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THE PARABOLIC p-LAPLACE EQUATION IN CARNOT GROUPS

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Abstract. By establishing a parabolic maximum principle, we show uniqueness of viscosity solutions to the parabolic p-Laplace equation and then examine the limit as t goes to infinity. Additionally, we explore the limit as p goes to infinity.

1. Background and motivation

In [7], uniqueness of viscosity solutions to a class of fully nonlinear subelliptic equations in Carnot groups was established. The key tool used was the Carnot Group Maximum Principle, which is a sub-Riemannian analog of the Euclidean version of [10]. In particular, the following theorem was proved:

Theorem 1.1. [7] *Let Ω be a bounded domain in a Carnot group and let $v\colon \partial\Omega \to \mathbf{R}$ be a continuous function. Then for $1 < p < \infty$, the Dirichlet problem*

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ u = v & \text{on } \partial\Omega, \end{cases}$$

has a unique viscosity solution u_p.

It is natural to ask if this result can be extended to parabolic equations in Carnot groups. Namely, our main goal is to prove the following conjecture:

Conjecture 1.2. *Let Ω be a bounded domain in a Carnot group and let $T > 0$. Let $\psi \in C(\overline{\Omega})$ and let $g \in C(\Omega \times [0,T))$ Then the Cauchy–Dirichlet problem*

$$\begin{cases} u_t - \Delta_p u = 0 & \text{in } \Omega \times (0,T), \\ u(x,0) = \psi(x) & \text{on } \overline{\Omega}, \\ u(x,t) = g(x,t) & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

has a unique viscosity solution u.

In Section 2, we recall the fundamental properties of Carnot groups and key facts from calculus on Carnot groups. In Section 3, we discuss the relationship between various notions of solution to the parabolic p-Laplace equation. In Section 4, we

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establish the main properties of parabolic viscosity solutions, including the parabolic version of the Carnot Group Maximum Principle. In Section 5, we prove that parabolic viscosity solutions are unique. In Sections 6 and 7, we explore the asymptotic limits of the parabolic viscosity solutions as $t \to \infty$ and as $p \to \infty$. The authors would also like to thank the anonymous referee for their suggestions and advice.

2. Calculus on Carnot groups

We begin by denoting an arbitrary Carnot group in $\mathbf{R}^N$ by G and its corresponding Lie Algebra by g. Recall that g is nilpotent and stratified, resulting in the decomposition

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

for appropriate vector spaces that satisfy the Lie bracket relation $[V_1, V_j] = V_{1+j}$. The Lie Algebra g is associated with the group G via the exponential map $\exp\colon g \to G$. Since this map is a diffeomorphism, we can choose a basis for g so that it is the identity map. Denote this basis by

$$X_1, X_2, \ldots, X_{n_1}, Y_1, Y_2, \ldots, Y_{n_2}, Z_1, Z_2, \ldots, Z_{n_3}$$

so that

$$\begin{aligned} V_1 &= \operatorname{span}\{X_1, X_2, \ldots, X_{n_1}\}, \\ V_2 &= \operatorname{span}\{Y_1, Y_2, \ldots, Y_{n_2}\}, \\ V_3 \oplus V_4 \oplus \cdots \oplus V_l &= \operatorname{span}\{Z_1, Z_2, \ldots, Z_{n_3}\}. \end{aligned}$$

We endow g with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. Clearly, the Riemannian dimension of g (and so G) is $N = n_1 + n_2 + n_3$. However, we will also consider the homogeneous dimension of G, denoted $\mathcal{Q}$, which is given by

$$\mathcal{Q} = \sum_{i=1}^{l} i \cdot \dim V_i. \tag{2.1}$$

Before proceeding with the calculus, we recall the group and metric space properties. Since the exponential map is the identity, the group law is the Campbell–Hausdorff formula (see, for example, [8]). For our purposes, this formula is given by

$$p \cdot q = p + q + \tfrac{1}{2}[p, q] + R(p, q), \tag{2.2}$$

where $R(p, q)$ are terms of order 3 or higher. The identity element of G will be denoted by 0 and called the origin. There is also a natural metric on G, which is the Carnot–Carathéodory distance, defined for the points p and q as follows:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| \, dt,$$

where the set Γ is the set of all curves γ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t) \in V_1$. By Chow's theorem (see, for example, [4]) any two points can be connected by such a curve, which means $d_C(p, q)$ is an honest metric. Define a Carnot–Carathéodory ball of radius r centered at a point p_0 by

$$B(p_0, r) = \{p \in G \colon d_C(p, p_0) < r\}.$$

In addition to the Carnot–Carathéodory metric, there is a smooth (off the origin) gauge. This gauge is defined for a point $p = (\zeta_1, \zeta_2, \dots, \zeta_l)$ with $\zeta_i \in V_i$ by

$$\mathcal{N}(p) = \left(\sum_{i=1}^{l} \|\zeta_i\|^{\frac{2l!}{i}} \right)^{\frac{1}{2l!}} \tag{2.3}$$

and it induces a metric $d_{\mathcal{N}}$ that is bi-Lipschitz equivalent to the Carnot–Carathéodory metric and is given by

$$d_{\mathcal{N}}(p,q) = \mathcal{N}(p^{-1} \cdot q).$$

We define a gauge ball of radius r centered at a point p_0 by

$$B_{\mathcal{N}}(p_0, r) = \{p \in G \colon d_{\mathcal{N}}(p, p_0) < r\}.$$

In this environment, a smooth function $u \colon G \to \mathbf{R}$ has the horizontal derivative given by

$$\nabla_0 u = (X_1 u, X_2 u, \dots, X_{n_1} u)$$

and the symmetrized horizontal second derivative matrix, denoted by $(D^2u)^\star$, with entries

$$((D^2u)^\star)_{ij} = \tfrac{1}{2}(X_i X_j u + X_j X_i u)$$

for $i, j = 1, 2, \dots, n_1$. We also consider the semi-horizontal derivative given by

$$\nabla_1 u = (X_1 u, X_2 u, \dots, X_{n_1} u, Y_1 u, Y_2 u, \dots, Y_{n_2} u).$$

Using the above derivatives, we define the subelliptic p-Laplace operator for $1 < p < \infty$ by

$$\begin{aligned} \Delta_p f &= \operatorname{div}(\|\nabla_0 f\|^{p-2} \nabla_0 f) = \sum_{i=1}^{n_1} X_i(\|\nabla_0 f\|^{p-2} \nabla_0 f) \\ &= \Big(\|\nabla_0 f\|^{p-2} \operatorname{tr}((D^2 f)^\star) + (p-2) \|\nabla_0 f\|^{p-4} \langle (D^2 f)^\star \nabla_0 f, \nabla_0 f \rangle \Big). \end{aligned}$$

Given $T > 0$ and a function $u \colon G \times [0,T] \to \mathbf{R}$, we may define the analogous subparabolic p-Laplace operator by

$$u_t - \Delta_p u.$$

We recall that for any open set $\mathcal{O} \subset G$, the function f is in the horizontal Sobolev space $W^{1,p}(\mathcal{O})$ if f and $X_i f$ are in $L^p(\mathcal{O})$ for $i = 1, 2, \dots, n_1$. Replacing $L^p(\mathcal{O})$ by $L^p_{loc}(\mathcal{O})$, the space $W^{1,p}_{\text{loc}}(\mathcal{O})$ is defined similarly. The space $W^{1,p}_0(\mathcal{O})$ is the closure in $W^{1,p}(\mathcal{O})$ of smooth functions with compact support. In addition, we recall a function $u \colon G \to \mathbf{R}$ is $\mathcal{C}^2_{\text{sub}}$ if $\nabla_1 u$ and $X_i X_j u$ are continuous for all $i, j = 1, 2, \dots n_1$. Note that $\mathcal{C}^2_{\text{sub}}$ is not equivalent to (Euclidean) C^2. For spaces involving time, the space $C(t_1, t_2; X)$ consists of all continuous functions $u \colon [t_1, t_2] \to X$ with $\max_{t_1 \le t \le t_2} \|u(\cdot, t)\|_X < \infty$. A similar definition holds for $L^p(t_1, t_2; X)$.

Given an open box $\mathcal{O} = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N)$, we define the parabolic space $\mathcal{O}_{t_1,t_2}$ to be $\mathcal{O} \times [t_1, t_2]$. Its parabolic boundary is given by $\partial_{\text{par}} \mathcal{O}_{t_1,t_2} = (\overline{\mathcal{O}} \times \{t_1\}) \cup (\partial \mathcal{O} \times (t_1, t_2])$. Following [18], we define the function space $V^p(t_1, t_2; \mathcal{O}) = C(t_1, t_2; L^2(\mathcal{O})) \cap L^p(t_1, t_2; W^{1,p}(\mathcal{O}))$. For a function $u \in V^p(t_1, t_2; \mathcal{O})$, we have $t \mapsto \int_{\mathcal{O}} |u(x,t)|^2 \, dx$ is a continuous map in $[t_1, t_2]$, $\nabla_0 u(x,t)$ exists for almost every $t \in [t_1, t_2]$ and $\int_{t_1}^{t_2} \int_{\mathcal{O}} u^2 + |\nabla_0 u|^p \, dx \, dt$ is finite.

Finally, recall that if G is a Carnot group with homogeneous dimension $\mathcal{Q}$, then $G \times \mathbf{R}$ is again a Carnot group of homogeneous dimension $\mathcal{Q}+1$ where we have added an extra vector field $\frac{\partial}{\partial t}$ to the first layer of the grading. This allows us to give meaning to notations such as $W^{1,2}(\mathcal{O}_{t_1,t_2})$ and $\mathcal{C}^2_{\text{sub}}(\mathcal{O}_{t_1,t_2})$ where we consider $\nabla_0 u$ to be $\big(X_1u, X_2u, \ldots, X_{n_1}u, \frac{\partial u}{\partial t}\big)$.

3. Parabolic jets and viscosity solutions

3.1. Parabolic jets. Let S^k be the set of $k \times k$ symmetric matrices. We define the parabolic superjet of $u(p,t)$ at the point $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$, denoted $P^{2,+}u(p_0,t_0)$, by using triples $(a,\eta,X) \in \mathbf{R} \times V_1 \oplus V_2 \times S^{n_1}$ so that $(a,\eta,X) \in P^{2,+}u(p_0,t_0)$ if

$$\begin{aligned} u(p,t) \leq u(p_0,t_0) + a(t-t_0) + \langle \eta, \widehat{p_0^{-1}\cdot p}\rangle + \tfrac{1}{2}\langle X\overline{p_0^{-1}\cdot p}, \overline{p_0^{-1}\cdot p}\rangle \\ + o(|t-t_0| + |p_0^{-1}\cdot p|^2) \quad \text{as } (p,t) \to (p_0,t_0). \end{aligned}$$

We recall that $n_i = \dim V_i$ and define $\overline{p_0^{-1}\cdot p}$ as the first n_1 coordinates of $p_0^{-1}\cdot p$ and $\widehat{p_0^{-1}\cdot p}$ as the first n_1+n_2 coordinates of $p_0^{-1}\cdot p$. This definition is an extension of the superjet definition for subparabolic equations in the Heisenberg group [6]. We define the subjet $P^{2,-}u(p_0,t_0)$ by

$$P^{2,-}u(p_0,t_0) = -P^{2,+}(-u)(p_0,t_0).$$

We define the set theoretic closure of the superjet, denoted $\overline{P}^{2,+}u(p_0,t_0)$, by requiring $(a,\eta,X) \in \overline{P}^{2,+}u(p_0,t_0)$ exactly when there is a sequence $(a_n,p_n,t_n,u(p_n,t_n),\eta_n,X_n) \to (a,p_0,t_0,u(p_0,t_0),\eta,X)$ with the triple $(a_n,\eta_n,X_n) \in P^{2,+}u(p_n,t_n)$. A similar definition holds for the closure of the subjet.

We may also define jets using appropriate test functions. Given a function $u\colon \mathcal{O}_{t_1,t_2} \to \mathbf{R}$ we consider the set $\mathcal{A}u(p_0,t_0)$ given by

$$\mathcal{A}u(p_0,t_0) = \{\phi \in \mathcal{C}^2_{\text{sub}}(\mathcal{O}_{t_1,t_2})\colon u(p,t)-\phi(p,t) \leq u(p_0,t_0)-\phi(p_0,t_0) = 0 \;\forall (p,t) \in \mathcal{O}_{t_1,t_2}\}$$

consisting of all test functions that touch u from above at (p_0,t_0). We define the set of all test functions that touch from below, denoted $\mathcal{B}u(p_0,t_0)$, similarly.

The following lemma relates the test functions to jets. The proof is identical to Lemma 3.1 in [6], but uses the (smooth) gauge $\mathcal{N}(p)$ instead of Euclidean distance.

Lemma 3.1.

$$P^{2,+}u(p_0,t_0) = \{(\phi_t(p_0,t_0), \nabla\phi(p_0,t_0), (D^2\phi(p_0,t_0))^\star)\colon \phi \in \mathcal{A}u(p_0,t_0)\}.$$

3.2. Jet twisting. We recall that the set $V_1 = \operatorname{span}\{X_1, X_2, \ldots, X_{n_1}\}$ and notationally, we will always denote n_1 by n. The vectors X_i at the point $p \in G$ can be written as

$$X_i(p) = \sum_{j=1}^N a_{ij}(p)\frac{\partial}{\partial x_j}$$

forming the $n \times N$ matrix $\mathbf{A}$ with smooth entries $\mathbf{A}_{ij} = a_{ij}(p)$. By linear independence of the X_i, $\mathbf{A}$ has rank n. Similarly,

$$Y_i(p) = \sum_{j=1}^N b_{ij}(p)\frac{\partial}{\partial x_j}$$

forming the $n_2 \times N$ matrix $\mathbf{B}$ with smooth entries $\mathbf{B}_{ij} = b_{ij}$. The matrix $\mathbf{B}$ has rank n_2. The following lemma differs from [7, Corollary 3.2] only in that there is now a parabolic term. This term however, does not need to be twisted. The proof is then identical, as only the space terms need twisting.

Lemma 3.2. *Let $(a, \eta, X) \in \overline{P}^{2,+}_{\mathrm{eucl}} u(p,t)$. (Recall that $(\eta, X) \in \mathbf{R}^N \times S^N$.) Then*

$$(a, \mathbf{A} \cdot \eta \oplus \mathbf{B} \cdot \eta,\ \mathbf{A} X \mathbf{A}^T + \mathbf{M}) \in \overline{P}^{2,+} u(p,t).$$

Here the entries of the (symmetric) matrix $\mathbf{M}$ are given by

$$\mathbf{M}_{ij} = \begin{cases} \displaystyle\sum_{k=1}^{N} \sum_{l=1}^{N} \left(a_{il}(p) \frac{\partial}{\partial x_l} a_{jk}(p) + a_{jl}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \right) \eta_k, & i \neq j, \\ \displaystyle\sum_{k=1}^{N} \sum_{l=1}^{N} a_{il}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \eta_k, & i = j. \end{cases}$$

3.3. Viscosity solutions. We consider parabolic equations of the form

$$u_t + F(t, p, u, \nabla_1 u, (D^2 u)^\star) = 0 \tag{3.1}$$

for continuous and proper $F\colon [0,T] \times G \times \mathbf{R} \times g \times S^n \to \mathbf{R}$. [10] We recall that S^n is the set of $n \times n$ symmetric matrices (where $\dim V_1 = n$) and the derivatives $\nabla_1 u$ and $(D^2 u)^\star$ are taken in the space variable p. We then use the jets to define subsolutions and supersolutions to equation (3.1) in the usual way.

Definition 1. Let $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ be as above. The upper semicontinuous function u is a *viscosity subsolution* in $\mathcal{O}_{t_1,t_2}$ if for all $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ we have $(a, \eta, X) \in \overline{P}^{2,+} u(p_0, t_0)$ produces

$$a + F(t_0, p_0, u(p_0, t_0), \eta, X) \leq 0.$$

A lower semicontinuous function u is a *viscosity supersolution* in $\mathcal{O}_{t_1,t_2}$ if for all $(p_0, t_0) \in \mathcal{O}_{t_1,t_2}$ we have $(b, \nu, Y) \in \overline{P}^{2,-} u(p_0, t_0)$ produces

$$b + F(t_0, p_0, u(p_0, t_0), \nu, Y) \geq 0.$$

A continuous function u is a *viscosity solution* in $\mathcal{O}_{t_1,t_2}$ if it is both a viscosity subsolution and viscosity supersolution.

We also wish to define what [17] refers to as parabolic viscosity solutions. We first need to consider the set

$$\mathcal{A}^- u(p_0, t_0) = \{\phi \in \mathcal{C}^2(\mathcal{O}_{t_1,t_2})\colon u(p,t) - \phi(p,t) \leq u(p_0,t_0) - \phi(p_0,t_0) = 0 \text{ for } t < t_0\}$$

consisting of all functions that touch from above only when $t < t_0$. Note that this set is larger than $\mathcal{A}u$ and corresponds physically to the past alone playing a role in determining the present. We define $\mathcal{B}^- u(p_0, t_0)$ similarly. We then have the following definition.

Definition 2. An upper semicontinuous function u on $\mathcal{O}_{t_1,t_2}$ is a *parabolic viscosity subsolution* in $\mathcal{O}_{t_1,t_2}$ if $\phi \in \mathcal{A}^- u(p_0, t_0)$ produces

$$\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0), \nabla_1 \phi(p_0, t_0), (D^2 \phi(p_0, t_0))^\star) \leq 0.$$

An lower semicontinuous function u on $\mathcal{O}_{t_1,t_2}$ is a *parabolic viscosity supersolution* in $\mathcal{O}_{t_1,t_2}$ if $\phi \in \mathcal{B}^- u(p_0, t_0)$ produces

$$\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0), \nabla_1 \phi(p_0, t_0), (D^2 \phi(p_0, t_0))^\star) \geq 0.$$

A continuous function is a *parabolic viscosity solution* if it is both a parabolic viscosity supersolution and subsolution.

We have the following proposition whose proof is obvious.

Proposition 3.3. *Parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions.*

3.4. The Carnot Parabolic Maximum Principle. We wish to prove an analog of the Carnot Maximum Prinicple [7, Lemma 3.6] for viscosity solutions to parabolic partial differential equations. We formulate the Carnot Parabolic Maximum Principle:

Lemma 3.4. (Carnot Parabolic Maximum Principle) *Let u be a viscosity subsolution to the equation* (3.1) *and v be a viscosity supersolution to the equation* (3.1) *in the bounded parabolic set $\Omega \times (0,T)$ where Ω is a (bounded) domain and let τ be a positive real parameter. Let $\phi(p,q,t) = \varphi(p \cdot q^{-1}, t)$ be a C^2 function in the space variables p and q and a C^1 function in t. Suppose the local maximum*

$$M_\tau \equiv \max_{\overline{\Omega}\times\overline{\Omega}\times[0,T]} \{u(p,t) - v(q,t) - \tau\phi(p,q,t)\} \tag{3.2}$$

occurs at the interior point (p_τ, q_τ, t_τ) of the parabolic set $\Omega \times \Omega \times (0,T)$. Define the $n \times n$ matrix W by

$$W_{ij} = X_i(p)X_j(q)\phi(p_\tau, q_\tau, t_\tau).$$

Let the $2n \times 2n$ matrix $\mathfrak{W}$ be given by

$$\mathfrak{W} = \begin{pmatrix} 0 & \frac{1}{2}(W - W^T) \\ \frac{1}{2}(W^T - W) & 0 \end{pmatrix} \tag{3.3}$$

and let the matrix $\mathcal{W} \in S^{2N}$ be given by

$$\mathcal{W} = \begin{pmatrix} D^2_{pp}\phi(p_\tau, q_\tau, t_\tau) & D^2_{pq}\phi(p_\tau, q_\tau, t_\tau) \\ \\ D^2_{qp}\phi(p_\tau, q_\tau, t_\tau) & D^2_{qq}\phi(p_\tau, q_\tau, t_\tau) \end{pmatrix}. \tag{3.4}$$

Suppose

$$\lim_{\tau\to\infty} \tau\phi(p_\tau, q_\tau, t_\tau) = 0.$$

Then for each $\tau > 0$, there exists real numbers a_1 and a_2, symmetric matrices $\mathcal{X}_\tau$ and $\mathcal{Y}_\tau$ and vector $\Upsilon_\tau \in V_1 \oplus V_2$, namely $\Upsilon_\tau = \nabla_1(p)\phi(p_\tau, q_\tau, t_\tau)$, so that the following hold:

A) $(a_1, \tau\Upsilon_\tau, \mathcal{X}_\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau)$ *and* $(a_2, \tau\Upsilon_\tau, \mathcal{Y}_\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau)$.
B) $a_1 - a_2 = \phi_t(p_\tau, q_\tau, t_\tau)$.
C) *For any vectors $\xi, \epsilon \in V_1$, we have*

$$\begin{aligned} \langle \mathcal{X}_\tau\xi, \xi\rangle - \langle \mathcal{Y}_\tau\epsilon, \epsilon\rangle \leq{}& \tau\langle (D^2_p\phi)^\star(p_\tau, q_\tau, t_\tau)(\xi - \epsilon), (\xi - \epsilon)\rangle \\ &+ \tau\langle \mathfrak{W}(\xi \oplus \epsilon), (\xi \oplus \epsilon)\rangle + \tau\|\mathcal{W}\|^2\|\mathbf{A}(\hat{p})^T\xi \oplus \mathbf{A}(\hat{q})^T\epsilon\|^2. \end{aligned} \tag{3.5}$$

In particular,

$$\langle \mathcal{X}_\tau\xi, \xi\rangle - \langle \mathcal{Y}_\tau\xi, \xi\rangle \lesssim \tau\|\mathcal{W}\|^2\|\xi\|^2. \tag{3.6}$$

Proof. We first need to check that condition 8.5 of [10] is satisfied, namely that there exists an $r > 0$ so that for each M, there exists a C so that $b \leq C$ when $(b, \eta, X) \in P^{2,+}_{\text{eucl}} u(p,t)$, $|p - p_\tau| + |t - t_\tau| < r$, and $|u(p,t)| + \|\eta\| + \|X\| \leq M$ with a similar statement holding for $-v$. If this condition is not met, then for each $r > 0$, we have an M so that for all C, $b > C$ when $(b, \eta, X) \in P^{2,+}_{\text{eucl}} u(p,t)$, $|p - p_\tau| + |t - t_\tau| < r$, and $|u(p,t)| + \|\eta\| + \|X\| \leq M$. By Lemma 3.2, we conclude $(b, \hat{\eta}, \mathcal{X}) \in P^{2,+} u(p,t)$ for the appropriate vector $\hat{\eta}$ and matrix $\mathcal{X} \in S^n$. This contradicts the fact that u is a subsolution. A similar conclusion is reached for $-v$ and so we conclude that this condition holds. We may then apply Theorem 8.3 of [10] to get

$$(a_1, \tau D_p \phi(p_\tau, q_\tau, t_\tau), X_\tau) \in \overline{P}^{2,+}_{\text{eucl}} u(p_\tau, t_\tau),$$
$$(a_2, -\tau D_q \phi(p_\tau, q_\tau, t_\tau), Y_\tau) \in \overline{P}^{2,-}_{\text{eucl}} v(q_\tau, t_\tau),$$

and by the Carnot Group Maximum Principle [7, Lemma 3.6], we have

$$(a_1, \tau \Upsilon_\tau, \mathcal{X}_\tau) \in \overline{P}^{2,+} u(p_\tau, t_\tau),$$
$$(a_2, \tau \Upsilon_\tau, \mathcal{Y}_\tau) \in \overline{P}^{2,-} v(q_\tau, t_\tau),$$

for the vector $\Upsilon_\tau \in V_1 \oplus V_2$ defined above and matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau \in S^n$. The inequalities are from the Carnot Group Maximum Priniciple [7, Lemma 3.6]. □

Corollary 3.5. *Let $\phi(p,q,t) = \phi(p,q) = \varphi(p \cdot q^{-1})$ be independent of t and a non-negative function. Suppose $\phi(p,q) = 0$ exactly when $p = q$. Then*

$$\lim_{\tau \to \infty} \tau \phi(p_\tau, q_\tau) = 0.$$

In particular, if

$$\phi(p,q,t) = \frac{1}{m} \sum_{i=1}^{N} \left((p \cdot q^{-1})_i \right)^m \tag{3.7}$$

for some even *integer $m \geq 4$ where $(p \cdot q^{-1})_i$ is the i-th component of the Carnot group multiplication group law, then for the vector Υ_τ and matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau$, from the Lemma, we have*

A) $(a_1, \tau \Upsilon_\tau, \mathcal{X}_\tau) \in \overline{P}^{2,+} u(p_\tau, t_\tau)$ *and* $(a_1, \tau \Upsilon_\tau, \mathcal{Y}_\tau) \in \overline{P}^{2,-} v(q_\tau, t_\tau)$.

B) *The vector Υ_τ satisfies*

$$\|\Upsilon_\tau\| \sim \phi(p_\tau, q_\tau)^{\frac{m-1}{m}}.$$

C) *For any fixed vector $\xi \in V_1$, we have*

$$\langle \mathcal{X}_\tau \xi, \xi \rangle - \langle \mathcal{Y}_\tau \xi, \xi \rangle \lesssim \tau \|\mathcal{W}\|^2 \|\xi\|^2 \lesssim \tau (\phi(p_\tau, q_\tau))^{\frac{2m-4}{m}} \|\xi\|^2. \tag{3.8}$$

Proof. Note that the defintion of M_τ (3.2) makes it a decreasing function of τ and by the compactness the set $\overline{\Omega} \times \overline{\Omega} \times [0,T]$, we have M_τ is finite. Thus, $\lim_{\tau \to \infty} M_\tau = M < \infty$. Using the definition of M_τ yields

$$M_{\frac{\tau}{2}} \geq u(p_\tau, t_\tau) - v(q_\tau, t_\tau) - \frac{\tau}{2} \phi(p_\tau, q_\tau) = M_\tau + \frac{\tau}{2} \phi(p_\tau, q_\tau)$$

and so

$$0 \leq \tau \phi(p_\tau, q_\tau) \leq 2(M_{\frac{\tau}{2}} - M_\tau)$$

and the result follows by taking limits. Property A follows from the fact that $\phi(p,q)$ is independent of t, Property B follows from the definition of $\phi(p,q)$ and Υ_τ while Property C follows from the equation (3.4). □

3.5. An application to a class of equations.

Definition 3. We say the continuous, proper function F in the equation (3.1) is *admissable* if for each $t \in [0,T]$, there is the same function $\omega\colon [0,\infty] \to [0,\infty]$ with $\omega(0+) = 0$ so that F satisfies

$$F(t,q,r,\tau\Upsilon,Y) - F(t,p,r,\tau\Upsilon,X) \le \omega(d_C(p,q) + \tau\|\Upsilon(p,q)\|^2 + \|X-Y\|), \tag{3.9}$$

where $\tau \in \mathbf{R}^+$.

We now formulate the comparison principle for the following problem.

$$\begin{cases} u_t + F(t,p,u,\nabla_1 u,(D^2u)^\star) = 0 & \text{in } (0,T)\times\Omega, & (E)\\ u(p,t) = g(p,t), & p\in\partial\Omega,\ t\in[0,T), & (BC)\\ u(p,0) = \psi(p), & p\in\overline{\Omega}. & (IC)\end{cases} \tag{3.10}$$

Here, $\psi \in C(\overline{\Omega})$ and $g \in C(\overline{\Omega}\times[0,T))$. Note that this is the Carnot group version of the problem considered in [10]. We also adopt their definition that a subsolution $u(p,t)$ to Problem (3.10) is a viscosity subsolution to (E), $u(p,t) \le g(p,t)$ on $\partial\Omega$ with $0 \le t < T$ and $u(p,0) \le \psi(p)$ on $\overline{\Omega}$. Supersolutions and solutions are defined in an analogous matter.

Theorem 3.6. *Let Ω be a bounded domain in G. Let F be admissible. If u is a viscosity subsolution and v a viscosity supersolution to Problem* (3.10) *then $u \le v$ on $\Omega\times[0,T)$.*

Proof. Our proof follows that of [10, Thm. 8.2] and so we discuss only the main parts.

For $\varepsilon > 0$, we substitute $\tilde{u} = u - \frac{\varepsilon}{T-t}$ for u and prove the theorem for

$$u_t + F(t,p,u,\nabla_1 u,(D^2u)^\star) \le -\frac{\varepsilon}{T^2} < 0, \tag{3.11}$$

$$\lim_{t\uparrow T} u(p,t) = -\infty \quad \text{uniformly on} \quad \overline{\Omega}, \tag{3.12}$$

and take limits to obtain the desired result. Assume the maximum occurs at $(p_0,t_0) \in \Omega\times(0,T)$ with

$$u(p_0,t_0) - v(p_0,t_0) = \delta > 0.$$

Let $\phi(p,q,t)$ be as in the equation (3.7) with $m = 4$ and define

$$M_\tau = u(p_\tau,t_\tau) - v(q_\tau,t_\tau) - \tau\phi(p_\tau,q_\tau),$$

where (p_τ,q_τ,t_τ) is the maximum point in $\overline{\Omega}\times\overline{\Omega}\times[0,T)$ of $u(p,t) - v(q,t) - \tau\phi(p,q)$. By Corollary 3.5, we have

$$\lim_{\tau\to\infty} \tau\phi(p_\tau,q_\tau) = 0.$$

If $t_\tau = 0$, we have

$$0 < \delta \le M_\tau \le \sup_{\overline{\Omega}\times\overline{\Omega}} (\psi(p) - \psi(q) - \tau\phi(p,q))$$

leading to a contradiction for large τ. We therefore conclude $t_\tau > 0$ for large τ. Since $u \le v$ on $\partial\Omega\times[0,T)$ by the equation (BC) of problem (3.10), we conclude that for large τ, we have (p_τ, q_τ, t_τ) is an interior point. That is, $(p_\tau, q_\tau, t_\tau) \in \Omega \times \Omega \times (0, T)$. Using the Carnot Parabolic Maximum Principle, we obtain

$$(a, \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}^\tau) \in \overline{P}^{2,+}u(p_\tau, t_\tau),$$
$$(a, \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}^\tau) \in \overline{P}^{2,-}v(q_\tau, t_\tau),$$

satisfying the equations

$$a + F(t_\tau, p_\tau, u(p_\tau, t_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}^\tau) \leq -\frac{\varepsilon}{T^2},$$
$$a + F(t_\tau, q_\tau, v(q_\tau, t_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}^\tau) \geq 0.$$

Using the fact that F is proper and that $u(p_\tau, t_\tau) \geq v(q_\tau, t_\tau)$ (otherwise $M_\tau < 0$), we have

$$0 < \frac{\varepsilon}{T^2} \leq F(t_\tau, q_\tau, v(q_\tau, t_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{Y}^\tau) - F(t_\tau, p_\tau, u(p_\tau, t_\tau), \tau\Upsilon(p_\tau, q_\tau), \mathcal{X}^\tau)$$
$$\leq \omega(d_C(p_\tau, q_\tau) + C\tau(\varphi(p_\tau, q_\tau))^{\frac{3}{2}} + \|\mathcal{X}^\tau - \mathcal{Y}^\tau\|).$$

Using the equations (3.8) and (3.9), we arrive at a contradiction as $\tau \to \infty$. □

We then have the following corollary, showing the equivalence of parabolic viscosity solutions and viscosity solutions.

Corollary 3.7. *For admissable F, we have the parabolic viscosity solutions are exactly the viscosity solutions.*

Proof. By Proposition 3.3, parabolic viscosity sub(super-)solutions are viscosity sub(super-)solutions. To prove the converse, we will follow the proof of the subsolution case found in [17], highlighting the main details. Assume that u is not a parabolic viscosity subsolution. Let $\phi \in \mathcal{A}^- u(p_0, t_0)$ have the property that

$$\phi_t(p_0, t_0) + F(t_0, p_0, \phi(p_0, t_0), \nabla_1\phi(p_0, t_0), (D^2\phi(p_0, t_0))^\star) \geq \epsilon > 0$$

for a small parameter ϵ. We may assume p_0 is the origin. Let $r > 0$ and define $S_r = B_{\mathcal{N}}(r) \times (t_0 - r, t_0)$ and let ∂S_r be its parabolic boundary. Then the function

$$\tilde{\phi}_r(p, t) = \phi(p, t) + (t_0 - t)^{8l!} - r^{8l!} + (\mathcal{N}(p))^{8l!}$$

is a classical supersolution for sufficiently small r. We then observe that $u \leq \tilde{\phi}_r$ on ∂S_r but $u(0, t_0) > \tilde{\phi}(0, t_0)$. Thus, the comparison prinicple, Theorem 3.6, does not hold. Thus, u is not a viscosity subsolution. The supersolution case is identical and omitted. □

Remark 3.8. The above proof can be extended to any class of parabolic equations possessing a comparison principle for viscosity solutions. (cf. [13] or [14] for more detailed analysis)

4. Notions of solutions to the parabolic p-Laplace equation

Given a bounded domain $\Omega \subset G$ and $1 < p < \infty$, we consider the equation

$$u_t - \Delta_p u = 0 \tag{4.1}$$

in the domain $\Omega_T \equiv \Omega \times [0, T]$.

4.1. Weak solutions. We begin by briefly recalling the definition of and comparison principle for weak solutions to the equation (4.1).

Definition 4. Consider a domain Ω_T and let $u\colon \Omega_T \to \mathbf{R}$ be a continuous function so that $u \in V^p(t_1,t_2;\mathcal{O})$ whenever $\overline{\mathcal{O}}_{t_1,t_2} \subset \Omega_T$. Then u is an *p-parabolic solution* to the equation (4.1) in Ω_T if

$$\iint_{\Omega_T} -u\phi_t + \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle = 0$$

for all $\phi \in \mathrm{W}_0^{1,\mathrm{p}}(\Omega_T)$. The function u is an *weak parabolic subsolution* to the equation (4.1) in Ω_T if

$$\iint_{\Omega_T} -u\phi_t + \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle \le 0$$

for all non-negative $\phi \in \mathrm{W}_0^{1,\mathrm{p}}(\Omega_T)$ and the function u is an *weak parabolic supersolution* to the equation (4.1) in Ω_T if

$$\iint_{\Omega_T} -u\phi_t + \|\nabla_0 u\|^{p-2}\langle \nabla_0 u, \nabla_0 \phi\rangle \ge 0$$

for all non-negative $\phi \in \mathrm{W}_0^{1,\mathrm{p}}(\Omega_T)$.

We then have the following existence theorem, the proof of which uses a Galerkin method and follows as in the Euclidean setting (see [19, Lemma 3.2], [21], [9], [16], and [1, Theorem 1.7]).

Theorem 4.1. *Suppose θ is a continuous function on $\overline{\mathcal{O}}_{0,T}$. Then there exists a unique p-parabolic function u that is continuous in $\overline{\mathcal{O}}_{0,T}$ such that $u = \theta$ on $\partial_{\mathrm{par}}\mathcal{O}_{0,T}$. Moreover, if $\theta \in V^p(0,T;\mathcal{O})$, then so is u.*

Theorem 4.1 allows us to show the following comparison principle for weak solutions, the proof of which follows [19, Lemma 3.1] with the Euclidean gradient replaced by the horizontal gradient.

Theorem 4.2. *Suppose that u is a weak parabolic supersolution and v is a weak parabolic subsolution to (4.1) in $\mathcal{O}_{t_1,t_2}$. If u and $-v$ are lower semicontinuous on $\overline{\mathcal{O}}_{t_1,t_2}$ and $v \le u$ on $\partial_{\mathrm{par}}\mathcal{O}_{t_1,t_2}$, then $v \le u$ a.e. in $\mathcal{O}_{t_1,t_2}$.*

4.2. *p*-superparabolic functions. We have another notion of solution related to the comparison principle, as given by the following definition.

Definition 5. The function u is *p-superparabolic* if

(1) u is lower semicontinuous,
(2) u is finite in a dense subset of Ω_T,
(3) for each set $\mathcal{O}_{t_1,t_2}$ with closure in Ω_T, we have if h is continuous on $\overline{\mathcal{O}}_{t_1,t_2}$ and p-parabolic in $\mathcal{O}_{t_1,t_2}$ with $h \le u$ on $\partial_{\mathrm{par}}\mathcal{O}_{t_1,t_2}$, then $h \le u$ in $\mathcal{O}_{t_1,t_2}$.

A function is u is *p-subparabolic* if $-u$ is p-superparabolic.

We have the following theorem, whose proof follows similarly to that of [19, Lemma 4.4], [20, Theorem 7.2] and [15, Theorem 7.6].

Theorem 4.3. *Suppose that u is p-superparabolic and v is p-subparabolic in Ω_T. If*

$$\limsup_{q\to p} v(q) \le \liminf_{q\to p} u(q) \tag{4.2}$$

for every $p \in \partial_{\mathrm{par}}\Omega_T$ and if both sides of (4.2) are not simultaneously ∞ or $-\infty$, then $v \le u$ in Ω_T.

We are then able to conclude the following corollary ([15, Lemma 7.8]).

Corollary 4.4. *A function is p-parabolic if and only if it is both p-subparabolic and p-superparabolic.*

4.3. Viscosity solutions. Because of the singularity that occurs when $1 < p < 2$, we will have to modify the definition of viscosity solution. In particular, we must be cautious about the spacial gradient vanishing. We note while the definition below is also valid for $2 \le p < \infty$, the non-vanishing gradient condition is superfluous. (cf. [18]). Viscosity solutions are then defined as follows:

Definition 6. A function $u\colon \Omega_T \to \mathbf{R} \cup \{\infty\}$ is a *parabolic viscosity p-supersolution* if

(1) u is lower semicontinuous,
(2) u is finite in a dense subset of Ω_T,
(3) for $(p_0, t_0) \in \Omega_T$ and $\phi \in \mathcal{B}^- u(p_0, t_0)$ with $\nabla_0 \phi(p,t) \neq 0$ when $p \neq p_0$, we have

$$\limsup_{\substack{(p,t)\to(p_0,t_0)\\ p\neq p_0, t<t_0}} \phi_t(p,t) - \Delta_p \phi(p,t) \ge 0.$$

A function u is a *parabolic viscosity p-subsolution* if $-u$ is a parabolic viscosity p-supersolution. A function u is a *parabolic viscosity p-solution* if it is both a parabolic viscosity p-supersolution and a parabolic viscosity p-subsolution.

Remark 4.5. As noted above, once we have established a comparison principle for viscosity solutions to parabolic p-Laplace equations, the choice of using $\mathcal{B}^- u(p_0, t_0)$ or $\mathcal{B} u(p_0, t_0)$ in the definition above is an equivalent one.

5. Uniqueness of viscosity solutions

We state the following theorem:

Theorem 5.1. *Let $1 < p < \infty$ and let Ω be a bounded domain in a Carnot group. Let $\Omega_T = \Omega \times (0,T)$ be the corresponding parabolic domain. If u is a parabolic viscosity p-subsolution and v a parabolic viscosity p-supersolution in Ω_T such that $u \le v$ on the parabolic boundary, then $u \le v$ in Ω_T.*

Proof. The proof mirrors that of [18] and follows the flavor of [7]. We highlight the key details. We assume that $\sup_{\Omega_T}(u-v) > \sup_{\partial_{\mathrm{par}}\Omega_T}(u-v)$ and by replacing $v(q,t)$ with $v(q,t) + \varepsilon(T-t)^{-1}$, we may assume v is a strict supersolution and $v(q,t) \to \infty$ as $t \to T$.

We consider the function $\phi\colon G \times G \to \mathbf{R}$ given by

$$\phi(p,q) = \frac{1}{m} \sum_{i=1}^{N} ((p \cdot q^{-1})_i)^m$$

for some large positive even integer $m > \max\{4, \frac{p}{p-1}, p+2\}$. Here, $(p \cdot q^{-1})_i$ is the i-th component of $p \cdot q^{-1}$. Define the function

$$\psi_j(p,q,t,s) = j\phi(p,q) + \frac{j}{2}(t-s)^2.$$

Let (p_j, q_j, t_j, s_j) be the maximum point of $u(p,t) - v(q,s) - \psi_j(p,q,t,s)$ in $\overline{\Omega} \times \overline{\Omega} \times [0,T)$. For j sufficiently large, we have (p_j, q_j, t_j, s_j) occurs in the interior.

Suppose $p_j \neq q_j$. Then, the Carnot Parabolic Maximum Principle, Lemma 3.4 produces $(a, j\Upsilon_j, \mathcal{X}_j) \in \overline{P}^{2,+}u(p_j,t_j)$ and $(a, j\Upsilon_j, \mathcal{Y}_j) \in \overline{P}^{2,-}v(q_j,s_j)$. We then compute

$$0 < \varepsilon(T-s_j)^{-2} \leq a - \Big(\|j\Upsilon_j\|^{p-2}\operatorname{tr}(\mathcal{Y}_j) + (p-2)\|j\Upsilon_j\|^{p-4}\langle \mathcal{Y}_j j\Upsilon_j, j\Upsilon_j\rangle\Big)$$

and

$$0 \geq a - \Big(\|j\Upsilon_j\|^{p-2}\operatorname{tr}(\mathcal{X}_j) + (p-2)\|j\Upsilon_j\|^{p-4}\langle \mathcal{X}_j j\Upsilon_j, j\Upsilon_j\rangle\Big).$$

Subtracting, we obtain

$$\begin{aligned}0 < \varepsilon(T-s_j)^{-2} &\leq \|j\Upsilon_j\|^{p-2}\Big(\operatorname{tr}(\mathcal{X}_j) - \operatorname{tr}(\mathcal{Y}_j)\Big)\\ &\quad + (p-2)\|j\Upsilon_j\|^{p-4}\Big(\langle \mathcal{X}_j j\Upsilon_j, j\Upsilon_j\rangle - \langle \mathcal{Y}_j j\Upsilon_j, j\Upsilon_j\rangle\Big).\end{aligned} \tag{5.1}$$

Claim 5.2.

$$\operatorname{tr}(\mathcal{X}_j) - \operatorname{tr}(\mathcal{Y}_j) \lesssim j(\phi(p_j,q_j))^{\frac{2m-4}{m}}$$

Proof. Given the standard unit vectors e_k with every entry 0 except for the k-th entry which is equal to 1, we see that for any matrix A,

$$\operatorname{tr}(A) = \sum_k \langle Ae_k, e_k\rangle \tag{5.2}$$

and so via the equation (3.8)

$$\operatorname{tr}(\mathcal{X}_j) - \operatorname{tr}(\mathcal{Y}_j) = \sum_{k=1}^{n_1}\langle \mathcal{X}_j e_k, e_k\rangle - \sum_{k=1}^{n_1}\langle \mathcal{Y}_j e_k, e_k\rangle \lesssim \sum_{k=1}^{n_1} j(\phi(p_j,q_j))^{\frac{2m-4}{m}}\|e_k\|^2.$$

The claim follows. □

We next note that by Properties (ii) and (iii) of Corollary 3.5, we have

$$\langle \mathcal{X}_j j\Upsilon_j, j\Upsilon_j\rangle - \langle \mathcal{Y}_j j\Upsilon_j, j\Upsilon_j\rangle \lesssim j\phi(p_\tau,q_\tau)^{\frac{2m-4}{m}}\|j\Upsilon_j\|^2 = j^3\phi(p_\tau,q_\tau)^{\frac{4m-6}{m}}.$$

Combining this fact along with Claim 5.2 and the equation (5.1) then yields

$$0 < \varepsilon(T-s_j)^{-2} \lesssim j^{p-1}(\phi(p_j,q_j)^{\frac{1}{m}})^{p(m-1)-2}.$$

Since $m > p+2$, we have $(p(m-1)-2)(\frac{1}{m}) > p-1$. We arrive at a contradiction as $j \to \infty$.

Suppose $p_j = q_j$. By definition, we have for any (p,q,t,s),

$$u(p,t) - v(q,s) - \psi_j(p,q,t,s) \leq u(p_j,t_j) - v(q_j,s_j) - \psi_j(p_j,q_j,t_j,s_j)$$

and so when $p = p_j$ and $t = t_j$, we have

$$v(q,s) \geq v(q_j,s_j) + \psi_j(p_j,q_j,t_j,s_j) - \psi_j(p_j,q,t_j,s).$$

Defining the function $\beta^v(q,s)$ by

$$\beta^v(q,s) = v(q_j,s_j) + \psi_j(p_j,q_j,t_j,s_j) - \psi_j(p_j,q,t_j,s) - \varphi(q_j\cdot q^{-1})$$

we see that $v - \beta^v$ has a strict local minimum at (q_j,s_j). If $p_j = q_j$, we have

$$\beta^v(q,s) = v(q_j,s_j) + \frac{j}{2}(t_j-s_j)^2 - \frac{j}{2}(t_j-s)^2 - (j+1)\phi(q_j,q).$$

Using this definition of $\beta^v(q,s)$ with the non-divergence form of the p-Laplacian (2.4), we have

$$|\Delta_p \beta^v(q,s)| \lesssim \|\nabla_0\phi(q_j,q)\|^{p-2}\bigg|\operatorname{tr}(D^2\phi)^\star(q_j,q) + \|(D^2\phi)^\star(q_j,q)\|\ \bigg|.$$

From the equation (5.2), we have

$$|\operatorname{tr}(D^2\varphi)^\star(q_j,q)| \lesssim \|(D^2\varphi)^\star(q_j,q)\|$$

and so we conclude

$$\bigg|\operatorname{tr}(D^2\varphi)^\star(q_j,q) + \|(D^2\varphi)^\star(q_j,q)\|\ \bigg| \lesssim \varphi(q_j,q)^{\frac{m-2}{m}}$$

so that

$$|\Delta_p\beta(q)| \lesssim (\varphi(q_j,q)^{\frac{1}{m}})^{(m-1)(p-2)+(m-2)}.$$

Since $m > \frac{p}{p-1}$, we have

$$\lim_{\substack{q\to q_j\\ q\neq q_j}}(-\Delta_p\beta(q)) = 0.$$

Using this result along with the fact that v is a strict supersolution, we obtain

$$0 < \varepsilon(T-s_j)^{-2} \le \limsup_{\substack{(q,s)\to(q_j,s_j)\\ q\neq q_j, s<s_j}} \bigg(\beta^v_s(q,s) - \Delta_p\beta^v(q,s)\bigg) \le \limsup_{\substack{(q,s)\to(q_j,s_j)\\ q\neq q_j, s<s_j}} \beta^v_s(q,s) = j(t_j-s_j).$$

Using a symmetric argument, we have

$$0 \ge \liminf_{\substack{(p,t)\to(p_j,t_j)\\ p\neq p_j, t<t_j}} \bigg(\beta^u_t(p,t) - \Delta_p\beta^u(p,t)\bigg) \ge \liminf_{\substack{(p,t)\to(p_j,t_j)\\ p\neq p_j, t<t_j}} \beta^u_t(p,t) = j(t_j-s_j).$$

Subtracting these two inequalities yields

$$0 < \varepsilon(T-s_j)^{-2} \le j(t_j-s_j) - j(t_j-s_j) = 0$$

which is an obvious contradiction. □

We have the following relationship between notions of solution (cf. [18, Lemma 4.6]).

Lemma 5.3. *A p-superparabolic function is a parabolic viscosity p-supersolution. A p-subparabolic function is a parabolic viscosity p-subsolution. A p-parabolic function is a parabolic viscosity p-solution.*

Proof. We will assume that a p-superparabolic function $u(p,t)$ fails to be a parabolic viscosity p-supersolution at the point $(p_0,t_0) = (0,0)$. Let $Q_r = B_{\mathcal{N}}(0,r) \times (-r,0)$. Suppose there exists a test function $\phi(p,t)$ satisfying the hypotheses of condition (iii) in Definition 6 but whenever $(p,t) \in Q_r \cap \{p \neq 0\}$, we have

$$\phi_t(p,t) - \Delta_p\phi(p,t) < 0. \tag{5.3}$$

Let $\psi \in C_0^\infty(Q_r)$ be a non-negative function. Then we have

$$\begin{aligned}&-\iint_{Q_r} \|\nabla_0\phi\|^{p-2}\langle\nabla_0\phi,\nabla_0\psi\rangle\,dp\,dt\\ &= \lim_{R\to 0}\left(\iint_{Q_r\setminus\{\mathcal{N}(p)\le R\}} \psi(\Delta_p\phi)\,dp\,dt - \iint_{Q_r\setminus\{\mathcal{N}(p)\le R\}} \operatorname{div}(\psi\|\nabla_0\phi\|^{p-2}\nabla_0\phi)\,dp\,dt\right).\end{aligned}$$

For a horizontal vector field X, an easy calculation shows that $\operatorname{div} X = \operatorname{div}_{\text{eucl}} X$. We may then apply the Divergence Theorem to the second term and, along the fact that $\psi \in C_0^\infty(Q_r)$, obtain

$$\iint_{Q_r \setminus \{\mathcal{N}(p) \le R\}} \operatorname{div}(\psi \|\nabla_0 \phi\|^{p-2} \nabla_0 \phi)\, dp\, dt = \int_{-r}^{0} \oint_{\mathcal{N}(p)=R} \psi \|\nabla_0 \phi\|^{p-2} \langle \nabla_0 \phi, -n \rangle\, dS\, dt$$

where n is the outward Euclidean unit normal to the (smooth) surface $\{\mathcal{N}(p) = R\}$. Therefore, we have

$$\begin{aligned} &\lim_{R \to 0} \iint_{Q_r \setminus \{\mathcal{N}(p) \le R\}} \operatorname{div}(\psi \|\nabla_0 \phi\|^{p-2} \nabla_0 \phi)\, dp\, dt \\ &= \lim_{R \to 0} \int_{-r}^{0} \oint_{\mathcal{N}(p)=R} \psi \|\nabla_0 \phi\|^{p-2} \langle \nabla_0 \phi, -n \rangle\, dS\, dt = 0. \end{aligned}$$

Using the equation (5.3) and integration by parts

$$\begin{aligned} -\iint_{Q_r} \|\nabla_0 \phi\|^{p-2} \langle \nabla_0 \phi, \nabla_0 \psi \rangle\, dp\, dt &\ge \lim_{R \to 0} \iint_{Q_r \setminus \{\mathcal{N}(p) \le R\}} \psi \phi_t\, dp\, dt \\ &= \iint_{Q_r} \psi \phi_t\ dp\ dt = -\iint_{Q_r} \psi_t \phi\, dp\, dt. \end{aligned}$$

By Definition 4, we have that ϕ is a weak parabolic subsolution in Q_r and by [19, Lemma 4.2], ϕ is a p-subparabolic function. Let

$$m = \inf_{\partial_{\text{par}} Q_r} (u - \phi) > 0$$

and consider the p-subparabolic function $\Phi = \phi + \frac{m}{2}$. By Theorem 4.3, since $u > \Phi$ on $\partial_{\text{par}} Q_r$ by construction, we have $u \ge \Phi$ in Q_r. However,

$$\Phi(0,0) = \phi(0,0) + \frac{m}{2} = u(0,0) + \frac{m}{2} > u(0,0)$$

contradicting the comparison principle. □

This Lemma, along with Theorem 5.1, produces the following corollary.

Corollary 5.4. *A function is p-superparabolic if and only if it is a parabolic viscosity p-supersolution. A function is p-subparabolic if and only if it is a parabolic viscosity p-subsolution. A continuous function is p-parabolic if and only if it is a parabolic viscosity p-solution.*

6. Asymptotic limits as $t \to \infty$

We now focus our attention on the asymptotic limits of the parabolic viscosity solutions. We wish to show that for a fixed p, $1 < p < \infty$, we have the (unique) viscosity solution to $u_t - \Delta_p u = 0$ approaches the viscosity solution of $-\Delta_p u = 0$ as $t \to \infty$. Our proof follows that of [17, Theorem 2], the core of which hinges on the contruction of a parabolic test function from an elliptic one. Recall the definition of viscosity solution (Definition 6).

Theorem 6.1. *Let $1 < p < \infty$ and $u \in C(\overline{\Omega} \times [0,\infty))$ be a viscosity solution of*

$$\begin{cases} u_t - \operatorname{div}(||\nabla_0 u||^{p-2} \nabla_0 u) = 0 & \text{in } \Omega \times (0,\infty), \\ u(p,t) = g(p) & \text{on } \partial_{\text{par}}(\Omega \times (0,\infty)), \end{cases} \tag{6.1}$$

with $g\colon \overline{\Omega} \to \mathbf{R}$ *continuous and assuming that* $\partial\Omega$ *satisfies the property of positive geometric density (see* [17, p. 2909]*). Then* $u(p,t) \to U(p)$ *uniformly in* Ω *as* $t \to \infty$ *where* $U(p)$ *is the unique viscosity solution of* $-\Delta_p U = 0$ *with the Dirichlet boundary condition* $\lim_{q\to p} U(q) = g(p)$ *for all* $p \in \partial\Omega$.

We will also need the following lemma, the proof of which can be found in [11, p. 170] and only relies on the homogeneity of (6.1).

Lemma 6.2. *Let* u *be as in Theorem 6.1. Then for every* $(x,t) \in \Omega \times (0,\infty)$ *and for* $0 < h < t$*, we have*

$$|u(x,t-h) - u(x,t)| \le \frac{2||g||_{\infty,\Omega}}{p-2}\left(1-\frac{h}{t}\right)^{\frac{p-1}{2-p}}\frac{h}{t} \quad \text{when } 2<p<\infty,$$

$$|u(x,t-h) - u(x,t)| \le \frac{2||g||_{\infty,\Omega}}{2-p}\left(1+\frac{h}{t}\right)^{\frac{p-1}{2-p}}\frac{h}{t} \quad \text{when } 1<p<2.$$

Proof of Theorem 6.1. We consider only the $2 < p < \infty$ case. The other case is similar and omitted. Let u be a viscosity solution of (6.1). The results of [11, Chapter III] imply that the family $\{u(\cdot,t)\colon t \in (0,\infty)\}$ is equicontinuous. Since it is uniformly bounded due to the boundedness of g, Arzela–Ascoli's theorem yields that there exists a sequence $t_j \to \infty$ such that $u(\cdot,t_j)$ converge uniformly in $\overline{\Omega}$ to a function $U \in C(\overline{\Omega})$ for which $U(p) = g(p)$ for all $p \in \partial\Omega$. Since it is known from [7, Lemma 5.5] that the Dirichlet problem for the subelliptic p-Laplace equation possesses a unique solution, it is enough to show that U is a viscosity p-subsolution to $-\Delta_p U = 0$ on Ω. With that in mind, let $p_0 \in \Omega$ and choose $\phi \in \mathcal{C}^2_{\mathrm{sub}}(\Omega)$ such that $\nabla_0\phi(p) \neq 0$ and $0 = \phi(p_0) - U(p_0) < \phi(p) - U(p)$ for $p \in \Omega$, $p \neq p_0$. Using the uniform convergence, we can find a sequence $p_j \to p_0$ such that $u(\cdot,t_j) - \phi$ has a local maximum at p_j. Now define

$$\phi_j(p,t) = \phi(p) + C\left(\frac{t}{t_j}\right)^{\frac{p-1}{2-p}}\frac{t_j - t}{t_j},$$

where $C = 2||g||_{\infty,\Omega}/(p-2)$. Notice that $\phi_j(p,t) \in \mathcal{C}^2_{\mathrm{sub}}(\Omega\times(0,\infty))$ and that $\nabla_0\phi_j(p,t) = \nabla_0\phi(p) \neq 0$ for $p \neq p_0, p_j$. Then using Lemma 6.2,

$$\begin{aligned} u(p_j,t_j) - \phi_j(p_j,t_j) &= u(p_j,t_j) - \phi(p_j) \ge u(p,t_j) - \phi(p) \\ &\ge u(p,t) - \phi(p) - C\left(\frac{t}{t_j}\right)^{\frac{p-1}{2-p}}\frac{t_j - t}{t_j} \\ &= u(p,t) - \phi_j(p,t) \end{aligned}$$

for any $p \in \Omega$ and $0 < t < t_j$. Thus we have that ϕ_j is an admissible test function at (p_j,t_j) on $\Omega \setminus \{p_0\} \times [0,T]$ according to Definition 6. Therefore,

$$\limsup_{\substack{(p,t)\to(p_j,t_j)\\ p\neq p_j, t<t_j}} (\phi_j)_t(p,t) - \Delta_p\phi_j(p,t) \le 0.$$

This yields

$$\limsup_{\substack{p\to p_j\\ p\neq p_j}} (-\Delta_p\phi(p_j)) \le \frac{C}{t_j}.$$

Letting $j \to \infty$ yields the claim. □

7. The limit as $p \to \infty$

7.1. The equation. We consider the Cauchy problem

$$(7.1)\qquad \begin{cases} u_t - \operatorname{div}(||\nabla_0 u||^{p-2}\nabla_0 u) = f_p(x,t) & \text{in } \Omega \times (0,T), \\ u(x,t) = g(x) & \text{on } \Omega \times \{0\}, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $g\colon G \to \mathbf{R}$ is Lipschitz and satisfies

$$\operatorname{ess\,sup} |\nabla_0 g| \leq 1.$$

Since we are considering the limit as $p \to \infty$, we will also assume that $p \geq \mathcal{Q}+1$, with $\mathcal{Q}$ defined as in the equation (2.1).

We will follow the well-written exposition in Sections 2 and 3 of [3]. In particular, we assume that for $0 \leq t < T$, there is a constant C_1 so that

$$(7.2)\qquad \int_G |f_p| + \left|\frac{\partial}{\partial t} f_p\right| \leq C_1$$

for all $p \geq \mathcal{Q}+1$ and we have the following lemma whose proof is identical to Lemma 2.1 [3]) and omitted.

Lemma 7.1. *For each $\mathcal{T} > 0$, there exists a constant C_2 such that*

1. $\sup\limits_{G\times[0,\mathcal{T}]} |u_p| \leq C_2,$
2. $\displaystyle\int_0^{\mathcal{T}} \int_G \left(\frac{\partial}{\partial t} u_p\right)^2 dp\, dt \leq C_2,$
3. $\displaystyle\left(\int_0^{\mathcal{T}} \int_G |\nabla_0 u|^p \, dt\right)^{\frac{1}{p}} \leq C_2,$

for all $p > \mathcal{Q}+1$. The constant C_2 depends only on $C_1, \mathcal{T}, g$ and $\mathcal{Q}$.

The uniform bounds from Lemma 7.1 produce the following result:

Lemma 7.2. *Passing to a subsequence if necessary, there is a sequence $p_i \to \infty$ and a limiting function u so that*

(1) $u_{p_i} \to u$ *a.e. and in* $L^2(G \times (0,T))$,
(2) $\nabla_0 u_p \rightharpoonup \nabla_0 u$ *and* $\frac{\partial}{\partial t} u_p \rightharpoonup \frac{\partial}{\partial t} u$ *weakly in* $L^2(G \times (0,T))$,
(3) $\operatorname{ess\,sup}\limits_{G\times[0.T]} |u| \leq C_2,$
(4) $\displaystyle\int_0^T \int_G \frac{\partial}{\partial t} u^2 \, dp\, dt \leq C_2,$
(5) $\operatorname{ess\,sup}\limits_{G\times[0.T]} |\nabla_0 u| \leq 1.$

Proof. The first two follow from standard Sobolev embedding. The next two follow from Lemma 7.1. The last is Lemma 3.1 of [3] and has the same proof, which is omitted. □

Remark 7.3. Following the presentation of [3], we may actually consider a more general equation. Namely, we can replace the equation (7.1) with

$$(7.3)\qquad \begin{cases} u_t - \operatorname{div}(||\nabla_0 u||^{p-2}\nabla_0 u) = f_p(x,t) & \text{in } G \times (0,\infty), \\ u(x,t) = g(x) & \text{on } G \times \{0\} \end{cases}$$

under the additional assumption that g has compact support. It is well-known that the equation (7.3) has a unique weak solution $u_p \in V^p(0,\infty;G)$ and Lemma 7.2 easily extends to this generality. A complication arises due to the fact that in an arbitrary Carnot group, there is no explicit Barenblatt-type fundamental solution to the parabolic p-Laplace equation. Therefore, the method of proof for Lemma 2.2 in [3] does not carry through to Carnot groups. In order to follow the machinery of [3], one also needs the following hypothesis:

Hypothesis 7.4. [3, Lemma 2.2] *For each $\mathcal{T} > 0$, there exists a radius R so that*

$$\operatorname{supp}(u_p) \subset B(0,R) \times [0,\mathcal{T}]$$

for all $p \geq \mathcal{Q}+1$.

Note that Hypothesis 7.4 is satisfied by the equation (7.1).

7.2. The limit. We turn our attention to subdifferentials. The interested reader is directed to the thorough discussion in [12, Section 9.6]. We consider the real Hilbert space $L^2(G)$ and define the convex functionals

$$I_p = \begin{cases} \dfrac{1}{p}\displaystyle\int_G |\nabla_0 v|^p & v \in L^2(G),\ |\nabla_0 v| \in L^p(G), \\ \infty & \text{else} \end{cases}$$

for $1 < p < \infty$ and

$$I_\infty = \begin{cases} 0 & v \in L^2(G),\ |\nabla_0 v| \leq 1 \text{ a.e.}, \\ \infty & \text{else.} \end{cases}$$

We then say that for $1 < p \leq \infty, u \in D(\partial I_r)$ and $w \in \partial I_r[u]$ if

$$I_r[v] \geq I_r[u] + \int_G w(v-u)$$

for all $v \in L^2(G)$. We are able to rewrite the equation (7.1) (cf. [12, Thm 4, Sec. 9.6.3]) as

$$\begin{cases} f_p - \frac{\partial}{\partial t} u_p \in \partial I_p[u_p] & \text{for a.e. } 0 < t < T, \\ u_p = g & t = 0. \end{cases}$$

We then have the following theorem:

Theorem 7.5. *The function u from Lemma 7.2 is the unique solution to*

$$\begin{cases} f - \frac{\partial}{\partial t} u \in \partial I_\infty[u] & \text{for a.e. } t > 0, \\ u_p = g & t = 0, \end{cases}$$

where f is constructed as follows ([3, p. 313]): Select m distinct points $\{d_k\}_{k=1}^m \subset G$ and m non-negative smooth functions of time $\{h_k(t)\}_{k=1}^m$ to create the measure

$$f(p,t) = \sum_{k=1}^m h_k(t)\delta_{d_k}(p).$$

For each $k = 1, 2, \ldots, m$, and $\mathcal{Q}+1 \leq p < \infty$, let $d_k^p \colon G \to \mathbf{R}$ be a smooth non-negative function such that

$$\operatorname{supp}(d_k^p) \subset B(d_k, r_p)$$

and

$$\int_{B(d_k,r_p)} d_k^p = 1 \text{ as } p \to \infty$$

with $r_p \to 0$ *as* $p \to \infty$. *Note that we then have a smooth approximation of* f_p *given by*

$$\widetilde{f}_p(p,t) = \sum_{k=1}^{m} h_k(t) d_k^p(p)$$

satisfying the equation (7.2).

Proof. The existence proof is similar to that of [3, Theorem 3.2] and omitted. The uniqueness proof is identical to [3, Theorem 3.3]. □

7.3. Convergence. Let us recall the definition of Mosco convergence. (See, for example, [2].)

Definition 7. Let $\Psi(X)$ be the set of all lower semi-continuous convex functionals ϕ from a Hilbert space X into $(-\infty, \infty]$ with the property that $\phi \not\equiv \infty$. A sequence $\{\phi_n\}_{n=1}^{\infty}$ in $\Psi(X)$ converges to $\phi \in \Psi(X)$ on X in the sense of Mosco as $n \to \infty$ if the following hold:

(1) For all $u \in D(\phi)$, there exists a sequence u_n in X such that $u_n \to u$ strongly in X and $\phi_n(u_n) \to \phi(u)$.
(2) Let $\{u_k\}$ be a sequence in X such that $u_k \rightharpoonup u$ weakly in X as $k \to \infty$ and let $\{n_k\}$ be a sequence of $\{n\}$. Then, we have

$$\liminf_{k\to\infty} \phi_{n_k}(u_k) \geq \phi(u).$$

We have the following theorem, whose proof matches that of Theorem 2.2 in [2] and is omitted.

Theorem 7.6. *Let* p_n *be a sequence in* $(1, \infty)$ *such that* $p_n \to \infty$ *as* $n \to \infty$. *Then* $I_{p_n} \to I_\infty$ *on* $L^2(\Omega)$ *in the sense of Mosco as* $p_n \to \infty$.

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