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Unilaterally Supported Plates on Elastic Foundations by the Boundary Element Method

A boundary element solution is developed for the unilateral contact problem of a thin elastic plate resting on elastic homogeneous or nonhomogeneous subgrade. The reaction of the subgrade may depend linearly, or nonlinearly, on the deflection of the plate. The contact between the plate and the subgrade is unbonded. The subgrade surface is not necessarily plane, and miscontact between plate and subgrade due to initial gaps is also encountered. The solution procedure is based on the integral representation of the deflection for the biharmonic equation in which the unknown subgrade reaction is treated as loading term. The effectiveness of the proposed method is illustrated by several examples.

1 Introduction

In most investigations concerning plates supported on elastic foundation it is assumed that the bodies in contact (plate and subgrade) are bonded to each other and, consequently, compressive as well as tensile reactions are considered to be admissible. In this case the contact region is a priori known and the main effort is directed towards the evaluation of the deflection surface and the contact pressure.

However, for many foundation materials, the admission of tensile stresses across the interface separating the plate from the foundation is not realistic. When there is no bonding between plate and subgrade, regions of no contact develop beneath the plate under certain loading conditions and separation between the two bodies takes place at contours where the compressive pressure vanishes. Consequently, the contact region is unknown and the vanishing of the compressive stress provides the condition for the determination of the contact region.

Enormous work has been done for plates resting on elastic foundation with bonded contact between plate and subgrade. Since no attempt is made here to summarize the various researches in this area, we mention only the books of Selvadurai (1979) and Vlasov and Leontiev (1966) where extended literature on this subject is presented. On the other hand, relatively little work has been done for plates unilaterally supported on elastic foundation (Selvadurai, 1979). To the authors' knowledge, with regard to the unbonded plate contact, the majority of the presented methods are limited to axisymmetric problems (Weitsman, 1969; Pu and Hussain, 1970; Gladwell and Iyer, 1974). Problems involving a receding contact between an elastic layer and a half-space are analyzed by Keer et al. (1972) and Tsai et al. (1974) leading to the Fredholm integral equations related to the contact tractions. Solution for an infinite plate with unbonded contact on a Winkler foundation is given by Weitsman (1969), and for a circular plate by Weitsman (1970) and Hofmann (1938). An incremental numerical technique for the simulation of structural elements in receding/advancing contact (Mahmoud et al., 1986), the boundary integral equation method for the unilateral buckling of thin elastic plates (Bezine et al., 1985), variational methods (Kartvelishvili, 1976), and an attempt towards mixed finite elements (Panagiotopoulos and Talaslidis, 1980) have also been used.

In this paper, a boundary element solution is presented for the unilateral contact problem of a thin elastic plate resting on elastic homogeneous or nonhomogeneous foundation. The plate may have arbitrary shape and be subjected to any loading and boundary conditions. The subgrade model consists of closely spaced independent springs. The subgrade reaction may depend linearly (Winkler) or nonlinearly on the deflection. The subgrade surface is not necessarily plane, thus, miss contact between plate and subgrade due to initial gaps is also encountered. The solution procedure is based on the integral representation of the deflection which is established using the fundamental solution of the linear part of the governing operator, whereas the unknown subgrade reaction is included in the loading term. Application of the boundary element technique and Gauss integration for the domain integrals involving the unknown domain quantities yields a system of nonlinear algebraic equations from which the deflection surface is computed by an iterative process.

Actually the proposed method is not a pure boundary element method, since it requires discretization within the domain to determine the unknown field quantities. However, the number of the linear equations is still defined by the boundary discretization, thereby retaining most of the advantages over a possible pure domain discretization method. The domain discretization, in this work, is performed using Gauss inte-

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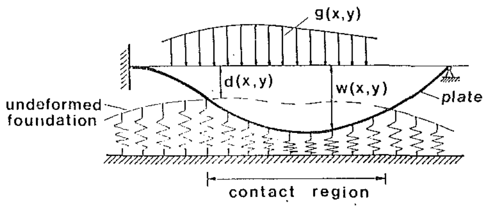

Fig. 1 Cross-section of deflected plate and foundation model

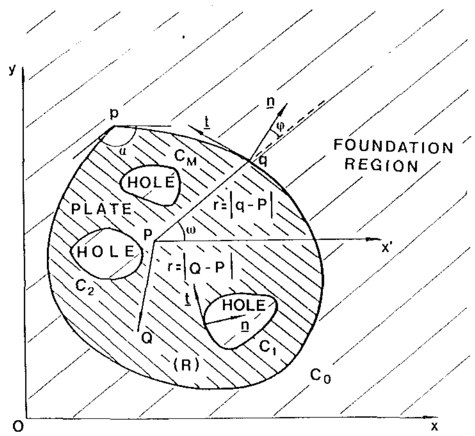

Fig. 2 Two-dimensional region occupied by the plate

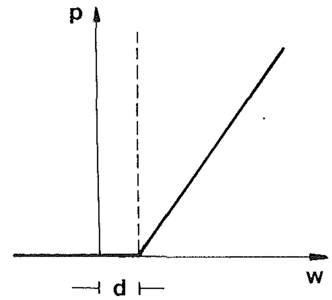

Fig. 3(*a*)

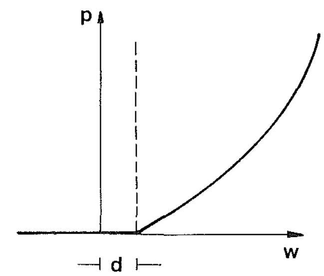

Fig. 3(*b*)

Fig. 3 Unilateral contact law $p=f(w-d)U(w-d)$ for a linear (*a*) and a nonlinear (*b*) Winkler-type spring

gration over regions of arbitrary shape (Katsikadelis and Sapountzakis, 1987; Katsikadelis, 1990) which renders the method very effective.

Several numerical examples are worked out to illustrate the effectiveness of the proposed method.

2 Governing Equations

Consider a thin elastic plate of thickness h occupying the two-dimensional multiply connected region R of the xy-plane, bounded by the $K+1$ curves C_0, C_1, C_2, ..., C_K and resting, in general, on a nonlinear Winkler-type elastic foundation (Figs. 1 and 2). The curves C_i $(i=0,\ 1,\ 2,\ \ldots,\ K)$ may be piecewise smooth, i.e., the boundary may have a finite number of corners. For unbonded contact between plate and subgrade, the interaction pressure at the interface is compressive and can be represented by the following relation:

$$p=f(w-d)U(w-d) \tag{1}$$

in which $f(w-d)$ is in general a nonlinear function of its argument $w-d$; $w=w(x,\ y)$ is the deflection of the plate; $d=d(x,\ y)$ is a function representing the initial gap between plate and subgrade (Fig. 1); and $U(w-d)$ is the unit step function defined as

$$U(w-d)=\begin{vmatrix} 0 & \text{if } w-d<0 \\ 1 & \text{if } w-d\geq 0 \end{vmatrix} \quad d\geq 0. \tag{2}$$

The particular case $f(w-d)=k(w-d)$, k being a constant denoting the subgrade reaction modulus, describes the conventional Winkler model (Fig. 3).

Assuming that there are no friction forces at the interface the deflection $w(P)$ at any point P: $(x,\ y)\epsilon R$ satisfies the following differential equation

$$D\nabla^4 w+f(w-d)U(w-d)=g \tag{3}$$

where $\nabla^4=(\nabla^2)=(\partial^2/\partial x^2+\partial^2/\partial y^2)^2$ is the biharmonic operator; $g=g(x,\ y)$ is the transverse loading; $D=\mathrm{Eh}^3/12(1-\nu^2)$ is the flexural rigidity, E and ν being the elastic constants of the material of the plate.

Moreover, the deflection w must satisfy the following boundary conditions on the boundary $C=U_{i=0}^{i=K}C_i$ of the plate

$$\alpha_1 w+\alpha_2 Vw=\alpha_3 \tag{4a}$$

$$\beta_1\frac{\partial w}{\partial n}+\beta_2 Mw=\beta_3 \tag{4b}$$

where $\alpha_i=\alpha_i(p)$, $\beta_i=\beta_i(p)$, $p\epsilon C\,(i=1,2,3)$ are given functions specified on the boundary C and M, V are differential operators defined in intrinsic coordinates as (Katsikadelis, 1982)

$$M=-D\left[\nabla^2+(\nu-1)\left(\frac{\partial^2}{\partial s^2}+\kappa\frac{\partial}{\partial n}\right)\right] \tag{5a}$$

$$V=-D\left[\frac{\partial}{\partial n}\nabla^2-(\nu-1)\frac{\partial}{\partial s}\left(\frac{\partial^2}{\partial s\partial n}-\kappa\frac{\partial}{\partial s}\right)\right] \tag{5b}$$

in which $\kappa=\kappa(s)$ is the curvature of the boundary; $\partial/\partial s$ and $\partial/\partial n$ denote differentiation with respect to the arc length s and the outward normal n to the boundary, respectively. The quantities Mw and Vw represent the bending moment and the effective shearing force along the boundary. The boundary conditions (4*a*,*b*) are the most general linear boundary conditions for the plate problem. It is apparent that all kinds of conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately the functions $\alpha_i(s)$ and $\beta_i(s)$ (e.g., for the clamped egde it is $\alpha_1=\beta_1=1$, $\alpha_2=\alpha_3=\beta_2=\beta_3=0$, for the simply supported edge it is $\alpha_1=\beta_2=1$, $\alpha_2=\alpha_3=\beta_1=\beta_3=0$).

In case of free or transversely elastically restrained edges, the boundary conditions (4*a*,*b*) must be supplemented by the corner condition

$$c_{1k}w+c_{2k}[\![Tw]\!]_k=c_{3k},\ c_{2k}\neq 0 \tag{6}$$

where c_{ik} $(i=1,\ 2,\ 3)$ are specified functions at the corner point p_k and T is the operator (Katsikadelis, 1982)

$$T=D(1-\nu)\left(\frac{\partial^2}{\partial s\partial n}-\kappa\frac{\partial}{\partial s}\right). \tag{7}$$

Therefore, Tw is the twisting moment along the boundary and $[\![Tw]\!]_k$ is its jump of discontinuity at the corner point p_k.

3 Solution Procedure

For any pair of functions w and v which are four times continuously differentiable inside R and three times continuously differentiable on the boundary C, the following reciprocal identity, known also as Rayleigh-Green identity, is valid (Duff and Naylor, 1966):

$$\iint_R (v\nabla^4 w - w\nabla^4 v)d\sigma = \int_C \left(v\frac{\partial}{\partial n}\nabla^2 w - w\frac{\partial}{\partial n}\nabla^2 v - \frac{\partial v}{\partial n}\nabla^2 w + \frac{\partial w}{\partial n}\nabla^2 v\right)ds, \quad (8)$$

application of relation (8) for the function w satisfying Eq. (3) and the function

$$v = \frac{1}{8\pi D} r^2 lnr, \; r = |P-Q| \quad (9)$$

which is a particular singular solution of the equation

$$D\nabla^4 v = \delta(P-Q) \;\; P:(x,y), \; Q:(\xi,\eta)\epsilon R. \quad (10)$$

The following integral representation for the deflection w is obtained

$$w(P) = -\frac{1}{2\pi D}\iint_R \Lambda_4(r) f(w-d)U(w-d)d\sigma + \frac{1}{2\pi D}\iint_R \Lambda_4(r) g d\sigma - \frac{1}{2\pi}\int_C [\Lambda_1(r)\Omega + \Lambda_2(r)X + \Lambda_3(r)\Phi + \Lambda_4(r)\Psi]ds \quad (11)$$

where the kernels $\Lambda_i(r)$, $(i=1, 2, 3, 4)$ are given as

$$\Lambda_1(r) = -\frac{\cos\varphi}{r} \qquad \Lambda_2(r) = lnr + 1 \quad (12a,b)$$

$$\Lambda_3(r) = -\frac{1}{4}(2rlnr + r)\cos\varphi \quad \Lambda_4(r) = \frac{1}{4}r^2 lnr \quad (12c,d)$$

and the following notation has been used

$$\Omega = w, \; X = \frac{\partial w}{\partial n}, \; \Phi = \nabla^2 w, \; \Psi = \frac{\partial}{\partial n}\nabla^2 w. \quad (13)$$

Notice that for the line integral it is $r = |P-q|$, whereas for the domain integrals, it is $r = |P-Q|$, $P, Q\epsilon R$, $q\epsilon C$; $\varphi = \widehat{\mathbf{r},\mathbf{n}}$ is the angle between the direction of $\mathbf{r}$ and the normal $\mathbf{n}$ to the boundary at point q.

Application of the operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ to Eq. (11) results in the integral representation of the Laplacian as

$$\nabla^2 w(P) = -\frac{1}{2\pi D}\iint_R \Lambda_2(r) f(w-d)U(w-d)d\sigma + \frac{1}{2\pi D}\iint_R \Lambda_2(r) g d\sigma - \frac{1}{2\pi}\int_C [\Lambda_1(r)\Phi + \Lambda_2(r)\Psi]ds. \quad (14)$$

Equation (11) involves five unknown quantities, i.e., the deflection w inside the domain R and the boundary quantities $\Omega = \Omega(s)$, $X = X(s)$, $\Phi = \Phi(s)$, $\Psi = \Psi(s)$. Four additional equations are established using the boundary equation method (Katsikadelis and Armenakas, 1989). According to this method the boundary conditions ($4a,b$) by virtue of Eqs. ($5a,b$) and notation (13) can be written as

$$\alpha_1\Omega - D\alpha_2\left[\Psi - (\nu-1)\frac{\partial}{\partial s}\left(\frac{\partial X}{\partial s} - \kappa\frac{\partial \Omega}{\partial s}\right)\right] = \alpha_3 \quad (15)$$

$$\beta_1 X - D\beta_2\left[\Phi + (\nu-1)\left(\frac{\partial^2\Omega}{\partial s^2} + \kappa X\right)\right] = \beta_3. \quad (16)$$

Moreover, letting point $P \rightarrow p\epsilon C$ in Eqs. (11) and (14), the following two boundary integral equations are derived

$$\alpha\Omega = -\frac{1}{D}\iint_R \Lambda_4 f(w-d)U(w-d)d\sigma + \frac{1}{D}\iint_R \Lambda_4 g d\sigma - \int_C (\Lambda_1\Omega + \Lambda_2 X + \Lambda_3\Phi + \Lambda_4\Psi)ds \quad (17)$$

$$\alpha\Phi = -\frac{1}{D}\iint_R \Lambda_2 f(w-d)U(w-d)d\sigma + \frac{1}{D}\iint_R \Lambda_2 g d\sigma - \int_C (\Lambda_1\Phi + \Lambda_2\Psi)ds \quad (18)$$

where α is the angle between the tangents at point p (see Fig. 2). Relations (11), (15), (16), (17), and (18) constitute a set of five simultaneous equations which can be solved to yield the deflection w of the plate.

The stress resultants at a point P inside R are obtained by direct differentiation of Eq. (11) using the relations

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \quad M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}$$

$$Q_x = -D\frac{\partial\nabla^2 w}{\partial x} \quad Q_y = -D\frac{\partial\nabla^2 w}{\partial y} \quad (19)$$

while the stress resultants M_n, M_t, M_{nt}, V_n along the boundary are obtained from relations

$$M_n = -D\left[\Phi + (\nu-1)\left(\frac{\partial^2\Omega}{\partial s^2} + \kappa X\right)\right]$$

$$M_t = -D\left[\nu\Phi - (\nu-1)\left(\frac{\partial^2\Omega}{\partial s^2} + \kappa X\right)\right]$$

$$M_{nt} = D(1-\nu)\left(\frac{\partial X}{\partial s} + \kappa\frac{\partial\Omega}{\partial s}\right)$$

$$V_n = -D\left[\Psi - (\nu-1)\left(\frac{\partial^2 X}{\partial s^2} - \frac{\partial\kappa}{\partial s}\frac{\partial\Omega}{\partial s} - \kappa\frac{\partial^2\Omega}{\partial s^2}\right)\right]. \quad (20)$$

The indicated derivatives in Eqs. (19) are given by Eqs. (A1) of the Appendix.

4 Numerical Solution

An analytic solution of the system of simultaneous equations, which form relations (11), (15), (16), (17), and (18), is out of the question. However, a numerical solution is feasible. The differential equations are treated using the finite difference method, the boundary integrals using the boundary element method, and the domain integrals using the finite sector method (Katsikadelis, 1990). Thus, using constant boundary elements to approximate the unknown boundary quantitites, unevenly spaced finite difference to approximate the derivatives, and a collocation technique, the following system of simultaneous algebraic equations is established:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{A}_{14} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{0} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{A}_{44} \end{bmatrix} \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{X} \\ \mathbf{\Phi} \\ \mathbf{\Psi} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{C}_3 \\ \mathbf{C}_4 \end{bmatrix}^{[p]} \quad (21a)$$

$$\mathbf{w} = \mathbf{B}_5 + \mathbf{C}_5\mathbf{p} + [\mathbf{A}_{51} \;\; \mathbf{A}_{52} \;\; \mathbf{A}_{53} \;\; \mathbf{A}_{54}][\mathbf{\Omega} \;\; \mathbf{X} \;\; \mathbf{\Phi} \;\; \mathbf{\Psi}]^T \quad (21b)$$

where

$$\mathbf{\Omega} = [\Omega_1 \;\; \Omega_2 \; \cdots \; \Omega_N]^T \quad \mathbf{X} = [X_1 \;\; X_2 \; \cdots \; X_N]^T$$

$$\mathbf{\Phi} = [\Phi_1 \;\; \Phi_1 \; \cdots \; \Phi_N]^T \quad \mathbf{\Psi} = [\Psi_1 \;\; \Psi_2 \; \cdots \; \Psi_N]^T \quad (22)$$

are the values of the unknown boundary quantities at the nodal points of the N boundary elements

$$\mathbf{w}=[w_1\ w_2\ \cdots\ w_M]^T$$

and

$$\mathbf{p}=[p_1\ p_2\ \cdots\ p_M]^T \tag{23}$$

are the values of the deflection w and the subgrade reaction $p=f(w-d)U(w-d)$ at the M Gauss integration points inside the domain R.

The elements of the constant matrices $\mathbf{A}_{kl}$ ($k=1, 2, 3, 4, 5$, $l=1, 2, 3, 4$), $\mathbf{B}_m$ ($m=1, 2, 3, 4, 5$), $\mathbf{C}_m$ ($m=3, 4, 5$) are given by Eqs. (A3) in the Appendix.

The integrals in the expressions for the coefficients $(A_{31})_{ij}$ and $(A_{32})_{ij}$ have been obtained using the relation $\cos\varphi ds=rd\omega$ (Katsikadelis and Armenakas, 1984).

Equations (21a,b) are linear with respect to the boundary quantities $\boldsymbol{\Omega}$, $\mathbf{X}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ which can be readily eliminated from Eqs. (21b). Thus, solving Eq. (21a) for $\boldsymbol{\Omega}$, $\mathbf{X}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ and substituting them into Eqs. (21b), we obtain the following equations:

$$\mathbf{w}=\mathbf{H}\mathbf{p}+\mathbf{G} \tag{24}$$

where

$$\mathbf{H}=\mathbf{C}_5+[\mathbf{A}_{51}\ \mathbf{A}_{52}\ \mathbf{A}_{53}\ \mathbf{A}_{54}]\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{A}_{14}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{0}\\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34}\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{A}_{44}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{C}_3\\ \mathbf{C}_4\end{bmatrix} \tag{25a}$$

$$\mathbf{G}=\mathbf{B}_5+[\mathbf{A}_{51}\ \mathbf{A}_{52}\ \mathbf{A}_{53}\ \mathbf{A}_{54}]\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{A}_{14}\\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{0}\\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} & \mathbf{A}_{34}\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{43} & \mathbf{A}_{44}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\\ \mathbf{B}_3\\ \mathbf{B}_4\end{bmatrix}. \tag{25b}$$

Equations (24) constitute a system of nonlinear algebraic equations which can be solved numerically to yield the values of the deflection at the internal Gauss points. Back substitution into Eqs. (21a) gives the values of the boundary quantities Ω, X, Φ, Ψ at the nodal points. Subsequently, using the discretized form of Eq. (11), the deflection at any point P within the plate is computed. That is,

$$w(P)=\frac{1}{2\pi D}\sum_{k=1}^{M}(C_5)_{Pk}f(w_k-d_k)U(w_k-d_k)+(B_5)_P$$
$$+\sum_{j=1}^{N}[(A_{51})_{Pj}\Omega_j(A_{52})_{Pj}X_j+(A_{53})_{Pj}\Phi_j+(A_{54})_{Pj}\Psi_j]. \tag{26}$$

The solution of Eq. (24) for the numerical examples presented in the next section has been accomplished iteratively by employing the two-term acceleration method (Isaacson and Keller, 1966).

An initial vector, say $\mathbf{w}^{(o)}=\mathbf{0}$, is assumed. Using this vector and Eq. (1), the values of the subgrade reaction $\mathbf{p}^{(o)}$ at the M Gauss points inside the domain R are obtained. Introducing the vector $\mathbf{p}^{(o)}$ into Eq. (24), a vector $\mathbf{w}^{(1)}$ is computed. Subsequently, the vector $\mathbf{w}^{(k)}$, $k>2$ is obtained from Eq. (24) as

$$\mathbf{w}^{(k)}=\mathbf{H}\mathbf{p}^{(k-1)}+\mathbf{G} \tag{27}$$

where

$$p_i^{(k-1)}=p(\alpha w_i^{(k-1)}+\beta w_i^{(k-2)}),\ \alpha+\beta=1,\ i=1, 2, \ldots, M. \tag{28}$$

The procedure converges to the solution vector $\mathbf{w}$ by choosing appropriately the weight factors α and β. For an example problem, the region of the permissible values of the parameters

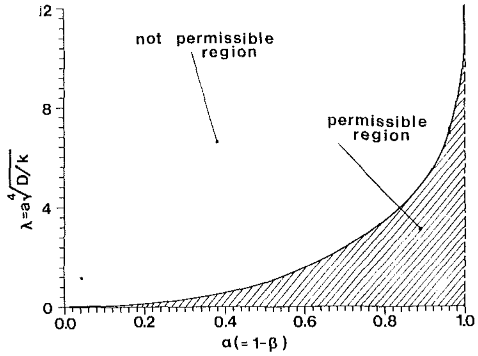

Fig. 4 Permissible values of the weight factors α and β for the convergence of the two-term acceleration method for a clamped or a simply supported rectangular plate with ratio $b/a=1.2$

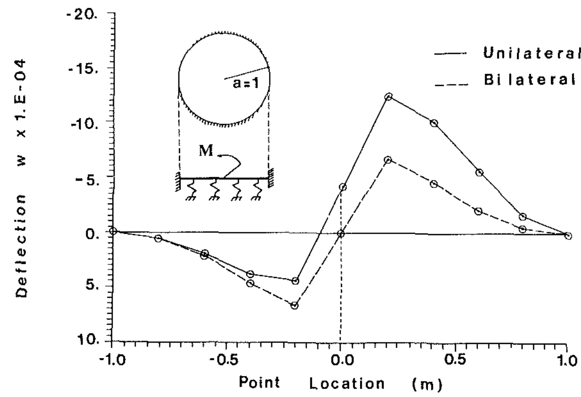

Fig. 5 Deflections along the diameter of a clamped circular plate ($D=192.3077$)

α and β was investigated (Fig. 4). The convergence depends on the mechanical and geometrical properties of the plate and the subgrade. Moreover, the optimum values were observed on the line separating the permissible and not permissible regions.

It should be mentioned that the kernels

$$\frac{\partial^2\Lambda_4(r)}{\partial x^2},\ \frac{\partial^2\Lambda_4(r)}{\partial y^2},\ \frac{\partial^2\Lambda_4(r)}{\partial x\partial y},\ \frac{\partial\Lambda_2(r)}{\partial x},\ \frac{\partial\Lambda_2(r)}{\partial y}$$

$r=|P-Q|$, P, $Q\epsilon R$ involved in the domain integrals (Eqs. (A1) of the Appendix) exhibit a singularity at $P=Q$ and special care must be taken for their evaluation. This singularity is extracted before employing the Gauss integration using the following technique.

In general, these kernels can be written in the form

$$F(P,Q)=R(P,Q)+S(P,Q) \tag{29}$$

where $R(P,Q)$ and $S(P,Q)$ are the regular and singular parts of the function $F(P,Q)$, respectively. Thus, the domain integrals can be written as

$$\iint_R F(P,Q)h(Q)d\sigma_Q=\iint_R R(P,Q)h(Q)d\sigma_Q$$
$$+\iint_R[h(Q)-h(P)]S(P,Q)d\sigma_Q$$
$$+h(P)\iint_R S(P,Q)d\sigma_Q. \tag{30}$$

With the assumption that $\partial h/\partial r$ is bounded, it is

$$\lim_{P\to Q}[h(Q)-h(P)]S(P,Q)=0.$$

Consequently, the first two-domain integrals in the right-hand

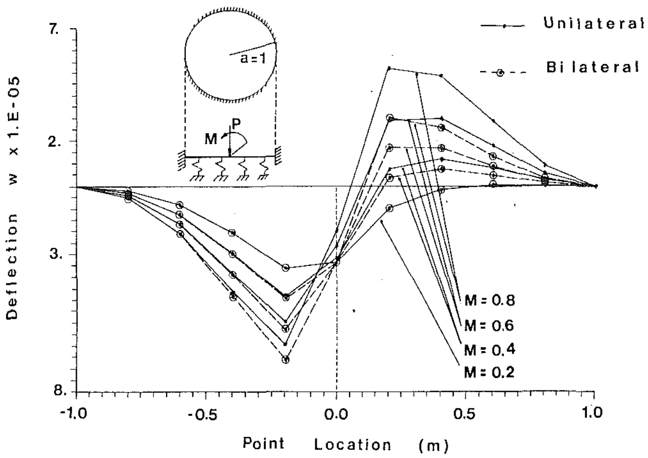

Fig. 6 Deflections along the diameter of a clamped circular plate ($D = 192.3077$)

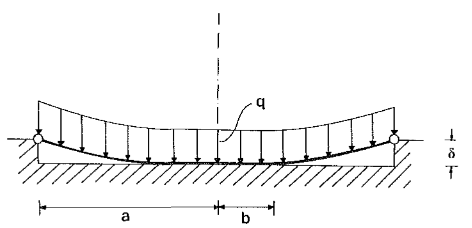

Fig. 7 Simply supported circular plate resting on an absolutely rigid foundation with initial gap ($a = 2.5$, $D = 175$, $\delta = 0.00037$)

side of Eq. (30) are regular. Finally, the third domain integral involving the singular part $S(P,Q)$ can be converted into a line integral on the boundary C of the plate (Nerantzaki and Katsikadelis, 1988).

5 Numerical Examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the range of applications of the developed method.

In all the examples treated, the numerical results have been obtained using 60 constant boundary elements with parabolic approximation of their geometry and 100 Gauss nodal points by dividing the interior of the plate into 4 sectors on each of which a 25 point Gauss-Radau integration is performed.

The worked-out examples are:

1 a clamped circular plate with unit radius loaded by a unit concentrated moment $M = 1$ at its center and resting on a tensionless foundation with $\lambda = a/\sqrt[4]{D/k} = 3$. In Fig. 5 the deflections of the plate along its diameter are compared with the corresponding values of the deflection surface of the plate resting on a bilateral Winkler foundation with the same subgrade reaction modulus. Moreover, in Fig. 6 the deflections along the diameter of the plate loaded by a unit concentrated load and a concentrated moment at its center and resting on a tensionless foundation with $\lambda = a/\sqrt[4]{D/k} = 11$ are compared with the corresponding values of the deflection surface of the plate resting on a bilateral Winkler foundation having the same subgrade reaction modulus.

2 a uniformly loaded circular plate, as shown in Fig. 7, simply supported along the edge and resting on an absolutely rigid foundation with initial gap δ. The radius b of the contact area of the aforementioned plate obtained by this method, $b = 0.78$, is in very good agreement with the corresponding value obtained from an analytical solution, $b = 0.76$ (Hofmann, 1938).

3 a clamped and a simply supported rectangular plate with sides $a = 5.0$ and $b = 6.0$, loaded by a concentrated load $P = 1$ at its center and resting on a tensionless linear foundation. In Table 1 the deflections of the plate along the center line parallel to the x-axis are presented as compared with the corresponding values of the plate resting on a Winkler foundation having the same subgrade reaction modulus. Moreover, in Fig. 8 the deflection contours of the clamped rectangular plate for various values of the parameter λ are presented.

4 a simply supported rectangular plate with sides $a = 5.0$ and $b = 6.0$ loaded by a concentrated load $P = 1$ at its center and unilaterally supported on a nonhomogeneous or a nonlinear foundation. In Table 2 the deflections of the plate along the center line parallel to the x-axis are presented as compared, wherever possible, with the corresponding values of the plate bilaterally supported.

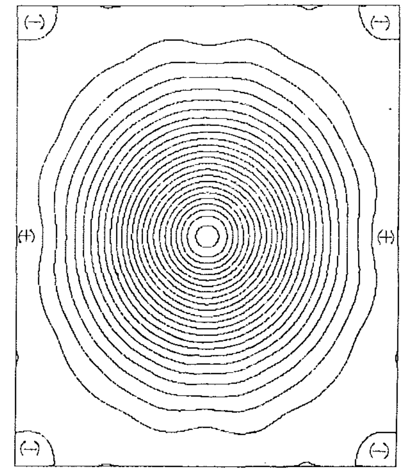

Fig. 8(a)

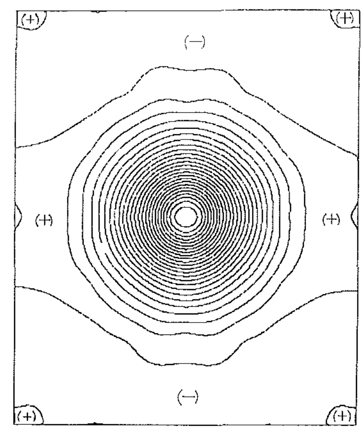

Fig. 8(b)

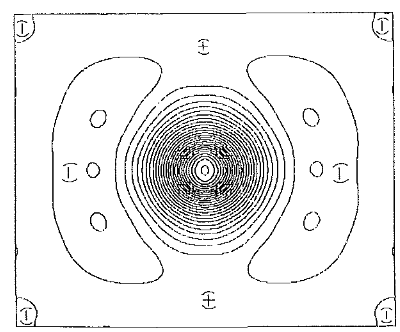

Fig. 8(c)

Fig. 8 Deflection contours of a clamped rectangular plate ($D = 192.3077$) resting on a tensionless linear foundation with subgrade reaction modulus (a) $\lambda = a/\sqrt[4]{D/k} = 3$ ($\Delta w = 0.000033$), (b) $\lambda = a/\sqrt[4]{D/k} = 5$ ($\Delta w = 0.000033$), (c) $\lambda = a/\sqrt[4]{D/k} = 7$ ($\Delta w = 0.0000027$).

Table 1 Deflections of $\overline{w}=w/(Pa^2/D)$ of a rectangular plate resting on a Winkler foundation

y/b	x/a	Clamped $\lambda=a/\sqrt[4]{D/k}=7$ Unilateral	Clamped $\lambda=a/\sqrt[4]{D/k}=7$ Bilateral Katsikadelis (1982)	Simply supported $\lambda=a/\sqrt[4]{D/k}=11$ Unilateral	Simply supported $\lambda=a/\sqrt[4]{D/k}=11$ Bilateral Katsikadelis (1982)
	0.0	0.2561E-02	0.2551E-02	0.1041E-02	0.1033E-02
	0.2	0.1201E-02	0.1174E-02	0.2181E-03	0.2123E-03
0.0	0.4	0.2091E-03	0.2325E-03	−0.8269E-04	−0.1161E-04
	0.6	−0.5934E-04	−0.1732E-04	−0.1332E-03	−0.5615E-05
	0.8	−0.4633E-04	−0.2083E-04	−0.8593E-04	0.3589E-06

Table 2 Deflections $\overline{w}=w/(Pa^2/D)$ of a simply supported rectangular plate resting on nonhomogeneous and on nonlinear foundations

y/b	x/a	Nonhomogeneous $f=16DEw\exp 0.1(x^2+y^2)$ Unilateral	Nonhomogeneous $f=16DEw\exp 0.1(x^2+y^2)$ Bilateral Katsikadelis and Sapountzakis (1987)	Nonlinear $f=w^{1/3}$ Unilateral	Nonlinear $f=10w^{1/3}$ Unilateral
	0.0	0.4912E-02	0.4900E-02	0.1972E-01	0.3314E-02
	0.2	0.3074E-02	0.3066E-02	0.1670E-01	0.1624E-02
0.0	0.4	0.1204E-02	0.1205E-02	0.1188E-01	0.1761E-03
	0.6	0.2668E-03	0.2745E-03	0.7234E-02	−0.2678E-03
	0.8	−0.2718E-05	0.5091E-05	0.3318E-02	−0.2312E-03

6 Concluding Remarks

A boundary element solution is developed for the unilateral contact problem of a thin elastic plate resting on elastic foundation. The main conclusions drawn from this investigation are the following:

1 Plates of arbitrary shape subjected to any type of boundary conditions and loading can be analyzed.

2 The subgrade reaction may depend linearly or nonlinearly on the deflection of the plate.

3 Miscontact between plate and subgrade due to initial gaps is also encountered.

4 The method is well suited for computer-aided analysis.

5 The iterative method converges. The convergence is slow for high values of the parameter λ.

6 The difference between the deflections of unilaterally and bilaterally supported plates increases with the eccentricity of the load, with the parameter λ and decreases with the distance from the boundary.

7 The use of the fundamental solution of the linear part of the governing operator alleviates the method from the computational difficulties arising from the use of special functions (Kelvin, Hankel).

8 The method retains most of the advantages of a BEM solution over a pure domain discretization method.

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APPENDIX

Derivatives of the Integral Representation for the Deflection of the Plate

$$\frac{\partial^2 w(P)}{\partial x^2}=-\frac{1}{2\pi D}\int\int_R \frac{\partial^2\Lambda_4(r)}{\partial x^2}f(w-d)U(w-d)d\sigma + \frac{1}{2\pi D}\int\int_R \frac{\partial^2\Lambda_4(r)}{\partial x^2}g d\sigma - \frac{1}{2\pi}\int_C\left[\frac{\partial^2\Lambda_1(r)}{\partial x^2}\Omega + \frac{\partial^2\Lambda_2(r)}{\partial x^2}X+\frac{\partial^2\Lambda_3(r)}{\partial x^2}\Phi+\frac{\partial^2\Lambda_4(r)}{\partial x^2}\Psi\right]ds \quad (A1a)$$

$$\frac{\partial^2 w(P)}{\partial y^2}=-\frac{1}{2\pi D}\int\int_R \frac{\partial^2\Lambda_4(r)}{\partial y^2}f(w-d)U(w-d)d\sigma + \frac{1}{2D}\int\int_R \frac{\partial^2\Lambda_4(r)}{\partial y^2}g d\sigma - \frac{1}{2\pi}\int_C\left[\frac{\partial^2\Lambda_1(r)}{\partial y^2}\Omega + \frac{\partial^2\Lambda_2(r)}{\partial y^2}X+\frac{\partial^2\Lambda_3(r)}{\partial y^2}\Phi+\frac{\partial^2\Lambda_4(r)}{\partial y^2}\Psi\right]ds \quad (A1b)$$

$$\frac{\partial^2 w(P)}{\partial x\partial y}=-\frac{1}{2\pi D}\int\int_R \frac{\partial^2\Lambda_4(r)}{\partial x\partial y}f(w-d)U(w-d)d\sigma + \frac{1}{2\pi D}\int\int_R \frac{\partial^2\Lambda_4(r)}{\partial x\partial y}g d\sigma - \frac{1}{2\pi}\int_C\left[\frac{\partial^2\Lambda_1(r)}{\partial x\partial y}\Omega + \frac{\partial^2\Lambda_2(r)}{\partial x\partial y}X+\frac{\partial^2\Lambda_3(r)}{\partial x\partial y}\Phi+\frac{\partial^2\Lambda_4(r)}{\partial x\partial y}\Psi\right]ds \quad (A1c)$$

$$\frac{\partial \nabla^2 w(P)}{\partial x} = -\frac{1}{2\pi D}\int\int_R \frac{\partial \Lambda_2(r)}{\partial x} f(w-d)U(w-d)d\sigma + \frac{1}{2\pi D}\int\int_R \frac{\partial \Lambda_2(r)}{\partial x} g d\sigma - \frac{1}{2\pi}\int_C \left[\frac{\partial \Lambda_1(r)}{\partial x}\Phi + \frac{\partial \Lambda_2(r)}{\partial x}\Psi\right] ds \quad (A1d)$$

$$\frac{\partial \nabla^2 w(P)}{\partial y} = -\frac{1}{2\pi D}\int\int_R \frac{\partial \Lambda_2(r)}{\partial y} f(w-d)U(w-d)d\sigma + \frac{1}{2\pi D}\int\int_R \frac{\partial \Lambda_2(r)}{\partial y} g d\sigma - \frac{1}{2\pi}\int_C \left[\frac{\partial \Lambda_1(r)}{\partial y}\Phi + \frac{\partial \Lambda_2(r)}{\partial y}\Psi\right] ds \quad (A1e)$$

where

$$\frac{\partial^2 \Lambda_1}{\partial x^2} = -\frac{2}{r^3}\cos(2\omega-\varphi) \quad (A2a)$$

$$\frac{\partial^2 \Lambda_1}{\partial y^2} = \frac{2}{r^3}\cos(2\omega-\varphi) \quad (A2b)$$

$$\frac{\partial^2 \Lambda_1}{\partial x \partial y} = -\frac{2}{r^3}\sin(2\omega-\varphi) \quad (A2c)$$

$$\frac{\partial^2 \Lambda_2}{\partial x^2} = \frac{1}{r^2}(\sin^2\omega - \cos^2\omega) \quad (A2d)$$

$$\frac{\partial^2 \Lambda_2}{\partial y^2} = \frac{1}{r^2}(\cos^2\omega - \sin^2\omega) \quad (A2e)$$

$$\frac{\partial^2 \Lambda_2}{\partial x \partial y} = -\frac{\sin^2\omega}{r^2} \quad (A2f)$$

$$\frac{\partial^2 \Lambda_3}{\partial x^2} = \frac{\sin\varphi\cos\omega\sin\omega}{r} - \frac{\cos\varphi}{2r} \quad (A2g)$$

$$\frac{\partial^2 \Lambda_3}{\partial y^2} = -\frac{\sin\varphi\cos\omega\sin\omega}{r} - \frac{\cos\varphi}{2r} \quad (A2h)$$

$$\frac{\partial^2 \Lambda_3}{\partial x \partial y} = -\frac{\sin\varphi\cos2\omega}{2r} \quad (A2i)$$

$$\frac{\partial^2 \Lambda_4}{\partial x^2} = \frac{1}{2}lnr + \frac{1}{4} + \frac{1}{2}\cos^2\omega \quad (A2j)$$

$$\frac{\partial^2 \Lambda_4}{\partial y^2} = \frac{1}{2}lnr + \frac{1}{4} + \frac{1}{2}\sin^2\omega \quad (A2k)$$

$$\frac{\partial^2 \Lambda_4}{\partial x \partial y} = \frac{1}{4}\sin2\omega \quad (A2l)$$

$$\frac{\partial \Lambda_1}{\partial x} = -\frac{\cos(\omega-\varphi)}{r^2} \quad \frac{\partial \Lambda_1}{\partial y} = -\frac{\sin(\omega-\varphi)}{r^2} \quad (A2m,n)$$

$$\frac{\partial \Lambda_2}{\partial x} = -\frac{\cos\omega}{r} \quad \frac{\partial \Lambda_2}{\partial y} = -\frac{\sin\omega}{r} \quad (A2o,p)$$

in which $\omega = \mathbf{x}\hat{,}\mathbf{r}$ is the angle between the x-axis and the vector $\mathbf{r}$ and $\varphi = \mathbf{r}\hat{,}\mathbf{n}$ is the angle between the vector $\mathbf{r}$ and the outward normal $\mathbf{n}$ (see Fig. 2).

Elements of the Matrices A, B, C

$$(A_{11})_{i,i-1} = -(\alpha_2)_i s_i\left(-\frac{\partial \kappa_i}{\partial s}s_i + 2\kappa_i\right) \quad (A3a)$$

$$(A_{11})_{ii} = (\alpha_1)_i/[(\nu-1)De_i] + (\alpha_1)_i(s_{i-1}+s_i)\left[(s_{i-1}-s_i)\frac{\partial \kappa_i}{\partial s} + 2\kappa_i\right] \quad (A3b)$$

$$(A_{11})_{i,i+1} = -(\alpha_2)_i s_{i-1}\left(\frac{\partial \kappa_i}{\partial s}s_{i-1} + 2\kappa_i\right) \quad (A3c)$$

$$(A_{12})_{i,i-1} = 2(\alpha_2)_i s_i \quad (A3d)$$

$$(A_{12})_{ii} = -2(\alpha_2)_i(s_{i-1}+s_i) \quad (A3e)$$

$$(A_{12})_{i,i+1} = 2(\alpha_2)_i s_{i-1} \quad (A3f)$$

$$(A_{14})_{ii} = -(\alpha_2)_i/[(\nu-1)De_i] \quad (A3g)$$

$$(A_{21})_{i,i-1} = -2(\beta_2)_i s_i \quad (A3h)$$

$$(A_{21})_{ii} = 2(\beta_2)_i(s_{i-1}+s_i) \quad (A3i)$$

$$(A_{21})_{i,i+1} = -2(\beta_2)_i s_{i-1} \quad (A3j)$$

$$(A_{22})_{ii} = (\beta_1)_i/[(\nu-1)De_i] - (\beta_2)_i\kappa_i/e_i \quad (A3k)$$

$$(A_{23})_{ii} = -(\beta_2)_i/[(\nu-1)e_i] \quad (A3l)$$

$$(A_{31})_{ij} = -\int_j d\omega_{iq} + \alpha\delta_{ij} \quad (A3m)$$

$$(A_{32})_{ij} = \int_j (lnr_{iq}+1)ds_q \quad (A3n)$$

$$(A_{33})_{ij} = -\frac{1}{4}\int_j r_{iq}^2(2lnr_{iq}+1)d\omega_{iq} \quad (A3o)$$

$$(A_{34})_{ij} = \frac{1}{4}\int_j r_{iq}^2 lnr_{iq} ds_q \quad (A3p)$$

$$(A_{43})_{ij} = (A_{31})_{ij} \quad (A3q)$$

$$(A_{44})_{ij} = (A_{32})_{ij} \quad (A3r)$$

$$(A_{51})_{lj} = \frac{1}{2\pi}\int_j d\omega_{lq} + \alpha\delta_{lj} \quad (A3s)$$

$$(A_{52})_{lj} = -\frac{1}{2\pi}\int_j (lnr_{lq}+1)ds_q \quad (A3t)$$

$$(A_{53})_{lj} = \frac{1}{8\pi}\int_j r_{lq}^2(2lnr_{lq}+1)d\omega_{lq} \quad (A3u)$$

$$(A_{54})_{lj} = -\frac{1}{8\pi}\int_j r_{lq}^2 lnr_{lq} ds_q \quad (A3v)$$

$$(B_1)_i = \alpha_3(s_i)/[De_i(\nu-1)] \quad (A4a)$$

$$(B_2)_i = \beta_3(s_i)/[De_i(\nu-1)] \quad (A4b)$$

$$(B_3)_i = \frac{1}{4D}\int\int_R r_{iQ}^2 lnr_{iQ} g_Q d\sigma_Q \quad (A4c)$$

$$(B_4)_i = \frac{1}{D}\int\int_R (lnr_{iQ}+1)g_Q d\sigma_Q \quad (A4d)$$

$$(B_5)_l = \frac{1}{8\pi D}\int\int_R r_{lQ}^2 lnr_{lQ} g_Q d\sigma_Q \quad (A4e)$$

$$(C_3)_i = \frac{1}{4D}C_m r_{im}^2 lnr_{im} \quad (A4f)$$

$$(C_4)_i = \frac{1}{D}C_m(lnr_{im}+1) \quad (A4g)$$

$$(C_5)_l = -\frac{1}{8\pi D}C_m r_{lm}^2 lnr_{lm} \quad (A4h)$$

where $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$, l, $m = 1, 2, \ldots, M$; s_{i-1}, s_i are the arc lengths between the nodal points $i-1$, i, and i, $i+1$, respectively; $e_i = 1/[s_{i-1}s_i(s_{i-1}+s_i)]$; $r_{iP} = |p_i - P|$; $P \in R$; $r_{iq} = |p_i - q|$, $q \in j$-element; ω_{iq} = is the angle between the x-axis and the line r_{ij} (see Fig. 2); $(\alpha_n)_i$ and $(\beta_n)_i$ ($n = 1, 2, 3$) are the values of the functions $\alpha_n(s)$ and $\beta_n(s)$, respectively, at the nodal point i; the symbol $\int_j$ indicates integration over the j-element; C_m are the modified weight factors of the Gauss integration on the domain R (Katsikadelis, 1990).