigma a S^2 \tag{3.61}$$

where choosing either the elementary or solitonic cases described in §3.5.3 above gives:

$$\varsigma = \begin{cases} +1 & \text{elementary} \\ -1 & \text{solitonic} \end{cases} \tag{3.62}$$

for the constant $\varsigma$ seen in (3.61).

The source term $S$ on the right hand side of these equations also depends on whether the solitonic or elementary case is being considered.

$$S = \begin{cases} \left(e^{a\phi/2 - dA + C}\right) C' & \text{elementary} \\ \lambda \left(e^{a\phi/2 - \tilde{d}B}\right) r^{-\tilde{d}-1} & \text{solitonic} \end{cases} \tag{3.63}$$

## 3.6. *p*-Brane Solutions

The $p$-brane equations found above in §3.5.4, (3.58)–(3.61) are still completely intractable. In order to make any progress towards a solution, further simplifying assumptions will be made.

### 3.6.1. Linearity Conditions

A linearity condition is chosen to link $A'$ and $B'$ such that:

$$dA' + \tilde{d}B' = 0 \tag{3.64}$$

When this is applied to the equations of motion (3.58)–(3.61), (3.58) and (3.59) become identical. This leaves three independent equations:

$$\nabla^2 \phi = -\frac{1}{2}\varsigma a S^2 \tag{3.65}$$

$$\nabla^2 A = \frac{\tilde{d}}{2(D-2)} S^2 \tag{3.66}$$

$$d(D-2)\left(A'\right)^2 + \frac{1}{2}\tilde{d}(\phi')^2 = \frac{1}{2}\tilde{d}S^2 \tag{3.67}$$

where the left-hand side of (3.65) and (3.66) has been written using the Laplacian $\nabla^2$, as defined in Box 3.5.

To further simplify (3.67), another linearity condition can be imposed which links $A'$ and $\phi'$:

$$\phi' = \frac{-\varsigma a(D-2)}{\tilde{d}} A', \qquad a^2 = \Delta - \frac{2d\tilde{d}}{(D-2)} \tag{3.68}$$

This is written in terms of the constant $a$ which quantifies the coupling of the scalar field. In (3.68), $a$ is written in terms of another constant $\Delta$. The equation of motion (3.67) can now be written:

$$\frac{\Delta(\phi')^2}{a^2} = S^2 \tag{3.69}$$

This can now be inserted into the other equation for $\phi$, (3.65) to form:

$$\nabla^2 \phi + \frac{\varsigma\Delta}{2a}(\phi')^2 = 0 \tag{3.70}$$

It will be useful to define the Laplacian, $\nabla^2\phi$ for isotropic scalar functions $\phi = \phi(r)$:

$$\nabla^2\phi = \phi'' + (\tilde{d}+1)r^{-1}\phi'$$

This form of $\nabla^2$ can be tested by returning to the familiar arena of 3 dimensional, spherically symmetric systems. A sample function $\phi = 1/r$ gives:

$$\phi = \frac{1}{r}, \qquad \phi' = \frac{-1}{r^2}, \qquad \phi'' = \frac{2}{r^3}$$

so, for a point particle with $\tilde{d} = 1$, substituting into the definition given above, we see that this is the correct form of $\nabla^2$:

$$\nabla^2\phi = \frac{2}{r^3} + \frac{2}{r}\left(\frac{-1}{r^2}\right) = 0$$

**Box 3.5:** The Laplacian

The harmonic functions are the set of functions $H$ that are solutions to Laplace's equation:

$$\nabla^2\varphi = 0$$

where $\varphi$ is any function. In our system, where we have specified spherical symmetry, the form of the Harmonic functions are know:

$$H = 1 + \frac{k}{r^d}$$

where $r$ is the radial coordinate, $k$ is a (positive) constant and $d$ counts the dimension of the spherical system.

**Box 3.6:** Harmonic Functions

### 3.6.2. Harmonic Functions

The equation of motion found above in §3.6.1, (3.70) can be rewritten in the form of a Laplace equation:

$$\nabla^2\left(e^{\frac{\varsigma\Delta}{2a}\phi}\right) = 0 \tag{3.71}$$

The solution to Laplace's equation in this form in a Harmonic function. Harmonic functions are very briefly introduced in Box 3.6 above. As we are working in a system with spherical symmetry (as determined by the $p$-brane ansatz in §3.5 above), the solution can be written:

$$e^{\frac{\varsigma\Delta}{2a}\phi} = H = 1 + \frac{k}{r^{\tilde{d}}} \tag{3.72}$$

so the ansatz (3.36) can be written:

$$\mathrm{d}s^2 = H^{\frac{-4\tilde{d}}{\Delta(D-2)}}\mathrm{d}x^\mu\mathrm{d}x^\nu\eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}}\mathrm{d}y^m\mathrm{d}y^n\delta_{mn} \tag{3.73}$$

In the electric case, the function $e^{C(r)}$ also needs to be determined using a relationship derived from (3.69) and (3.63):

$$\frac{\partial}{\partial r} e^{C(r)} = \frac{-\sqrt{\Delta}}{a} e^{\frac{-1}{2}a\phi + dA(r)} \phi' \tag{3.74}$$

which gives:

$$e^{C(r)} = \frac{2}{\sqrt{\Delta}} H^{-1} \tag{3.75}$$

### 3.6.3. Eleven Dimensional Supergravity

The solutions found above in §3.6.2 can now be applied to the supergravity system in §3.2, (3.1). This specifies the number of spacetime dimensions: $D = 11$ and also the dimensionality of the field strength: $n = 4$. Writing the field strength as a four-form gives $d = n - 1 = 3$ and $\tilde{d} = D - n - 1 = 6$.

Also, as there is no scalar field in (3.1), the scalar $\phi$ can be safely truncated by setting $a = 0$. Therefore, (3.68) can be used to find $\Delta$:

$$\Delta = a^2 + \frac{2d\tilde{d}}{(D-2)} = 0 + \frac{2 \times 3 \times 6}{11 - 2} = 4 \tag{3.76}$$

These constants can now be inserted into (3.72) and (3.73) to find the form of the functions $A(r)$, $B(r)$ and $C(r)$:

$$\begin{aligned}
e^{A(r)} &= \left(1 + \frac{k}{r^6}\right)^{-\frac{1}{3}} \\
e^{B(r)} &= \left(1 + \frac{k}{r^6}\right)^{\frac{1}{6}} &&= e^{-\frac{1}{2}A(r)} \\
e^{C(r)} &= \left(1 + \frac{k}{r^6}\right)^{-1} &&= e^{3A(r)}
\end{aligned} \tag{3.77}$$

allowing the line element (3.36) and gauge potential to be written:

$$\begin{aligned}
\mathrm{d}s^2 &= \left(1 + \frac{k}{r^6}\right)^{-\frac{2}{3}} \mathrm{d}x^\mu \mathrm{d}x^\nu \eta_{\mu\nu} + \left(1 + \frac{k}{r^6}\right)^{\frac{1}{3}} \mathrm{d}y^m \mathrm{d}y^n \delta_{mn} \\
A_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda} \left(1 + \frac{k}{r^6}\right)^{-1}
\end{aligned} \tag{3.78}$$

# 4. Sample Problem: The Orbiting Braneprobe

## 4.1. Introduction

The supergravity theory explored in §3 will now be applied to a sample problem. We shall look at a situation with one heavy brane forming a background in which a light test brane orbits. It will be assumed that the test brane or braneprobe is light enough that it does not perturb the heavy background brane at all. A classical analogy for this situation might be an electron orbiting a black hole.

In §4.2, we shall start by looking at the Nambu-Goto action, which is the action of a test brane in a gravitational background. Then, in §4.3, another term will be added to investigate electrical interaction between the test brane and the background, forming the braneprobe action.

In §4.4, the Lagrangian formalism will be used to find equations governing the orbit of the test brane. Finally, in §4.5, the properties of the braneprobe's orbit will be revealed.

## 4.2. The Nambu-Goto Action

In the more familiar territory of point particles in general relativity, equations of motion are found by minimising the length of a particle's worldline. The generalisation of this principle to higher dimensional structures such as $p$-branes gives the Nambu-Goto action:

$$I_{\mathrm{NG}} = \int \mathrm{d}^{p+1}\xi \sqrt{-\gamma} \tag{4.1}$$

Several new terms have been introduced. The integral is over coordinates $\xi^i$, which are the coordinates on the brane. The coordinates $\xi$ are embedded in the background such that $\xi^i = \xi^i(x)$ or similarly $x^\mu = x^\mu(\xi)$. There is also a new metric $\gamma_{ij}$ (written in (4.1) without indices to denote the determinant), which is the induced metric on the brane's worldvolume. $\gamma_{ij}$ is related to the spacetime metric $g_{\mu\nu}$ as:

$$\gamma_{ij}(\xi) = \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j} g_{\mu\nu}(x) \tag{4.2}$$

The conventions followed in this section for naming coordinates and indices are outlined in Box 4.1. The variables introduced in §3 will continue to be used here, so $p$ counts the dimensionality of the brane and $d = p + 1$ is the dimension of the brane's worldvolume.

In this section, we shall have two sets of coodinates:

$x^\mu$ for the background spacetime

$\xi^i$ for coordinates on the brane

The spacetime coordinates $x^\mu$ will be written with Greek indices and the brane coordinates $\xi$ will be written with Latin indices.

**Box 4.1:** Naming Coordinates

### 4.2.1. Variation of the Action

The action (4.1) can be varied to find the equations of motion:

$$\begin{aligned}\delta\left(I_{\mathrm{NG}}\right) &= \int \mathrm{d}^d\xi\ \delta\left(\sqrt{-\gamma}\right) \\ &= \int \mathrm{d}^d\xi\ \frac{-1}{2\sqrt{-\gamma}}\delta\left(\gamma\right) \\ &= \int \mathrm{d}^d\xi\ \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\delta\left(\gamma_{ij}\right)\end{aligned} \tag{4.3}$$

where the definitions in Box 2.8 have been used to vary the determinant of the metric. Proceeding by varying the definition of $\gamma_{ij}$ given in (4.2):

$$\begin{aligned}\delta\left(\gamma_{ij}\right) &= \delta\left(\partial_i x^\mu\, \partial_j x^\nu\, g_{\mu\nu}\right) \\ &= \partial_i\delta\left(x^\mu\right)\, \partial_j x^\nu\, g_{\mu\nu} + \partial_i x^\mu\, \partial_j\delta\left(x^\nu\right)\, g_{\mu\nu} + \partial_i x^\mu\, \partial_j x^\nu\, \delta\left(g_{\mu\nu}\right) \\ &= 2\partial_i\delta\left(x^\mu\right)\, \partial_j x^\nu\, g_{\mu\nu} + \partial_i x^\mu\, \partial_j x^\nu\, \delta\left(x^\lambda\right)\partial_\lambda g_{\mu\nu} \\ \gamma^{ij}\delta\left(\gamma_{ij}\right) &= 2\gamma^{ij}\partial_i\delta\left(x^\mu\right)\, \partial_j x^\nu\, g_{\mu\nu} + \gamma^{ij}\partial_i x^\lambda\, \partial_j x^\nu\, \delta\left(x^\mu\right)\partial_\mu g_{\lambda\nu}\end{aligned} \tag{4.4}$$

Inserting this under the integral in (4.3) gives:

$$\begin{aligned}\delta\left(I_{\mathrm{NG}}\right) &= \int \mathrm{d}^d\xi\ \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\delta\left(\gamma_{ij}\right) \\ &= \int \mathrm{d}^d\xi\ \left[\sqrt{-\gamma}\gamma^{ij}\partial_i\delta\left(x^\mu\right)\, \partial_j x^\nu\, g_{\mu\nu} + \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i x^\lambda\, \partial_j x^\nu\, \delta\left(x^\mu\right)\partial_\mu g_{\lambda\nu}\right] \\ &= \int \mathrm{d}^d\xi\ \left[-\partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_j x^\nu\, g_{\mu\nu}\right) + \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i x^\lambda\, \partial_j x^\nu\, \partial_\mu g_{\lambda\nu}\right]\delta\left(x^\mu\right)\end{aligned} \tag{4.5}$$

where a partial integration has been applied to the first term as explained previously in Box 3.3. Setting $\delta\left(I_{\mathrm{NG}}\right)$ to zero retrieves the equation of motion:

$$\begin{aligned}0 &= \partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_j x^\nu\, g_{\mu\nu}\right) - \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i x^\lambda\, \partial_j x^\nu\, \partial_\mu g_{\lambda\nu} \\ &= \partial_i\left(\sqrt{-\gamma}\gamma^{ij}\, \partial_j x^\mu\right) g_{\mu\nu} + \sqrt{-\gamma}\gamma^{ij}\, \partial_j x^\nu\, \partial_i x^\lambda\, \partial_\lambda g_{\mu\nu} - \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\, \partial_i x^\lambda\, \partial_\mu g_{\lambda\nu}\, \partial_j x^\nu \\ &= \partial_i\left(2\sqrt{-\gamma}\gamma^{ij}\, \partial_j x^\mu\right) g_{\mu\nu} + \sqrt{-\gamma}\gamma^{ij}\, \partial_i x^\lambda\, \partial_j x^\nu\left(2\partial_\lambda g_{\mu\nu} - \partial_\mu g_{\lambda\nu}\right) \\ &= \partial_i\left(2\sqrt{-\gamma}\gamma^{ij}\, \partial_j x^\mu\right) g_{\mu\nu}g^{\mu\alpha} + \sqrt{-\gamma}\gamma^{ij}\, \partial_i x^\lambda\, \partial_j x^\nu\, g^{\mu\alpha}\left(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu}\right)\end{aligned} \tag{4.6}$$

A Christoffel symbol (defined in Box 4.2) can be identified in the second term and the first term can be rewritten using a covariant derivative $\nabla_i$:

$$\partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_j x^\alpha\right) = \sqrt{-\gamma}\gamma^{ij}\nabla_i\partial_j x^\alpha \tag{4.7}$$

The covariant derivative is defined in Box 4.3 and the validity of the equality (4.7) is shown more thoroughly in Appendix A.2.

Now the equation of motion can be written:

$$\begin{aligned}0 &= \sqrt{-\gamma}\gamma^{ij}\, \nabla_i\partial_j x^\alpha + \sqrt{-\gamma}\gamma^{ij}\, \Gamma^\alpha_{\lambda\nu}\, \partial_i x^\lambda\, \partial_j x^\nu \\ 0 &= \nabla_i\partial_j x^\alpha + \Gamma^\alpha_{\lambda\nu}\, \partial_i x^\lambda\, \partial_j x^\nu\end{aligned} \tag{4.8}$$

Christoffel symbols or connection coefficients are used to define the covariant derivative (see Box 4.3 below). It quantifies the effects of a curved space by relating vectors in the tangent spaces of nearby points.

The Christoffel Symbol is usually defined [15] in terms of a metric $g_{\mu\nu}$ as:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left( \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right)$$

It is important to note that it is not a tensor. Despite being written using tensor notation, it does not behave as a tensor under general coodinate transforms. To reflect its status as a symbol rather than a proper tensor, it is often seen written as:

$$\Gamma^{\lambda}_{\mu\nu} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$$

**Box 4.2:** Christoffel Symbols

The covariant derivative differentiates tensors, taking into account the curvature and the dynamics of the basis vectors on which the tensor is defined. The covariant derivative of a contravariant tensor is defined as [15]:

$$\nabla_i V^j = \partial_i V^j + \Gamma^j_{ik} V^k$$

where the second term includes a Christoffel symbol $\Gamma$ as defined in Box 4.2. The covariant derivative is sometimes referred to as a semicolon derivative and written:

$$V^j_{\ ;i} = V^j_{\ ,i} + \Gamma^j_{ik} V^k$$

where the partial derivative in the first term has been written with a comma. Whilst this notion does save a little space, in this report we shall prefer to write the derivatives with $\nabla$ and $\partial$.

**Box 4.3:** The Covariant Derivative

### 4.2.2. Equations of Motion

As the equation of motion (4.8) found in §4.2.1 above describes the motion of a brane in a purely gravitational background, it could be seen as a generalisation of the geodesic equation for point particles. The geodesic equation can be written [17]:

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}t^2} + \Gamma^{\lambda}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0 \tag{4.9}$$

where in the second term the Christoffel Symbol defined in Box 4.2 is used.

It can be seen that in the limit of a brane having zero size and becoming a point particle, the covariant derivative in (4.8) tends to a simple partial derivative and the coordinates $\xi$ all disappear apart from the time, $\xi^0 = t$. Therefore, in this limit, (4.8) and (4.9) are indeed equivalent.

## 4.3. The Braneprobe Action

A term can be added to the Nambu-Goto action given in §4.2 to allow the coupling of the brane to the electromagnetic field. This term is written:

$$\mathcal{I}_{\mathcal{Q}} = \frac{Q}{(p+1)!} \int \left(\partial_{i_1} x^{\mu_1} \ldots \partial_{i_{p+1}} x^{\mu_{p+1}}\right) A_{\mu_1 \ldots \mu_{p+1}} \, \mathrm{d}\xi^{\mu_1} \wedge \ldots \wedge \mathrm{d}\xi^{\mu_{p+1}} \tag{4.10}$$

where $Q$ gives the charge on the brane and $A_{[d]}$ is the usual $d$-form gauge potential (where $d = p + 1$).

By writing this coupling term and requiring $A_{[d]}$ to support the whole of the brane's $d$ dimensional worldvolume, we have chosen to use the elementary case from §3.5.3.

This coupling term (4.10) is added to the term in the Nambu-Goto action (4.1) to give the braneprobe action:

$$\begin{aligned} \mathcal{I}_{\mathrm{B}} &= \mathcal{I}_{\mathrm{NG}} + \mathcal{I}_{\mathcal{Q}} \\ &= T \int \mathrm{d}^d\xi \sqrt{-\gamma} + \frac{Q}{d!} \int \left(\partial_{i_1} x^{\mu_1} \ldots \partial_{i_d} x^{\mu_d}\right) A_{\mu_1 \ldots \mu_d} \, \mathrm{d}\xi^{\mu_1} \wedge \ldots \wedge \mathrm{d}\xi^{\mu_d} \end{aligned} \tag{4.11}$$

as seen in Stelle [16, eq. 7.11]. The constant $T$ is added to the Nambu-Goto action as a generalised unit tension with units [energy]/[length]$^p$.

This constant would be a fundamental parameter of the theory governing the amplitude and frequency of the vibrations on the brane. However, in this project, we are assuming that the branes are not vibrating so the constant $T$ will be largely ignored.

In Stelle [16, eq. 7.11], there is another term in the braneprobe action of the form:

$$e^{\frac{1}{2}\varsigma a \cdot \phi} \tag{4.12}$$

which describes the coupling of the electromagnetic field to the scalar (or dilaton) field. For the purposes of this report we will simply ignore this term.

### 4.3.1. Variation of the Action

The action (4.11) can now be varied to find the equations of motion. The gravitational term from the Nambu-Goto action and the coupling term can be dealt with separately:

$$\delta\left(\mathcal{I}_{\mathrm{B}}\right) = \delta\left(\mathcal{I}_{\mathrm{NG}}\right) + \delta\left(\mathcal{I}_{\mathcal{Q}}\right) \tag{4.13}$$

allowing the result for the first term to be taken directly from the result of the variation of the Nambu-Goto action in §4.2.1, (4.8):

$$\delta\left(\mathcal{I}_{\mathrm{NG}}\right) = -T \int \mathrm{d}^d\xi \, \sqrt{-\gamma}\gamma^{ij} g_{\mu\alpha} \left(\nabla_i \partial_j x^\alpha + \Gamma^\alpha_{\lambda\nu} \, \partial_i x^\lambda \, \partial_j x^\nu\right) \delta\left(x^\mu\right) \tag{4.14}$$

The coupling term is fairly simple to vary, but it is important to note that the gauge potential $A_{[d]} = A_{[d]}(x)$ and so also needs to be included in the variation. Usually, the $\xi$ terms would also have to be included, as $\xi = \xi(x)$, but as we are assuming that the brane is flat (see §3.5) these terms can be safely ignored.

Removing the $\xi$ terms but taking care to remain faithful to the antisymmetric wedge product allows the coupling term (4.10) to be rewritten as:

$$\delta\left(\mathcal{I}_Q\right) = \frac{Q}{d!}\int \mathrm{d}^d\xi\, \epsilon^{i_1\ldots i_d}\, \delta\left(\partial_{i_1}x^{\mu_1}\ldots\partial_{i_d}x^{\mu_d}\, A_{\mu_1\ldots\mu_d}\right) \tag{4.15}$$

Applying the variational derivative using the product rule to the $d$-fold product of $x^\mu$ gives $d$ terms and one extra term is picked up from the variation of $A_{[d]}(x)$:

$$\begin{aligned}
\delta\left(\mathcal{I}_Q\right) = \frac{Q}{d!}\int \mathrm{d}^d\xi\, \epsilon^{i_1\ldots i_d}\, \Big[&\partial_{i_1}x^{\mu_1}\ldots\partial_{i_d}x^{\mu_d}\,\delta\left(A_{\mu_1\ldots\mu_d}\right) \\
&- \partial_{i_1}x^{\mu_1}\ldots\partial_{i_d}x^{\mu_d}\,\left(\partial_{\mu_1}A_{\sigma\mu_2\ldots\mu_d} + \partial_{\mu_2}A_{\mu_1\sigma\mu_3\ldots\mu_d} + \cdots + \partial_{\mu_d}A_{\mu_1\ldots\mu_{d-1}\sigma}\right)\delta\left(x^\sigma\right)\Big] \\
= \frac{Q}{d!}\int \mathrm{d}^d\xi\, \epsilon^{i_1\ldots i_d}\,&\partial_{i_1}x^{\mu_1}\ldots\partial_{i_d}x^{\mu_d}\,\partial_{[\sigma}A_{\mu_1\ldots\mu_d]}\,\delta\left(x^\sigma\right) \\
= \frac{Q}{d!}\int \mathrm{d}^d\xi\, \epsilon^{i_1\ldots i_d}\,&\partial_{i_1}x^{\mu_1}\ldots\partial_{i_d}x^{\mu_d}\,F_{\sigma\mu_1\ldots\mu_d}\,\delta\left(x^\sigma\right)
\end{aligned} \tag{4.16}$$

where the $d$-fold sum of all the differentials of $A_{[d]}$ in the first line are written simply into the antisymmetrising bracket on the indices in the second line. Finally, this antisymmetric derivative of the gauge potential is rewritten simply as the field strength $F$.

The contributions from the two terms $\delta\left(\mathcal{I}_{\mathrm{NG}}\right)$ and $\delta\left(\mathcal{I}_Q\right)$ can be combined to give the equations of motion in §4.3.2.

### 4.3.2. Equations of Motion

The variation performed in §4.2.1 and §4.3.1 allows the equations of motion for the braneprobe to be written:

$$T\sqrt{-\gamma}\gamma^{ij}\left[\nabla_i\partial_j x^\alpha + \Gamma^\alpha_{\mu\nu}\,\partial_i x^\mu\,\partial_j x^\nu\right] = \frac{Q}{d!}\epsilon^{i_1\ldots i_d}\,\partial_{i_1}x^{\mu_1}\ldots\partial_{i_d}x^{\mu_d}\,F^\sigma{}_{\mu_1\ldots\mu_d} \tag{4.17}$$

In §4.2.2, the left-hand side of this equation was compared to the geodesic equation for a point particle (4.9). The right-hand side can be compared to the case of a point particle in a simple magnetic field:

$$f = q\dot{x}\times B \tag{4.18}$$

where the force $f$ is equal to the charge times the cross product of the particle's velocity $\dot{x}$ with the magnetic force $B$. In the equation of motion (4.17) we see the generalised charge $Q$ multiplying a antisymmetrised product of the coordinate differentials $\partial_i x^\mu$ with the field strength $F$.

Solving these equations of motion directly is possible, but is not straightforward. Therefore, in §4.4 below, the Lagrangian formulation is used to investigate the properties of the braneprobe's orbit. A valuable extension to this project would be to confirm these results by direct solutions of the equations of motion.

## 4.4. The Lagrangian Approach

We shall use the Lagrangian formulation to find the conjugate momenta via the relation:

$$p_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{4.19}$$

The Lagrangian for our braneprobe action can be read directly from (4.11):

$$\mathcal{L} = T\sqrt{-\gamma} + \frac{Q}{d!}\,\epsilon^{i_1 \ldots i_d}\,\partial_{i_1} x^{\mu_1} \ldots \partial_{i_d} x^{\mu_d}\, A_{\mu_1 \ldots \mu_d} \tag{4.20}$$

To simplify the situation in our sample problem, it will be assumed that the test brane is uncharged. Thus $Q = 0$, reducing the Lagrangian to:

$$\mathcal{L} = T\sqrt{-\gamma} \tag{4.21}$$

Choosing this uncharged case simplifies the calculations substantially. In future work, the charged case should also be analysed.

As seen in (4.19), in order to find the conjugate momenta, we need to take the differential of the Lagrangian $\mathcal{L}$:

$$\begin{aligned} \partial \mathcal{L} &= T\,\partial(\sqrt{-\gamma}) \\ &= T\,\frac{-1}{2\sqrt{-\gamma}}\partial(\gamma) \\ &= \frac{1}{2}T\sqrt{-\gamma}\gamma^{ij}\,\partial(\gamma_{ij}) \end{aligned} \tag{4.22}$$

where as before, the definitions in Box 2.8 have been used to vary the determinant of the metric. To proceed with this expression, some further simplifications are required in the form of gauge and coordinate choices. This will give an expression for $\gamma_{ij}$ which can then be inserted into (4.22).

### 4.4.1. Gauge and Coordinate Choices

As in §3.5, the coordinate range will be split into the directions on the brane's worldvolume and directions transverse to the worldvolume:

$$x^\mu = (x^{\underline{\mu}}, y^{\underline{m}}) \tag{4.23}$$

where the underlined Greek and Latin coordinates have ranges:

$$\begin{aligned} \underline{\mu}, \underline{\nu}, \ldots &= 0 \ldots p \\ \underline{m}, \underline{n}, \ldots &= d \ldots (D-1) \end{aligned}$$

A gauge choice will be made such that the spatial parts of the two sets of coordinates $x^{\underline{\mu}}$ and $\xi^i$ are equal:

$$x^{\underline{\mu}} = \xi^i \qquad \text{for} \qquad \underline{\mu}, \xi = 1, \ldots, p \tag{4.24}$$

The time components $x^0$ and $\xi^0$ will not be set equal, and we shall label them explicitly as $t$ and $\tau$. $t$ is the proper time, and $\tau$ is the time kept by the emperor's heartbeat as he rides around on his brane.

$$\begin{aligned} x^0 &= t \\ \xi^0 &= \tau \end{aligned} \tag{4.25}$$

To simplify the situation, it will be assumed that the braneprobe's orbit is in a single plane. Then, by choosing to work in polar coordinates, all of the transverse coordinates can be set to zero except two which shall be called $r$ and $\theta$:

$$\begin{aligned} y^d &= r \\ y^{d+1} &= \theta \\ y^{\tilde{m}} &= 0, \qquad \tilde{m} = (d+2), \ldots, D-1 \end{aligned} \tag{4.26}$$

As before in §3.5, (3.36), the spacetime metric $g_{\mu\nu}$ will be split and is defined by the line element:

$$\begin{aligned} \mathrm{d}s^2 &= e^{2A(r)} \mathrm{d}x^{\underline{\mu}} \mathrm{d}x^{\underline{\nu}} \eta_{\underline{\mu}\underline{\nu}} + e^{2B(r)} \mathrm{d}y^{\underline{m}} \mathrm{d}y^{\underline{n}} \delta_{\underline{mn}} \\ &= e^{2A(r)} \mathrm{d}x^{\underline{\mu}} \mathrm{d}x^{\underline{\nu}} \eta_{\underline{\mu}\underline{\nu}} + e^{2B(r)} \left( \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \right) \end{aligned} \tag{4.27}$$

where the functions $A(r)$ and $B(r)$ are as defined in §3.5. This definition of the spacetime metric can now be used to find the components of the induced metric.

### 4.4.2. Components of the Induced Metric $\gamma_{ij}$

With the definition of the spacetime metric above in (4.27), the relationship between the spacetime metric and the induced metric given in (4.2) can be used:

$$\begin{aligned} \gamma_{ij} &= \partial_i x^\mu \; \partial_j x^\nu \; g_{\mu\nu} \\ &= e^{2A(r)} \; \partial_i x^{\underline{\mu}} \; \partial_j x^{\underline{\nu}} \; \eta_{\underline{\mu}\underline{\nu}} + e^{2B(r)} \left( \partial_i r \, \partial_j r + r^2 \, \partial_i \theta \, \partial_j \theta \right) \end{aligned} \tag{4.28}$$

where as before, $\partial_i$ is written as shorthand for $\partial / \partial \xi^i$.

As noted above in (4.25), the time components $x^0$ and $\xi^0$ are treated differently to the spatial components. To separate these, we introduce some new indices $\underline{\tilde{\mu}}$ that range only over the spatial part of the worldvolume:

$$\left\{ 0, \underline{\tilde{\mu}} \right\} = \underline{\mu} \qquad \Rightarrow \qquad \underline{\tilde{\mu}} = 1, \ldots, p \tag{4.29}$$

With these new indices, the sums on $\underline{\mu}$ and $\underline{\nu}$ in (4.28) can be written as:

$$\gamma_{ij} = -e^{2A(r)} \; \partial_i t \; \partial_j t + e^{2A(r)} \; \partial_i x^{\underline{\tilde{\mu}}} \; \partial_j x^{\underline{\tilde{\nu}}} \; \delta_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}} + e^{2B(r)} \left( \partial_i r \, \partial_j r + r^2 \, \partial_i \theta \, \partial_j \theta \right) \tag{4.30}$$

where the components of $\eta_{\underline{\mu}\underline{\nu}}$ have been entered:

$$\eta_{00} = -1 \qquad \text{and} \qquad \eta_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}} = \delta_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}} \tag{4.31}$$

The $\gamma_{00}$ component can be written out explicitly:

$$\gamma_{00} = -e^{2A(r)}\,\dot{t}^2 + e^{2A(r)}\,\dot{x}^{\underline{\tilde{\mu}}}\,\dot{x}^{\underline{\tilde{\nu}}}\,\delta_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}} + e^{2B(r)}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{4.32}$$

where the dotted terms (e.g. $\dot{t}$, $\dot{x}$) are derivatives with respect to the proper time $\tau$.

We can now introduce another simplification by setting $\dot{x}^{\underline{\tilde{\mu}}} = 0$. The physical interpretation of this is that the test brane is static. It does not vibrate or rotate, but remains flat and parallel to the heavy background brane. The $\gamma_{00}$ component can therefore be written:

$$\gamma_{00} = -e^{2A(r)}\,\dot{t}^2 + e^{2B(r)}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{4.33}$$

To look at the other components of $\gamma_{ij}$ another set of coordinates is introduced. In the same way as (4.29), coordinates $\tilde{i}$ are defined:

$$\left\{0, \tilde{i}\right\} = i \qquad \Rightarrow \qquad \tilde{i} = 1, \ldots, p \tag{4.34}$$

and the corresponding components of $\gamma_{ij}$ are written:

$$\begin{aligned}
\gamma_{\tilde{i}\tilde{j}} &= -e^{2A(r)}\,\partial_{\tilde{i}}t\,\partial_{\tilde{j}}t + e^{2A(r)}\,\partial_{\tilde{i}}x^{\underline{\tilde{\mu}}}\,\partial_{\tilde{j}}x^{\underline{\tilde{\nu}}}\,\delta_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}} + e^{2B(r)}\left(\partial_{\tilde{i}}r\,\partial_{\tilde{j}}r + r^2\,\partial_{\tilde{i}}\theta\,\partial_{\tilde{j}}\theta\right) \\
&= 0 + e^{2A(r)}\,\frac{\partial\xi^{\tilde{k}}}{\partial\xi^{\tilde{i}}}\,\frac{\partial\xi^{\tilde{l}}}{\partial\xi^{\tilde{j}}}\,\delta_{\tilde{k}\tilde{l}} + 0 \\
&= e^{2A(r)}\,\delta^{\tilde{k}}_{\tilde{i}}\,\delta^{\tilde{l}}_{\tilde{j}}\,\delta_{\tilde{k}\tilde{l}} \\
&= e^{2A(r)}\,\delta_{\tilde{i}\tilde{j}}
\end{aligned} \tag{4.35}$$

where all the differentials $\partial_{\tilde{i}}t$, $\partial_{\tilde{i}}r$ and $\partial_{\tilde{i}}\theta$ are zero due to the independence of the directions on and transverse to the worldvolume. Also the gauge choice (4.24) has been used to relate the coordinates $x$ and $\xi$.

The induced metric $\gamma_{ij}$ can mow be written:

$$\gamma_{ij} = e^{2A(r)}\,\mathrm{diag}\left(\frac{\gamma_{00}}{e^{2A(r)}}, 1, 1, \ldots, 1\right) \tag{4.36}$$

with the component $\gamma_{00}$ given in (4.33) above. The determinant $\gamma$ of the induced metric can be seen from (4.36):

$$\gamma = \gamma_{00}\,e^{2pA(r)} \tag{4.37}$$

and differential of $\gamma_{ij}$ will also be useful:

$$\begin{aligned}
\partial\gamma_{ij} &= \partial\left(e^{2A(r)}\,\mathrm{diag}\left(\frac{\gamma_{00}}{e^{2A(r)}}, 1, 1, \ldots, 1\right)\right) \\
&= \partial(\gamma_{00})
\end{aligned} \tag{4.38}$$

where all the $e^{2A(r)}$ are constant, so they do not contribute.

These relations found in (4.36), (4.37) and (4.38) can now be used to find expressions for the momenta, see §4.4.3 below.

### 4.4.3. Conjugate Momenta

As the motion of the test brane has been compressed into a single plane, we are left with just three components of the relativistic momentum vector: the linear and angular momenta and the energy. To find these, the components of $\gamma_{ij}$ specified above in §4.4.2 are inserted into the differential of the Lagrangian (4.22):

$$\begin{aligned}\partial\mathcal{L} &= \frac{1}{2}T\sqrt{-\gamma}\gamma^{ij}\,\partial(\gamma_{ij})\\ &= \frac{1}{2}T\sqrt{-\gamma}\gamma^{00}\,\partial(\gamma_{00})\\ &= T\frac{e^{2pA(r)}}{2\sqrt{-\gamma}}\partial\Big(-e^{2A(r)}\,\dot{t}^2 + e^{2B(r)}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\Big)\end{aligned} \tag{4.39}$$

where the $\gamma_{00}$ given in (4.33) has also been used.

The **angular momentum** $p_\theta$ is found as:

$$\begin{aligned}p_\theta &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}}\\ &= T\frac{e^{2pA(r)}}{2\sqrt{-\gamma}}\,2e^{2B(r)}r^2\dot{\theta}\\ &= T\frac{e^{2(pA(r)+B(r))}}{\sqrt{-\gamma}}r^2\dot{\theta}\end{aligned} \tag{4.40}$$

which is similar to the form of the classical expression for angular momentum: $L = mr^2\dot{\theta}$. Also, as $\dot{\theta}$ rather than $\theta$ appears in the Lagrangian, it follows that the angular momentum is a conserved quantity.

The **linear momentum** $p_r$ is found similarly as:

$$\begin{aligned}p_r &= \frac{\partial\mathcal{L}}{\partial\dot{r}}\\ &= T\frac{e^{2pA(r)}}{2\sqrt{-\gamma}}\,2e^{2B(r)}\dot{r}\\ &= T\frac{e^{2(pA(r)+B(r))}}{\sqrt{-\gamma}}\dot{r}\end{aligned} \tag{4.41}$$

which is similar to the form of the classical expression: $p = m\dot{r}$.

The **energy** $E$ is given by:

$$\begin{aligned}E = -p_t &= -\frac{\partial\mathcal{L}}{\partial\dot{t}}\\ &= -T\frac{e^{2pA(r)}}{2\sqrt{-\gamma}}\left(-2e^{2A(r)}\dot{t}\right)\\ &= T\frac{e^{2(p+1)A(r)}}{\sqrt{-\gamma}}\dot{t}\end{aligned} \tag{4.42}$$

There is no classical analogy that can be drawn in this case, but as the Lagrangian is independent of $\dot{t}$, it follows that energy is also a conserved quantity of the system.

## 4.5. Braneprobe Orbits

### 4.5.1. The Mass-Shell Condition

To proceed and find an equation of radial motion for the braneprobe, we shall use the Mass-Shell condition:

$$\mu^2 = p_a p_b g^{ab} \tag{4.43}$$

where the indices $a, b$ now sum over the different momentum components, which in this case is $t$, $r$ and $\theta$. The form of the line element in (4.27) allows the spacetime metric $g_{ab}$ to be written:

$$g_{ab} = \mathrm{diag}\left(-e^{2A(r)},\ e^{2B(r)},\ r^2 e^{2B(r)}\right) \tag{4.44}$$

Inserting the components of momentum found in (4.40), (4.41) and (4.42) above gives an expression for the rest mass $\mu$:

$$\begin{aligned}
\mu^2 &= -e^{-2A(r)} p_t^2 + e^{-2B(r)} \left(p_r^2 + r^{-2} p_\theta^2\right) \\
&= \left(\frac{T}{\sqrt{-\gamma}}\right)^2 \left( - e^{-2A(r)} e^{4(p+1)A(r)} \dot{t}^2 + e^{-2B(r)} e^{4(pA(r)+B(r))} \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \right) \\
&= e^{4pA(r)} \left(\frac{T}{\sqrt{-\gamma}}\right)^2 \left( - e^{2A(r)} \dot{t}^2 + e^{2B(r)} \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \right) \\
&= \gamma_{00} \left(e^{2pA(r)} \frac{T}{\sqrt{-\gamma}}\right)^2 \\
&= e^{4pA(r)} \frac{T^2}{-\gamma} \frac{\gamma}{e^{2pA(r)}} \\
&= -e^{2pA(r)} T^2
\end{aligned} \tag{4.45}$$

where the $\gamma_{00}$ component was recognised from (4.32), in the last step, the definition (4.37) of the determinant $\gamma$ was used.

This definition (4.45) of the of the rest mass can be insrted back into the mass-shell condition with the constants of motion $L = p_\theta$ and $E = p_t$ to give:

$$\begin{aligned}
0 &= \mu^2 + e^{-2A(r)} p_t^2 - e^{-2B(r)} \left(p_r^2 + r^{-2} p_\theta^2\right) \\
&= -e^{2pA(r)} T^2 + e^{-2A(r)} E^2 - e^{-2B(r)} \left( T^2 \frac{e^{4(pA(r)+B(r))}}{-\gamma} \dot{r}^2 + \frac{L^2}{r^2} \right) \\
&= -e^{2pA(r)} T^2 + e^{-2A(r)} E^2 - T^2 \frac{e^{2(pA(r)+B(r))}}{-\gamma_{00}} \dot{r}^2 - e^{-2B(r)} \frac{L^2}{r^2}
\end{aligned} \tag{4.46}$$

This links $r$ and $\dot{r}$, forming an equation of radial motion. To make any further progress, a specific case from §3.5 will have to be chosen. This will give a value for the constant $p$ and expressions for the functions $A(r)$ and $B(r)$ which can be inserted to allow the properties of the motion to be studied.

### 4.5.2. An Electric Two-Brane

We shall choose to look at the specific case of a two-brane with electric type charge. This gives the value $p = 2$ and expressions for $A(r)$ and $B(r)$ can be taken from §3.6.3:

$$e^{A(r)} = \left(1 + \frac{k}{r^6}\right)^{-\frac{1}{3}}, \qquad A(r) = -2B(r) \tag{4.47}$$

To find a specific expression for $\gamma_{00}$, these definitions (4.47) are inserted into (4.33). Also, the $\dot{t}$ and $\dot{\theta}$ in (4.33) are rewritten in terms of the constants $E$ and $L$ using the expressions (4.42) and (4.40). This gives:

$$\begin{aligned}\gamma_{00} &= e^{-6A(r)}\frac{E^2}{T^2}\gamma_{00} + e^{-A(r)}\left(\dot{r}^2 - r^2\frac{L^2}{T^2r^4}\gamma_{00}e^{-2A(r)}\right)\\ &= e^{-A(r)}\dot{r}^2\left(1 + e^{-3A(r)}\frac{L^2}{T^2r^2} - e^{-6A(r)}\frac{E^2}{T^2}\right)^{-1}\end{aligned} \tag{4.48}$$

This $\gamma_{00}$ term can then be inserted into (4.46) to give:

$$\begin{aligned}\dot{r}^4 &= \left(e^{2A(r)} - e^{-4A(r)}\frac{E^2}{T^2} + e^{-A(r)}\frac{1}{r^2}\frac{L^2}{T^2}\right)\left(1 - e^{-6A(r)}\frac{E^2}{T^2} + e^{-3A(r)}\frac{1}{r^2}\frac{L^2}{T^2}\right)\\ \dot{r}^2 &= e^{A(r)}\left(e^{-6A(r)}\frac{E^2}{T^2} - 1 - e^{-3A(r)}\frac{1}{r^2}\frac{L^2}{T^2}\right)\end{aligned} \tag{4.49}$$

In taking the square root to retrieve $\dot{r}^2$ from $\dot{r}^4$, the positive roots have been chosen to be physically meaningful. This is explained further below.

To find the properties of the orbit (following Misner, Thorne & Wheeler [17]), we look at the behaviour of $\dot{r}^2$ at large $r$. The function $e^{A(r)}$ tends simply to 1:

$$\lim_{r\to\infty}\left[\left(1 + \frac{k}{r^6}\right)^{-\frac{1}{3}}\right] = 1 \tag{4.50}$$

which then can be used to find:

$$\lim_{r\to\infty}\dot{r}^2 = \frac{E^2}{T^2} - 1 \tag{4.51}$$

Therefore, for $\left(E^2/T^2\right) > 1$ we have open or unbound orbits and for $\left(E^2/T^2\right) < 1$ we have closed or bound orbits.

Choosing negative roots in (4.49) above would have produced an unphysical result: a test brane with large energy would be captured but one with small energy would escape. By choosing the positive sign in (4.49), we have a result that satisfies physical intuition: a test brane with large energy is not captured, but one with small energy forms bound orbits.

# 5. Conclusions

In §2, Kaluza-Klein theory is investigated. The success of the theory is seen in §2.2 as electromagnetism is found as a consequence of pure gravity in higher dimensions. However, in §2.3, the theory's failings are also seen. An extra scalar term is found which conflicts with experiment.

Some of the mathematics and ideas of Kaluza-Klein theory are used in §3 to outline the basics of supergravity. A toy system is constructed as a truncation of the full action integral in §3.4, and this integral is varied to find equations of motion. Simplifications are introduced in §3.5 to allow these equations to be solved in certain specific cases.

In §4, the sample problem of the braneprobe is introduced. In §4.2 and §4.3 various action integrals are considered that describe the motion of a test brane in gravitational and electromagnetic backgrounds. For simplicity, we choose to only look at motion confined to a single plane. After choosing an appropriate gauge and system of coordinates, these action integrals are then varied to find equations of motion. These equations could be solved directly to retrieve equations governing the orbit of the test brane.

However, in §4.4, the Lagrangian formalism is used instead to find the energy and angular momentum of as uncharged test brane. These constants are then inserted into the mass-shell condition to find an equation describing the radial motion of the brane. A possible extension to this project would be to use the equations of motion to independently derive this equation of radial motion and check that it is consistent. A further check for consistency would be to take time derivatives of the energy and angular momentum and confirm that they are indeed constant.

In §4.5 the equation of radial motion is further investigated and limits are taken to determine the nature of the test brane's orbit. It was found that the test brane can form both open and closed orbits depending on the amount of energy it is carrying. The threshold energy is found to be $E^2/T^2 = 1$.

The only feature of the test brane's orbit that was properly investigated was whether it was open or closed. A more thorough investigation should also look at the effects of angular momentum on the shape of the orbit, perhaps hoping to find a relationship with the eccentricity of the orbit. In this way, one could check whether supergravity reduces to an appropriate Newtonian limit.

In this report, only uncharged test branes are considered. A simple extension would be to also investigate the charged case and see how different amounts of charge affect the motion. It would also be instructive to generalise the results to branes of different dimensionality.

# Appendices

# A. Extended Calculations

## A.1. Rewriting the Supergravity Action

In §3.3.1, when rewriting the supergravity action (3.1) in form language, it is necessary to show the equivalence of:

$$-\sqrt{-g}\frac{1}{48}\int \mathrm{d}^{11}x F_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4} = -\frac{1}{2}\int F_{[n]}\wedge{}^\star F_{[n]} \tag{A.1}$$

Starting with the definition of a form:

$$T_{[n]} = \frac{1}{n!}T_{\mu_1\mu_2\ldots\mu_n}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\ldots\wedge\mathrm{d}x^{\mu_n} \tag{A.2}$$

the product $F_{[n]}\wedge{}^\star F_{[n]}$ can be written as:

$$\begin{aligned}
F_{[n]}\wedge{}^\star F_{[n]} &= \frac{1}{4!}F_{\mu_1\ldots\mu_4}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_4}\left({}^\star\left[\frac{1}{4!}F_{\mu_1\ldots\mu_4}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_4}\right]\right)\\
&= \frac{1}{4!\,4!\,7!}F_{\mu_1\ldots\mu_4}F^{\nu_1\ldots\nu_4}\varepsilon_{\nu_1\ldots\nu_4\mu_5\mu_11}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_4}\wedge\mathrm{d}x^{\nu_5}\wedge\ldots\wedge\mathrm{d}x^{\nu_{11}}\\
&= H_{[11]}
\end{aligned} \tag{A.3}$$

where the definition of the Hodge dual given in Box 3.2 has been used and $H_{[11]}$ is an arbitrary 11 form:

$$H_{[11]} = \frac{1}{11!}H_{\mu_1\ldots\mu_{11}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{11}} \tag{A.4}$$

The components $H_{\mu_1\ldots\mu_{11}}$ can be written as:

$$H_{\mu_1\ldots\mu_{11}} = \frac{11!}{4!\,4!\,7!}F_{\mu_1\ldots\mu_4}F^{\nu_1\ldots\nu_4}\varepsilon_{\nu_1\ldots\nu_4\mu_5\mu_{11}} \tag{A.5}$$

so:

$$\begin{aligned}
F_{[n]}\wedge{}^\star F_{[n]} &= \int H_{[11]}\\
&= \frac{1}{11!}\int \mathrm{d}^{11}x\ \epsilon^{\mu_1\ldots\mu_{11}}H_{\mu_1\ldots\mu_{11}}\\
&= \frac{11!}{4!\ 4!\ 11!\ 7!}\int \mathrm{d}^{11}x\,(-\sqrt{-g})\ \epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F^{\nu_1\ldots\nu_4}\epsilon_{\nu_1\ldots\nu_4\mu_5\mu_{11}}\\
&= \frac{\sqrt{-g}}{4!\ 4!\ 7!}\int \mathrm{d}^{11}x\ 4!\ \ 7!\ \ \delta^{\mu_1}_{\nu_1}\delta^{\mu_3}_{\nu_3}\delta^{\mu_3}_{\nu_3}\delta^{\mu_4}_{\nu_4}\ F_{\mu_1\ldots\mu_4}F^{\nu_1\ldots\nu_4}\\
&= \frac{\sqrt{-g}}{4!}\int \mathrm{d}^{11}x\ \delta^{\mu_1}_{\nu_1}\delta^{\mu_3}_{\nu_3}\delta^{\mu_3}_{\nu_3}\delta^{\mu_4}_{\nu_4}\ F_{\mu_1\ldots\mu_4}F^{\nu_1\ldots\nu_4}\\
&= \frac{\sqrt{-g}}{24}\int \mathrm{d}^{11}x F_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4}\\
\frac{1}{2}F_{[n]}\wedge{}^\star F_{[n]} &= \frac{\sqrt{-g}}{48}\int \mathrm{d}^{11}x F_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4} \qquad \square
\end{aligned} \tag{A.6}$$

## A.2. The Covariant Derivative on an Induced Metric

In §4.2.1, (4.7), the covariant derivative on an induced metric is used to simplify the equation of motion:

$$\partial_i \left(\sqrt{-\gamma}\gamma^{ij}\partial_j x^\alpha\right) = \sqrt{-\gamma}\gamma^{ij}\nabla_i\partial_j x^\alpha \tag{A.7}$$

To show this equality, we start from the definition of a covariant derivative acting on a contravariant tensor as given in Box 4.3.

$$\nabla_i \zeta^j = \partial_i \zeta^j + \Gamma^j_{ik}\,\zeta^k \tag{A.8}$$

here, it is important to note that $\Gamma^j_{ik}$ is the Christoffel connection of the induced metric $\gamma_{ij}$ rather than the spacetime metric $g_{\mu\nu}$. In this calculation, we use the same index conventions as in §4, described in Box 4.1.

Inserting the vector $(\sqrt{-\gamma}\gamma^{ij}\partial_j x^\alpha)$ as $\zeta^i$ in the definition (A.8) gives:

$$\nabla_a \left(\sqrt{-\gamma}\gamma^{bj}\partial_j x^\alpha\right) = \partial_a \left(\sqrt{-\gamma}\gamma^{bj}\partial_j x^\alpha\right) + \Gamma^b_{ac}\sqrt{-\gamma}\gamma^{cj}\partial_j x^\alpha \tag{A.9}$$

and expanding the covariant derivative on the left-hand side using the product rule:

$$\begin{aligned}\nabla_a \left(\sqrt{-\gamma}\gamma^{bj}\partial_j x^\alpha\right) &= \left(\nabla_a\sqrt{-\gamma}\right)\gamma^{bj}\partial_j x^\alpha + \sqrt{-\gamma}\left(\nabla_a\gamma^{bj}\right)\partial_j x^\alpha + \sqrt{-\gamma}\gamma^{bj}\left(\nabla_a\partial_j x^\alpha\right)\\ &= \left(\partial_a\sqrt{-\gamma}\right)\gamma^{bj}\partial_j x^\alpha + \sqrt{-\gamma}\gamma^{bj}\left(\nabla_a\partial_j x^\alpha\right)\\ &= \Gamma^i_{ia}\sqrt{-\gamma}\gamma^{bj}\partial_j x^\alpha + \sqrt{-\gamma}\gamma^{bj}\left(\nabla_a\partial_j x^\alpha\right)\end{aligned} \tag{A.10}$$

where one of the defining identities of the Christoffel symbol has been used:

$$\Gamma^i_{ij} = \frac{1}{\sqrt{-\gamma}}\partial_j\sqrt{-\gamma} \tag{A.11}$$

Now inserting (A.10) back into (A.9) and setting the indices $a, b \to i$ gives:

$$\partial_i \left(\sqrt{-\gamma}\gamma^{ij}\partial_j x^\alpha\right) = \sqrt{-\gamma}\gamma^{ij}\,\nabla_i\partial_j x^\alpha \tag{A.12}$$

as required.

# B. Cadabra

## B.1. Introducing Cadabra

Cadabra is a software package for computer algebra written by Kasper Peeters [19, 20]. Cadabra is specifically designed for use in field theory problems and tensor calculus. In this document, we will show cadabra's input and output as follows:

```
Input to Cadabra
```

Output from Cadabra.

In §B.2, Cadabra is used to dimensionally reduce the Ricci Scalar using the process:

$$\hat{g}_{\mu\nu} \quad \rightarrow \quad \Gamma^{\lambda}_{\mu\nu} \quad \rightarrow \quad \hat{R}$$

Cadabra was also used to check many of the other calculations included in this report.

## B.2. Dimensional Reduction in Kaluza-Klein Gravity

### B.2.1. Initialisation

We start by defining two sets of indices: Greek for the full worldvolume and Latin for the flat subspace.

```
{\mu,\nu,\rho,\sigma,\kappa,\lambda,\eta,\chi#}::Indices(full, position=independent).
{m,n,p,q,r,s,t,u,v,m#}::Indices(subspace, position=independent, parent=full).
```

Assigning list property Indices to $\{\mu,\ \nu,\ \rho,\ \sigma,\ \kappa,\ \lambda,\ \eta,\ \chi\#\}$.
Assigning list property Indices to $\{m,\ n,\ p,\ q,\ r,\ s,\ t,\ u,\ v,\ m\#\}$.

We also define some other useful objects: a derivative ($\partial$), metrics $g$ for the $(D+1)$ worldvolume and $h$ for the flat space and a Kronecker Delta ($\delta$). Also, we define the field strength $F$ to be anti symmetric.

```
\partial{#}::PartialDerivative.
{g_{\mu \nu}, h_{m n}}::Metric.
{g^{\mu \nu}, h^{m n}}::InverseMetric.
{g_{\mu? \nu?}, g^{\mu? \nu?}, h_{m n}, h^{m n}}::Symmetric.
{\delta^{\mu?}_{\nu?}, \delta_{\mu?}^{\nu?}}::KroneckerDelta.
F_{m n}::AntiSymmetric.
```

Assigning property PartialDerivative to $\partial\#$.
Assigning property Metric to $g_{\mu\nu}$, $h_{mn}$.
Assigning property InverseMetric to $g^{\mu\nu}$, $h^{mn}$.
Assigning property Symmetric to $g_{\mu?\nu?}$, $g^{\mu?\nu?}$, $h_{mn}$, $h^{mn}$.
Assigning property KroneckerDelta to $\delta^{\mu?}{}_{\nu?}$, $\delta_{\mu?}{}^{\nu?}$.
Assigning property AntiSymmetric to $F_{mn}$.

### B.2.2. Expanding the Riemann Tensor

We will proceed by expanding the Riemann tensor in terms of the metric. We will convert Riemann tensor to Christoffel symbols with:

$$R^{\lambda}_{\mu\nu\kappa} = -\partial_{\kappa}\Gamma^{\lambda}_{\mu\nu} + \partial_{\nu}\Gamma^{\lambda}\mu\kappa + \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\kappa} - \Gamma^{\lambda}_{\alpha\kappa}\Gamma^{\alpha}_{\mu\nu}$$

and convert the Christoffel symbols to metrics with:

$$\Gamma^{\lambda}_{\mu\nu} = (1/2) * g^{\lambda\kappa}(\partial_{\nu}g_{\kappa\mu} + \partial_{\mu}g_{\kappa\nu} - \partial_{\kappa}g_{\mu\nu})$$

```
RtoG:= R^{\lambda?}_{\mu?\nu?\kappa?} ->
 - \partial_{\kappa?}{ \Gamma^{\lambda?}_{\mu?\nu?} }
 + \partial_{\nu?}{ \Gamma^{\lambda?}_{\mu?\kappa?} }
 - \Gamma^{\eta}_{\mu?\nu?} \Gamma^{\lambda?}_{\kappa?\eta}
 + \Gamma^{\eta}_{\mu?\kappa?} \Gamma^{\lambda?}_{\nu?\eta};
```

$$RtoG \ := R^{\lambda?}{}_{\mu?\nu?\kappa?} \rightarrow (-\partial_{\kappa?}\Gamma^{\lambda?}{}_{\mu?\nu?} + \partial_{\nu?}\Gamma^{\lambda?}{}_{\mu?\kappa?} - \Gamma^{\eta}{}_{\mu?\nu?}\Gamma^{\lambda?}{}_{\kappa?\eta} + \Gamma^{\eta}{}_{\mu?\kappa?}\Gamma^{\lambda?}{}_{\nu?\eta});$$

```
Gtog:= \Gamma^{\lambda?}_{\mu?\nu?} ->
  (1/2) * g^{\lambda?\kappa} (
        \partial_{\nu?}{ g_{\kappa\mu?} }
      + \partial_{\mu?}{ g_{\kappa\nu?} }
      - \partial_{\kappa}{ g_{\mu?\nu?} } );
```

$$Gtog \ := \Gamma^{\lambda?}{}_{\mu?\nu?} \rightarrow \frac{1}{2} g^{\lambda?\kappa}(\partial_{\nu?}g_{\kappa\mu?} + \partial_{\mu?}g_{\kappa\nu?} - \partial_{\kappa}g_{\mu?\nu?});$$

Now select the $R_{m4n4}$ component and do the substitution $R \rightarrow \Gamma$ and $\Gamma \rightarrow g$. After each substitution, will expand brackets and apply the product rule:

```
Rm4n4 := g_{m1 m} R^{m1}_{4 n 4} + g_{4 m} R^{4}_{4 n 4};
@substitute!(%)( @(RtoG) ):
@substitute!(%)( @(Gtog) ):
@distribute!(%): @prodrule!(%):
@distribute!(%): @prodsort!(%);
```

$$Rm4n4 \ := g_{m1m}R^{m1}{}_{4n4} + g_{4m}R^{4}{}_{4n4};$$

$$\begin{aligned}
Rm4n4 \ := & -\frac{1}{2}\,\partial_4 g^{m1\kappa}\,\partial_n g_{\kappa 4}\,g_{m1m} - \frac{1}{2}\,\partial_{4n}g_{\kappa 4}\,g_{m1m}g^{m1\kappa} - \frac{1}{2}\,\partial_4 g_{\kappa n}\,\partial_4 g^{m1\kappa}\,g_{m1m} \\
& - \frac{1}{2}\,\partial_{44}g_{\kappa n}\,g_{m1m}g^{m1\kappa} + \frac{1}{2}\,\partial_4 g^{m1\kappa}\,\partial_\kappa g_{4n}\,g_{m1m} + \frac{1}{2}\,\partial_{4\kappa}g_{4n}\,g_{m1m}g^{m1\kappa} \\
& + \frac{1}{2}\,\partial_4 g_{\kappa 4}\,\partial_n g^{m1\kappa}\,g_{m1m} + \frac{1}{2}\,\partial_{n4}g_{\kappa 4}\,g_{m1m}g^{m1\kappa} + \frac{1}{2}\,\partial_4 g_{\kappa 4}\,\partial_n g^{m1\kappa}\,g_{m1m} \\
& + \frac{1}{2}\,\partial_{n4}g_{\kappa 4}\,g_{m1m}g^{m1\kappa} - \frac{1}{2}\,\partial_\kappa g_{44}\,\partial_n g^{m1\kappa}\,g_{m1m} - \frac{1}{2}\,\partial_{n\kappa}g_{44}\,g_{m1m}g^{m1\kappa} \\
& - \frac{1}{4}\,\partial_\eta g_{\mu 4}\,\partial_n g_{\kappa 4}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} - \frac{1}{4}\,\partial_4 g_{\mu\eta}\,\partial_n g_{\kappa 4}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} \\
& + \frac{1}{4}\,\partial_\mu g_{4\eta}\,\partial_n g_{\kappa 4}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} - \frac{1}{4}\,\partial_4 g_{\kappa n}\,\partial_\eta g_{\mu 4}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} \\
& - \frac{1}{4}\,\partial_4 g_{\mu\eta}\,\partial_4 g_{\kappa n}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} + \frac{1}{4}\,\partial_4 g_{\kappa n}\,\partial_\mu g_{4\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} \\
& + \frac{1}{4}\,\partial_\kappa g_{4n}\,\partial_\eta g_{\mu 4}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} + \frac{1}{4}\,\partial_4 g_{\mu\eta}\,\partial_\kappa g_{4n}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} \\
& - \frac{1}{4}\,\partial_\mu g_{4\eta}\,\partial_\kappa g_{4n}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} + \frac{1}{4}\,\partial_4 g_{\kappa 4}\,\partial_\eta g_{\mu n}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} \\
& + \frac{1}{4}\,\partial_4 g_{\kappa 4}\,\partial_n g_{\mu\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} - \frac{1}{4}\,\partial_4 g_{\kappa 4}\,\partial_\mu g_{n\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} \\
& + \frac{1}{4}\,\partial_4 g_{\kappa 4}\,\partial_\eta g_{\mu n}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} + \frac{1}{4}\,\partial_4 g_{\kappa 4}\,\partial_n g_{\mu\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{\mu}g_{n\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} - \frac{1}{4}\,\partial_{\kappa}g_{44}\,\partial_{\eta}g_{\mu n}\,g^{\eta\kappa}g_{m1m}g^{m1\mu}\\
&- \frac{1}{4}\,\partial_{\kappa}g_{44}\,\partial_{n}g_{\mu\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu} + \frac{1}{4}\,\partial_{\kappa}g_{44}\,\partial_{\mu}g_{n\eta}\,g^{\eta\kappa}g_{m1m}g^{m1\mu}\\
&- \frac{1}{2}\,\partial_{4}g^{4\kappa}\,\partial_{n}g_{\kappa 4}\,g_{4m} - \frac{1}{2}\,\partial_{4n}g_{\kappa 4}\,g^{4\kappa}g_{4m} - \frac{1}{2}\,\partial_{4}g^{4\kappa}\,\partial_{4}g_{\kappa n}\,g_{4m}\\
&- \frac{1}{2}\,\partial_{44}g_{\kappa n}\,g^{4\kappa}g_{4m} + \frac{1}{2}\,\partial_{4}g^{4\kappa}\,\partial_{\kappa}g_{4n}\,g_{4m} + \frac{1}{2}\,\partial_{4\kappa}g_{4n}\,g^{4\kappa}g_{4m} + \frac{1}{2}\,\partial_{4}g_{\kappa 4}\,\partial_{n}g^{4\kappa}\,g_{4m}\\
&+ \frac{1}{2}\,\partial_{n4}g_{\kappa 4}\,g^{4\kappa}g_{4m} + \frac{1}{2}\,\partial_{4}g_{\kappa 4}\,\partial_{n}g^{4\kappa}\,g_{4m} + \frac{1}{2}\,\partial_{n4}g_{\kappa 4}\,g^{4\kappa}g_{4m} - \frac{1}{2}\,\partial_{\kappa}g_{44}\,\partial_{n}g^{4\kappa}\,g_{4m}\\
&- \frac{1}{2}\,\partial_{n\kappa}g_{44}\,g^{4\kappa}g_{4m} - \frac{1}{4}\,\partial_{\eta}g_{\mu 4}\,\partial_{n}g_{\kappa 4}\,g^{4\mu}g_{4m}g^{\eta\kappa} - \frac{1}{4}\,\partial_{4}g_{\mu\eta}\,\partial_{n}g_{\kappa 4}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&+ \frac{1}{4}\,\partial_{\mu}g_{4\eta}\,\partial_{n}g_{\kappa 4}\,g^{4\mu}g_{4m}g^{\eta\kappa} - \frac{1}{4}\,\partial_{4}g_{\kappa n}\,\partial_{\eta}g_{\mu 4}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&- \frac{1}{4}\,\partial_{4}g_{\mu\eta}\,\partial_{4}g_{\kappa n}\,g^{4\mu}g_{4m}g^{\eta\kappa} + \frac{1}{4}\,\partial_{4}g_{\kappa n}\,\partial_{\mu}g_{4\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&+ \frac{1}{4}\,\partial_{\kappa}g_{4n}\,\partial_{\eta}g_{\mu 4}\,g^{4\mu}g_{4m}g^{\eta\kappa} + \frac{1}{4}\,\partial_{4}g_{\mu\eta}\,\partial_{\kappa}g_{4n}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&- \frac{1}{4}\,\partial_{\mu}g_{4\eta}\,\partial_{\kappa}g_{4n}\,g^{4\mu}g_{4m}g^{\eta\kappa} + \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{\eta}g_{\mu n}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&+ \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{n}g_{\mu\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa} - \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{\mu}g_{n\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&+ \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{\eta}g_{\mu n}\,g^{4\mu}g_{4m}g^{\eta\kappa} + \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{n}g_{\mu\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&- \frac{1}{4}\,\partial_{4}g_{\kappa 4}\,\partial_{\mu}g_{n\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa} - \frac{1}{4}\,\partial_{\kappa}g_{44}\,\partial_{\eta}g_{\mu n}\,g^{4\mu}g_{4m}g^{\eta\kappa}\\
&- \frac{1}{4}\,\partial_{\kappa}g_{44}\,\partial_{n}g_{\mu\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa} + \frac{1}{4}\,\partial_{\kappa}g_{44}\,\partial_{\mu}g_{n\eta}\,g^{4\mu}g_{4m}g^{\eta\kappa};
\end{aligned}
$$

---

At this point, we realise how useful the computer can be in dealing with these long algebraic expressions. Some of the following equations contain thousands of terms and would take tens of pages to write out!

We now ask Cadabra to split the $\mu$ index into a $m$ part and the remaining 4 direction. After that, we remove $x^4$ derivatives of the gauge field and write the expression in canonical form:

```
@split_index!!(%){\mu,m1,4}:
@canonicalise!(%):
@substitute!(%)( \partial_{4}{A??} -> 0 ):
@substitute!(%)( \partial_{4 m?}{A??} -> 0 );
```

$$
\begin{aligned}
Rm4n4 \ := & -\frac{1}{2}\,\partial_{m1}g_{44}\,\partial_{n}g^{m1p}\,g_{mp} - \frac{1}{2}\,\partial_{nm1}g_{44}\,g_{mp}g^{m1p} - \frac{1}{4}\,\partial_{n}g_{4m1}\,\partial_{p}g_{4q}\,g^{m1p}g_{mr}g^{qr} \\
& - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{n}g_{4p}\,g^{m1p}g_{mq}g^{4q} - \frac{1}{4}\,\partial_{m1}g_{4p}\,\partial_{n}g_{44}\,g^{4m1}g_{mq}g^{pq} \\
& - \frac{1}{4}\,\partial_{n}g_{44}\,\partial_{m1}g_{44}\,g^{4m1}g_{mp}g^{4p} + \frac{1}{4}\,\partial_{m1}g_{4p}\,\partial_{n}g_{4q}\,g^{m1r}g_{mr}g^{pq} \\
& + \frac{1}{4}\,\partial_{m1}g_{4p}\,\partial_{n}g_{44}\,g^{4p}g_{mq}g^{m1q} + \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{n}g_{4p}\,g^{4p}g_{mq}g^{m1q} \\
& + \frac{1}{4}\,\partial_{n}g_{44}\,\partial_{m1}g_{44}\,g^{44}g_{mp}g^{m1p} + \frac{1}{4}\,\partial_{m1}g_{4n}\,\partial_{p}g_{4q}\,g^{m1p}g_{mr}g^{qr} \\
& + \frac{1}{4}\,\partial_{m1}g_{4n}\,\partial_{p}g_{44}\,g^{m1p}g_{mq}g^{4q} - \frac{1}{4}\,\partial_{m1}g_{4p}\,\partial_{q}g_{4n}\,g^{m1r}g_{mr}g^{pq} \\
& - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{p}g_{4n}\,g^{4p}g_{mq}g^{m1q} - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{p}g_{nq}\,g^{m1p}g_{mr}g^{qr} \\
& - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{p}g_{4n}\,g^{m1p}g_{mq}g^{4q} - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{n}g_{pq}\,g^{m1p}g_{mr}g^{qr} \\
& - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{n}g_{4p}\,g^{m1p}g_{mq}g^{4q} - \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{n}g_{4p}\,g^{4m1}g_{mq}g^{pq} \\
& - \frac{1}{4}\,\partial_{n}g_{44}\,\partial_{m1}g_{44}\,g^{4m1}g_{mp}g^{4p} + \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{p}g_{nq}\,g^{m1q}g_{mr}g^{pr} \\
& + \frac{1}{4}\,\partial_{m1}g_{44}\,\partial_{p}g_{4n}\,g^{4m1}g_{mq}g^{pq} - \frac{1}{2}\,\partial_{p}g_{44}\,\partial_{n}g^{4p}\,g_{4m} - \frac{1}{2}\,\partial_{np}g_{44}\,g^{4p}g_{4m} \\
& - \frac{1}{4}\,\partial_{n}g_{4p}\,\partial_{q}g_{4r}\,g^{4r}g_{4m}g^{pq} - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{n}g_{4q}\,g^{44}g_{4m}g^{pq} \\
& - \frac{1}{4}\,\partial_{p}g_{4q}\,\partial_{n}g_{44}\,g^{4q}g_{4m}g^{4p} - \frac{1}{4}\,\partial_{n}g_{44}\,\partial_{p}g_{44}\,g^{44}g_{4m}g^{4p} \\
& + \frac{1}{4}\,\partial_{p}g_{4q}\,\partial_{n}g_{4r}\,g^{4p}g_{4m}g^{qr} + \frac{1}{4}\,\partial_{p}g_{4q}\,\partial_{n}g_{44}\,g^{4p}g_{4m}g^{4q} \\
& + \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{n}g_{4q}\,g^{4p}g_{4m}g^{4q} + \frac{1}{4}\,\partial_{n}g_{44}\,\partial_{p}g_{44}\,g^{4p}g_{4m}g^{44} \\
& + \frac{1}{4}\,\partial_{p}g_{4n}\,\partial_{q}g_{4r}\,g^{4r}g_{4m}g^{pq} + \frac{1}{4}\,\partial_{p}g_{4n}\,\partial_{q}g_{44}\,g^{44}g_{4m}g^{pq} \\
& - \frac{1}{4}\,\partial_{p}g_{4q}\,\partial_{r}g_{4n}\,g^{4p}g_{4m}g^{qr} - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{q}g_{4n}\,g^{4p}g_{4m}g^{4q} \\
& - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{q}g_{nr}\,g^{4r}g_{4m}g^{pq} - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{q}g_{4n}\,g^{44}g_{4m}g^{pq} \\
& - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{n}g_{qr}\,g^{4q}g_{4m}g^{pr} - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{n}g_{4q}\,g^{44}g_{4m}g^{pq} \\
& - \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{n}g_{4q}\,g^{4q}g_{4m}g^{4p} - \frac{1}{4}\,\partial_{n}g_{44}\,\partial_{p}g_{44}\,g^{44}g_{4m}g^{4p} \\
& + \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{q}g_{nr}\,g^{4q}g_{4m}g^{pr} + \frac{1}{4}\,\partial_{p}g_{44}\,\partial_{q}g_{4n}\,g^{4q}g_{4m}g^{4p};
\end{aligned}
$$

### B.2.3. Inserting the Metric Ansatz

We can now retrieve our metric ansatz from §2.2:

$$\hat{g}_{MN} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu z} \\ \hat{g}_{\mu z} & \hat{g}_{zz} \end{pmatrix} \qquad \mu = 0 \dots D, \quad z = (D+1)$$

with

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + \mathcal{A}_\mu \mathcal{A}_\nu, \quad \hat{g}_{\mu z} = e^{2\beta\phi} \mathcal{A}_\mu, \quad \hat{g}_{zz} = e^{2\beta\phi}$$

```
g1 := g_{m n} -> e**{2 \alpha \phi} h_{m n} + e**{2 \beta \phi} A_{m} A_{n};
g2 := g^{m n} -> e**{-2 \alpha \phi} h^{m n};
g3 := g_{4 m} -> e**{2 \beta \phi} A_{m};
g4 := g^{4 m} -> - e**{-2 \alpha \phi} h^{m n} A_{n};
g5 := g_{4 4} -> e**{2 \beta \phi};
g6 := g^{4 4} -> e**{-2 \beta \phi} + e**{-2 \alpha \phi} h^{m n} A_{m} A_{n};
```

$$\begin{aligned}
&g1 := g_{mn} \to (e^{2\,\alpha\phi} h_{mn} + e^{2\,\beta\phi} A_m A_n);\\
&g2 := g^{mn} \to e^{(-2)\,\alpha\phi} h^{mn};\\
&g3 := g_{4m} \to e^{2\,\beta\phi} A_m;\\
&g4 := g^{4m} \to -e^{(-2)\,\alpha\phi} h^{mn} A_n;\\
&g5 := g_{44} \to e^{2\,\beta\phi};\\
&g6 := g^{44} \to (e^{(-2)\,\beta\phi} + e^{(-2)\,\alpha\phi} h^{mn} A_m A_n);
\end{aligned}$$

The inverse components were found by hand, and are tested by:

```
test := g_{\mu \nu} g^{\nu \mu};
@split_index!!(%){\mu, m, 4}:
@substitute!(%)( g_{m 4} -> g_{4 m}, g^{m 4} -> g^{4 m} );
@substitute!(%)( @(g1), @(g2), @(g3), @(g4), @(g5), @(g6) );
@distribute!(%); @prodsort!(%);
@collect_factors!(%);
@canonicalise!(%); @collect_terms!(%);
```

$$\begin{aligned}
&test := g_{\mu\nu} g^{\nu\mu};\\
&test := g_{mn} g^{nm} + g_{4m} g^{4m} + g_{4m} g^{4m} + g_{44} g^{44};\\
&test := (e^{2\,\alpha\phi} h_{mn} + e^{2\,\beta\phi} A_m A_n) e^{(-2)\,\alpha\phi} h^{nm} - e^{2\,\beta\phi} A_m e^{(-2)\,\alpha\phi} h^{mn} A_n\\
&\quad - e^{2\,\beta\phi} A_m e^{(-2)\,\alpha\phi} h^{mn} A_n + e^{2\,\beta\phi} (e^{(-2)\,\beta\phi} + e^{(-2)\,\alpha\phi} h^{mn} A_m A_n);\\
&test := e^{2\,\alpha\phi} h_{mn} e^{(-2)\,\alpha\phi} h^{nm} + e^{2\,\beta\phi} A_m A_n e^{(-2)\,\alpha\phi} h^{nm} - e^{2\,\beta\phi} A_m e^{(-2)\,\alpha\phi} h^{mn} A_n\\
&\quad - e^{2\,\beta\phi} A_m e^{(-2)\,\alpha\phi} h^{mn} A_n + e^{2\,\beta\phi} e^{(-2)\,\beta\phi} + e^{2\,\beta\phi} e^{(-2)\,\alpha\phi} h^{mn} A_m A_n;\\
&test := e^{(-2)\,\alpha\phi} e^{2\,\alpha\phi} h_{mn} h^{nm} + A_m A_n e^{(-2)\,\alpha\phi} e^{2\,\beta\phi} h^{nm} - A_m A_n e^{(-2)\,\alpha\phi} e^{2\,\beta\phi} h^{mn}\\
&\quad - A_m A_n e^{(-2)\,\alpha\phi} e^{2\,\beta\phi} h^{mn} + e^{(-2)\,\beta\phi} e^{2\,\beta\phi} + A_m A_n e^{(-2)\,\alpha\phi} e^{2\,\beta\phi} h^{mn};
\end{aligned}$$

$$test := h_{mn}h^{nm} + A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{nm} - A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn} - A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn} + 1 + A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn};$$

$$test := h_{mn}h^{mn} + A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn} - A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn} - A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn} + 1 + A_m A_n e^{(-2\,\alpha\phi+2\,\beta\phi)} h^{mn};$$

$$test := h_{mn}h^{mn} + 1;$$

---

Which yields the correct result. We also need to teach the program how to take the first and second derivatives of exponentials properly:

---

```
e1 := \partial_{m}{ e**{ 2 \beta \phi} }
  -> 2 \beta \partial_{m}{\phi} e**{ 2 \beta \phi};
e2 := \partial_{m}{ e**{ 2 \alpha \phi} }
  -> 2 \alpha \partial_{m}{\phi} e**{ 2 \alpha \phi};
e3 := \partial_{m}{ e**{-2 \beta \phi} }
  -> -2 \beta \partial_{m}{\phi} e**{-2 \beta \phi};
e4 := \partial_{m}{ e**{-2 \alpha \phi} }
  -> -2 \alpha \partial_{m}{\phi} e**{-2 \alpha \phi};
e5 := \partial_{m n}{ e**{ 2 \beta \phi} }
  -> 4 \beta \beta \partial_{m}{\phi} \partial_{n}{\phi} e**{2 \beta \phi} );
e6 := \partial_{m n}{ e**{ 2 \alpha \phi} }
  -> 4 \alpha \alpha \partial_{m}{\phi} \partial_{n}{\phi} e**{2 \alpha \phi} );
e7 := \partial_{m n}{ e**{-2 \beta \phi} }
  -> 4 \beta \beta \partial_{m}{\phi} \partial_{n}{\phi} e**{-2 \beta \phi} );
e8 := \partial_{m n}{ e**{-2 \alpha \phi} }
  -> 4 \alpha \alpha \partial_{m}{\phi} \partial_{n}{\phi} e**{-2 \alpha \phi} );
```

$$e1 := \partial_m e^{2\,\beta\phi} \to 2\,\beta\partial_m\phi\, e^{2\,\beta\phi};$$
$$e2 := \partial_m e^{2\,\alpha\phi} \to 2\,\alpha\partial_m\phi\, e^{2\,\alpha\phi};$$
$$e3 := \partial_m e^{(-2)\,\beta\phi} \to (-2)\,\beta\partial_m\phi\, e^{(-2)\,\beta\phi};$$
$$e4 := \partial_m e^{(-2)\,\alpha\phi} \to (-2)\,\alpha\partial_m\phi\, e^{(-2)\,\alpha\phi};$$
$$e5 := \partial_{mn} e^{2\,\beta\phi} \to 4\,\beta\beta\partial_m\phi\,\partial_n\phi\, e^{2\,\beta\phi};$$
$$e6 := \partial_{mn} e^{2\,\alpha\phi} \to 4\,\alpha\alpha\partial_m\phi\,\partial_n\phi\, e^{2\,\alpha\phi};$$
$$e7 := \partial_{mn} e^{(-2)\,\beta\phi} \to 4\,\beta\beta\partial_m\phi\,\partial_n\phi\, e^{(-2)\,\beta\phi};$$
$$e8 := \partial_{mn} e^{(-2)\,\alpha\phi} \to 4\,\alpha\alpha\partial_m\phi\,\partial_n\phi\, e^{(-2)\,\alpha\phi};$$

---

We can now proceed and insert the components of the metric into our Riemann tensor.

```
@substitute!(Rm4n4)( @(g1), @(g2), @(g3), @(g4), @(g5), @(g6) );
```

$$
\begin{aligned}
Rm4n4 := & -\frac{1}{2}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_n(e^{(-2)\,\alpha\phi}h^{m1p})\,(e^{2\,\alpha\phi}h_{mp}+e^{2\,\beta\phi}A_mA_p) \\
& -\frac{1}{2}\,\partial_{nm1}e^{2\,\beta\phi}\,(e^{2\,\alpha\phi}h_{mp}+e^{2\,\beta\phi}A_mA_p)e^{(-2)\,\alpha\phi}h^{m1p} \\
& -\frac{1}{4}\,\partial_n(e^{2\,\beta\phi}A_{m1})\,\partial_p(e^{2\,\beta\phi}A_q)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mr}+e^{2\,\beta\phi}A_mA_r)e^{(-2)\,\alpha\phi}h^{qr} \\
& +\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_n(e^{2\,\beta\phi}A_p)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{qr}A_r \\
& +\frac{1}{4}\,\partial_{m1}(e^{2\,\beta\phi}A_p)\,\partial_ne^{2\,\beta\phi}\,e^{(-2)\,\alpha\phi}h^{m1r}A_r(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{pq} \\
& -\frac{1}{4}\,\partial_ne^{2\,\beta\phi}\,\partial_{m1}e^{2\,\beta\phi}\,e^{(-2)\,\alpha\phi}h^{m1q}A_q(e^{2\,\alpha\phi}h_{mp}+e^{2\,\beta\phi}A_mA_p)e^{(-2)\,\alpha\phi}h^{pr}A_r \\
& +\frac{1}{4}\,\partial_{m1}(e^{2\,\beta\phi}A_p)\,\partial_n(e^{2\,\beta\phi}A_q)\,e^{(-2)\,\alpha\phi}h^{m1r}(e^{2\,\alpha\phi}h_{mr}+e^{2\,\beta\phi}A_mA_r)e^{(-2)\,\alpha\phi}h^{pq} \\
& -\frac{1}{4}\,\partial_{m1}(e^{2\,\beta\phi}A_p)\,\partial_ne^{2\,\beta\phi}\,e^{(-2)\,\alpha\phi}h^{pr}A_r(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{m1q} \\
& -\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_n(e^{2\,\beta\phi}A_p)\,e^{(-2)\,\alpha\phi}h^{pr}A_r(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{m1q} \\
& +\frac{1}{4}\,\partial_ne^{2\,\beta\phi}\,\partial_{m1}e^{2\,\beta\phi}\,(e^{(-2)\,\beta\phi}+e^{(-2)\,\alpha\phi}h^{qr}A_qA_r)(e^{2\,\alpha\phi}h_{mp} \\
& \qquad +e^{2\,\beta\phi}A_mA_p)e^{(-2)\,\alpha\phi}h^{m1p} \\
& +\frac{1}{4}\,\partial_{m1}(e^{2\,\beta\phi}A_n)\,\partial_p(e^{2\,\beta\phi}A_q)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mr}+e^{2\,\beta\phi}A_mA_r)e^{(-2)\,\alpha\phi}h^{qr} \\
& -\frac{1}{4}\,\partial_{m1}(e^{2\,\beta\phi}A_n)\,\partial_pe^{2\,\beta\phi}\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{qr}A_r \\
& -\frac{1}{4}\,\partial_{m1}(e^{2\,\beta\phi}A_p)\,\partial_q(e^{2\,\beta\phi}A_n)\,e^{(-2)\,\alpha\phi}h^{m1r}(e^{2\,\alpha\phi}h_{mr}+e^{2\,\beta\phi}A_mA_r)e^{(-2)\,\alpha\phi}h^{pq} \\
& +\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_p(e^{2\,\beta\phi}A_n)\,e^{(-2)\,\alpha\phi}h^{pr}A_r(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{m1q} \\
& -\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_p(e^{2\,\alpha\phi}h_{nq}+e^{2\,\beta\phi}A_nA_q)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mr} \\
& \qquad +e^{2\,\beta\phi}A_mA_r)e^{(-2)\,\alpha\phi}h^{qr} \\
& +\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_p(e^{2\,\beta\phi}A_n)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{qr}A_r \\
& -\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_n(e^{2\,\alpha\phi}h_{pq}+e^{2\,\beta\phi}A_pA_q)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mr} \\
& \qquad +e^{2\,\beta\phi}A_mA_r)e^{(-2)\,\alpha\phi}h^{qr} \\
& +\frac{1}{4}\,\partial_{m1}e^{2\,\beta\phi}\,\partial_n(e^{2\,\beta\phi}A_p)\,e^{(-2)\,\alpha\phi}h^{m1p}(e^{2\,\alpha\phi}h_{mq}+e^{2\,\beta\phi}A_mA_q)e^{(-2)\,\alpha\phi}h^{qr}A_r \\
& +\ldots
\end{aligned}
$$

### B.2.4. Simplifying

We can now start to simplify this expression by multiplying out brackets, applying the above definitions for the differentials of exponentials and collecting terms together.

```
@distribute!(%):
@substitute!(%)( @(e1), @(e2), @(e3), @(e4), @(e5), @(e6), @(e7), @(e8) ):
@prodrule!(%): @prodsort!(%): @distribute!(%):
@substitute!(%)( @(e1), @(e2), @(e3), @(e4), @(e5), @(e6), @(e7), @(e8) ):
@canonicalise!(%): @prodsort!(%): @collect_factors!(%):
@substitute!!(%)( h_{m1 m2} h^{m3 m2} -> \delta_{m1}^{m3} ):
@eliminate_kr!(%): @prodsort!(%): @collect_terms!(%);
```

$$
\begin{aligned}
Rm4n4 \;:=\; & 3\,\alpha\beta\partial_{m}\phi\,\partial_{n}\phi\,e^{2\,\beta\phi} - \beta\partial_{m1}\phi\,\partial_{n}h^{m1p}\,e^{2\,\beta\phi}h_{mp}\\
& + 3\,A_{m}A_{m1}\alpha\beta\partial_{n}\phi\,\partial_{p}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1p} - A_{m}A_{m1}\beta\partial_{p}\phi\,\partial_{n}h^{m1p}\,e^{(-2\,\alpha\phi+4\,\beta\phi)}\\
& - \partial_{m}\phi\,\partial_{n}\phi\,\beta^{2}e^{2\,\beta\phi} - A_{m}A_{m1}\partial_{n}\phi\,\partial_{p}\phi\,\beta^{2}e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1p}\\
& - \frac{1}{2}\,A_{m}\beta\partial_{n}A_{m1}\,\partial_{q}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1q} - \frac{1}{4}\,\partial_{n}A_{m1}\,\partial_{p}A_{m}\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1p}\\
& - \frac{1}{2}\,A_{m}A_{m1}A_{p}\beta\partial_{q}A_{r}\,\partial_{n}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1q}h^{pr}\\
& - \frac{1}{2}\,A_{m}A_{m1}A_{q}\beta\partial_{n}A_{p}\,\partial_{r}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1q}h^{pr}\\
& - \frac{1}{4}\,A_{m}A_{m1}\partial_{n}A_{p}\,\partial_{q}A_{r}\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1r}h^{pq} + A_{m}\beta\partial_{n}A_{q}\,\partial_{p}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{pq}\\
& + A_{m}A_{m1}A_{p}\beta\partial_{n}A_{r}\,\partial_{q}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1p}h^{qr}\\
& + \frac{1}{2}\,A_{q}\beta\partial_{m}A_{p}\,\partial_{n}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{pq} + \frac{1}{4}\,\partial_{m}A_{q}\,\partial_{n}A_{m1}\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1q}\\
& + \frac{1}{2}\,A_{m}A_{m1}A_{p}\beta\partial_{n}A_{r}\,\partial_{q}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1q}h^{pr}\\
& + \frac{1}{2}\,A_{m}A_{m1}A_{r}\beta\partial_{p}A_{q}\,\partial_{n}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1p}h^{qr}\\
& + \frac{1}{4}\,A_{m}A_{m1}\partial_{n}A_{p}\,\partial_{q}A_{r}\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1q}h^{pr}\\
& - \frac{1}{2}\,A_{m1}\beta\partial_{m}A_{q}\,\partial_{n}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1q}\\
& + \frac{1}{2}\,A_{m}\beta\partial_{m1}A_{n}\,\partial_{q}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1q} + \frac{1}{4}\,\partial_{m1}A_{n}\,\partial_{p}A_{m}\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1p}\\
& + A_{m}A_{m1}A_{n}A_{q}\partial_{p}\phi\,\partial_{r}\phi\,\beta^{2}e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1q}h^{pr}\\
& + \frac{1}{2}\,A_{m}A_{m1}A_{n}\beta\partial_{q}A_{r}\,\partial_{p}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1r}h^{pq}\\
& + \frac{1}{2}\,A_{m}A_{m1}A_{q}\beta\partial_{p}A_{n}\,\partial_{r}\phi\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1q}h^{pr}\\
& + \frac{1}{4}\,A_{m}A_{m1}\partial_{p}A_{n}\,\partial_{q}A_{r}\,e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1r}h^{pq}\\
& - \frac{1}{2}\,A_{p}\beta\partial_{m1}A_{n}\,\partial_{m}\phi\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1p} - \frac{1}{4}\,\partial_{m}A_{q}\,\partial_{m1}A_{n}\,e^{(-2\,\alpha\phi+4\,\beta\phi)}h^{m1q}\\
& - A_{m}A_{m1}A_{n}A_{q}\partial_{p}\phi\,\partial_{r}\phi\,\beta^{2}e^{(-4\,\alpha\phi+6\,\beta\phi)}h^{m1p}h^{qr}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} A_m A_{m1} A_n \beta \partial_q A_r \, \partial_p \phi \, e^{(-4\,\alpha\phi + 6\,\beta\phi)} h^{m1q} h^{pr} \\
&- \frac{1}{2} A_m A_{m1} A_q \beta \partial_p A_n \, \partial_r \phi \, e^{(-4\,\alpha\phi + 6\,\beta\phi)} h^{m1p} h^{qr} \\
&- \frac{1}{4} A_m A_{m1} \partial_p A_n \, \partial_q A_r \, e^{(-4\,\alpha\phi + 6\,\beta\phi)} h^{m1q} h^{pr} \\
&+ \frac{1}{2} A_{m1} \beta \partial_q A_n \, \partial_m \phi \, e^{(-2\,\alpha\phi + 4\,\beta\phi)} h^{m1q} - \alpha\beta \partial_{m1} \phi \, \partial_p \phi \, e^{2\,\beta\phi} h_{mn} h^{m1p} \\
&- \frac{1}{2} \beta \partial_{m1} \phi \, \partial_p h_{nm} \, e^{2\,\beta\phi} h^{m1p} - A_m A_n \alpha\beta \partial_p \phi \, \partial_q \phi \, e^{(-2\,\alpha\phi + 4\,\beta\phi)} h^{pq} + \ldots
\end{aligned}
$$

By applying further simplifications:

```
@substitute!(%)( \partial_{p}{h^{n m}} h_{q m}
  -> - \partial_{p}{h_{q m}} h^{n m} ): @canonicalise!(%):
@substitute!(%)( h_{m1 m2} h^{m3 m2} -> \delta_{m1}^{m3} ):
@eliminate_kr!(%);
```

Finally, replacing the derivative of the gauge potential with the field strength and choosing a convenient value of $\beta$:

$$\beta = -2\alpha$$

```
@substitute!(%)( \partial_{n}{A_{m}} -> F_{n m} ):
@prodsort!(%): @canonicalise!(%): @rename_dummies!(%):
@collect_terms!(%): @sumsort!(%);
@substitute!(%)( \beta -> -2 \alpha ):
@expand_power!(%): @prodsort!(%): @collect_factors!(%): @collect_terms!(%);
```

$$
\begin{aligned}
Rm4n4 \; := \; & F_{mp} F_{nq} e^{(-2\,\alpha\phi + 4\,\beta\phi)} h^{pq} - \partial_m \phi \, \partial_n \phi \, \beta^2 e^{2\,\beta\phi} + 2\,\alpha\beta \partial_m \phi \, \partial_n \phi \, e^{2\,\beta\phi} \\
& + \frac{1}{2} \beta \partial_p \phi \, \partial_m h_{nq} \, e^{2\,\beta\phi} h^{pq} + \frac{1}{2} \beta \partial_p \phi \, \partial_n h_{mq} \, e^{2\,\beta\phi} h^{pq} \\
& - \frac{1}{2} \beta \partial_p \phi \, \partial_q h_{mn} \, e^{2\,\beta\phi} h^{pq} - \alpha\beta \partial_p \phi \, \partial_q \phi \, e^{2\,\beta\phi} h_{mn} h^{pq};
\end{aligned}
$$

$$
\begin{aligned}
Rm4n4 \; := \; & F_{mp} F_{nq} e^{(-10)\,\alpha\phi} h^{pq} - 8\,\alpha^2 \partial_m \phi \, \partial_n \phi \, e^{(-4)\,\alpha\phi} \\
& - \alpha \partial_p \phi \, \partial_m h_{nq} \, e^{(-4)\,\alpha\phi} h^{pq} - \alpha \partial_p \phi \, \partial_n h_{mq} \, e^{(-4)\,\alpha\phi} h^{pq} \\
& + \alpha \partial_p \phi \, \partial_q h_{mn} \, e^{(-4)\,\alpha\phi} h^{pq} + 2\,\alpha^2 \partial_p \phi \, \partial_q \phi \, e^{(-4)\,\alpha\phi} h_{mn} h^{pq};
\end{aligned}
$$

Which is the required result.

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