Allocating Indivisible Items in Categorized Domains

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Abstract

We formulate a general class of allocation problems called *categorized domain allocation problems (CDAPs)*, where indivisible items from multiple categories are allocated to agents without monetary transfer and each agent gets at least one item per category. We focus on *basic CDAPs*, where the number of items in each category equals to the number of agents. We characterize serial dictatorships for basic CDAPs by a minimal set of three desired properties: strategy-proofness, non-bossiness, and category-wise neutrality. Then, we propose a natural extension of serial dictatorships called *categorical sequential allocation mechanisms (CSAMs)*, which allocate the items in multiple rounds: in each round, the active agent chooses an item from a designated category. We fully characterize the worst-case ordinal efficiency of CSAMs for optimistic and pessimistic agents. We believe that these constitute a promising first step towards theoretical foundations and applications of general CDAPs.

Introduction

Suppose we are organizing a seminar and must allocate 10 discussion topics and 10 dates to 10 students. Students have heterogeneous and combinatorial preferences over (topic, date) bundles: their preferences over the topics may depend on the date and vice versa, because she may prefer an early date if she gets an easy topic and may prefer a late date if she gets a hard topic.

This example illustrates a common setting for allocating multiple indivisible items, which we formulate as a *categorized domain*. A categorized domain contains multiple indivisible items, each of which belongs to one of the $p \geq 1$ categories. In *categorized domain allocation problems (CDAPs)*, we want to design a mechanism to allocate the items to agents without monetary transfer, such that each agent gets at least one item from each category. In the above example, there are two categories: topics and dates, and each agent (student) must get a topic and a date.

Many other allocation problems are CDAPs. For example, in cloud computing, agents have heterogeneous preferences over multiple types of items including CPU, memory, and storage[1] [15, 14, 1]; patients must be allocated multiple types of resources including surgeons, nurses, rooms, and equipments [17]; college students need to choose courses from multiple categories per semester, e.g. computer science courses, math courses, social science courses, etc.

The design and analysis of allocation mechanisms for non-categorized domains have been an active research area at the interface of computer science and economics. In computer science, allocation problems have been studied as *multi-agent resource allocation* [12]. In economics, allocation problems have been studied as *one-sided matching*, also known as *assignment problems* [26]. Previous research faces three main barriers.

• *Preference bottleneck:* When the number of items is not too small, it is impractical for the agents to express their preferences over all (exponential) bundles of items.

• *Computational bottleneck:* Even if the agents can express their preferences compactly using some preference language, computing an "optimal" allocation is often a hard combinatorial optimization problem.

• *Threats of agents' strategic behavior:* An agent may have incentive to report untruthfully to get a more preferred bundle. This may lead to a socially inefficient allocation.

Our Contributions. We initiate the study of mechanism design under the novel framework of CDAPs towards breaking the three aforementioned barriers. CDAPs naturally generalize classical non-categorized allocation problems, which are CDAPs with one category. CDAPs are our main conceptual contribution.

As a first step, we focus on *basic categorized domain allocation problems (basic CDAPs)*, where the number of items in each category is exactly the same as the number of agents, so that each agent gets exactly one item from each category. See e.g. the seminar-organization example. As we will show, mechanism design for basic CDAPs is already highly non-trivial.

Our technical contributions are two-fold. First, we characterizes *serial dictatorships* for any basic CDAPs

[1] Suppose each type contains discrete units of resources that are essentially indivisible for operational convenience.

with at least two categories by a minimal set of three axiomatic properties: *strategy-proofness*, *non-bossiness*, and *category-wise neutrality*. This helps us understand the possibility of designing strategy-proof mechanisms to overcome the third barrier, i.e. threats of agents' strategic behavior.

Second, to overcome the preference bottleneck and the computational bottleneck, and to go beyond serial dictatorships, we propose *categorical sequential allocation mechanisms (CSAMs)*, which are a large class of indirect mechanisms that naturally extend serial dictatorships [27], sequential allocation protocols [6], and the draft mechanism [11]. For n agents and p categories, a CSAM is defined by an ordering over all (agent, category) pairs: in each round, the active agent picks an item that has not been chosen yet from the designated category. CSAMs have low communication complexity and can be implemented in a distributed manner.

We completely characterize the worst-case ordinal efficiency of CSAMs, measured by agents' *ranks* of the bundles they receive, for any combination of two types of myopic agents: *optimistic* agents, who always choose the item in their top-ranked bundle that is still available, and *pessimistic* agents, who always choose the item that gives them best worst-case guarantee.[2] This characterization naturally leads to useful corollaries on worst-case efficiency of various CSAMs. For example, we show that while serial dictatorships with all-optimistic agents have the best worst-case utilitarian rank, they have the worst worst-case egalitarian rank. On the other hand, balanced CSAMs with all-pessimistic agents have good worst-case utilitarian rank.

Related Work and Discussions. We are not aware of previous work that explicitly formulates CDAPs. Previous work on multi-type resource allocation assumes that items of the same type are interchangeable, and agents have specific preferences, e.g. *Leontief preferences* [15] and threshold preferences [17]. CDAPs are more general as agents' preferences are only required to be rankings but not otherwise restricted.

From the modeling perspective, ignoring the categorical information, CDAPs become standard centralized multi-agent resource allocation problems. However, the categorical information opens more possibilities for designing natural allocation mechanisms such as CSAMs. More importantly, we believe that CDAPs provide a natural framework for cross-fertilization of ideas and techniques from other fields of preference representation and aggregation. For example, the combinatorial structure of categorized domains naturally allows agents to use graphical languages (e.g. *CP-nets* [4]) to represent their preferences, which is otherwise hard [7]. Approaches in *combinatorial voting* [10] can also be naturally considered in CDAPs.

Technically, one-sided matching problems are basic CDAPs with one category. Our characterization of serial dictatorships for basic CDAPs may look similar to characterizations of serial dictatorships and similar mechanisms for one-sided matching [27, 21, 22, 23, 13, 16]. However, our theorem is stronger as the category-wise neutrality used in our characterization is weaker than the neutrality used in previous work.

Our analysis of the worst-case ordinal efficiency of categorical sequential allocation mechanisms resembles the *price of anarchy* [18], which is defined for strategic and self-interested agents, with the presence of a social welfare function that numerically evaluates the quality of outcomes. Our theorem is also related to *distortion* in the voting setting [25, 5], which concerns the social welfare loss caused by agents reporting a ranking instead of a utility function. Nevertheless, our approach is significantly different because we focus on allocation problems for myopic agents, and we do not assume the existence of agents' cardinal preferences nor a social welfare function, even though our theorem can be easily extended to study worst-case social welfare loss given a social welfare function, as in Proposition 2 through 5.

Categorized Domain Allocation Problems

Definition 1 *A* categorized domain *is composed of $p \geq 1$ categories of indivisible items, denoted by $\{D_1, \ldots, D_p\}$. In a* categorized domain allocation problem (CDAP)*, we want to allocate the items to n agents without monetary transfer, such that each agent gets at least one item from each category.*

In a basic categorized domain *for n agents, for each $i \leq p$, $|D_i| = n$, $\mathfrak{D} = D_1 \times \cdots \times D_p$, and each agent's preferences are represented by a linear order over $\mathfrak{D}$. In a* basic categorized domain allocation problem (basic CDAP)*, we want to allocate the items to n agents without monetary tranfer, such that every agent gets exactly one item in each category.*

In this paper, we focus on basic categorized domains and basic CDAPs for *non-sharable* items [12], that is, each item can only be allocated to one agent. Therefore, for all $i \leq p$, we write $D_i = \{1, \ldots, n\}$. Each element in $\mathfrak{D}$ is called a *bundle*. For any $j \leq n$, let R_j denote a linear order over $\mathfrak{D}$ and let $P = (R_1, \ldots, R_n)$ denote the agents' *(preference) profile*. An *allocation* A is a mapping from $\{1, \ldots, n\}$ to $\mathfrak{D}$, such that $\bigcup_{j=1}^{n}[A(j)]_i = D_i$, where for any $j \leq n$ and $i \leq p$, $A(j)$ is the bundle allocated to agent j and $[A(j)]_i$ is the item in category i allocated to agent j. An *allocation mechanism* f is a mapping that takes a profile as input, and outputs an allocation. We use $f^j(P)$ to denote the bundle allocated to agent j by f for profile P.

We now define three desired axiomatic properties for allocation mechanisms. The first two properties are common in the literature [27], and the third is new.

• A direct mechanism f satisfies *strategy-proofness*, if no agent benefits from misreporting her preferences. That is, for any profile P, any agent j, and any linear order R'_j over $\mathfrak{D}$, $f^j(P) \succ_{R_j} f^j(R'_j, R_{-j})$, where R_{-j} is

[2]Similar types of agents have been studied in other social choice settings [8, 19, 9].

composed of preferences of all agents except agent j.

• f satisfies *non-bossiness*, if no agent is *bossy*. An agent is bossy if she can report differently to change the bundles allocated to some other agents without changing her own allocation. That is, for any profile P, any agent j, and any linear order R'_j over $\mathfrak{D}$, $[f^j(P) = f^j(R'_j, R_{-j})] \Rightarrow [f(P) = f(R'_j, R_{-j})]$.

• f satisfies *category-wise neutrality*, if after applying a permutation over the items in a given category, the allocation is also permuted in the same way. That is, for any profile P, any category i, and any permutation M_i over D_i, we have $f(M_i(P)) = M_i(f(P))$, where for any bundle $\vec{d} \in \mathfrak{D}$, $M_i(\vec{d}) = (M_i([\vec{d}]_i), [\vec{d}]_{-i})$.

When there is only one category, category-wise neutrality degenerates to the traditional neutrality for one-sided matching [27]. When $p \geq 2$, category-wise neutrality is much weaker than the traditional neutrality.

A *serial dictatorship* is defined by a linear order $\mathcal{K}$ over $\{1, \ldots, n\}$ such that agents choose items in turns according to $\mathcal{K}$. A truthful agent chooses her top-ranked bundle that is still available in each step.

Example 1 Let $n = 3$ and $p = 2$. $\mathfrak{D} = \{1, 2, 3\} \times \{1, 2, 3\}$. Agents' preferences are as follows.
$R_1 = [12 \succ 21 \succ 32 \succ 33 \succ 31 \succ 22 \succ 23 \succ 13 \succ 11]$
$R_2 = [32 \succ 12 \succ 21 \succ 13 \succ 33 \succ 11 \succ 31 \succ 23 \succ 22]$
$R_3 = [13 \succ 12 \succ 11 \succ 22 \succ 32 \succ 21 \succ 33 \succ 31 \succ 23]$

Suppose the agents are truthful. Let $\mathcal{K} = [1 \rhd 2 \rhd 3]$. In the first round of the serial dictatorship, agent 1 chooses 12; in the second round, agent 2 cannot choose 32 or 12 because item 2 in D_2 is unavailable, so she chooses 21; in the final round, agent 3 chooses 33. $\square$

An Axiomatic Characterization

Theorem 1 *For any $p \geq 2$ and $n \geq 2$, an allocation mechanism for basic categorized domain is strategy-proof, non-bossy, and category-wise neutral if and only if it is a serial dictatorship. Moreover, the three axioms are minimal for characterizing serial dictatorships.*

Proof sketch: It is easy to check that any serial dictatorship satisfies strategy-proofness, non-bossiness, and category-wise neutrality. We prove the converse by four lemmas. The first three lemmas are standard and the last one (Lemma 4) is novel, whose proof is more involved and heavily depends on the categorical structure. **Due to the space constraint, most proofs are omitted. All missing proofs can be found in the supplementary material.**

The first lemma resembles *strong monotonicity* in voting theory: for all strategy-proof and non-bossy mechanism f and all profile P, if each agent j reports differently without enlarging the set of bundles ranked above $f^j(P)$, then the allocation does not change.

Lemma 1 *Let f be a strategy-proof and non-bossy allocation mechanism. For any pair of profiles P and P' such that for all $j \leq n$, $\{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R'_j} f^j(P)\} \subseteq \{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R_j} f^j(P)\}$, we have $f(P') = f(P)$.*

For any linear order R over $\mathfrak{D}$ and any bundle $\vec{d} \in \mathfrak{D}$, we say a linear order R' is a *pushup* of $\vec{d}$ from R, if R' can be obtained from R by raising the position of $\vec{d}$ without changing the orders of other bundles. The second lemma states that for any strategy-proof and non-bossy mechanism f, if an agent reports differently by only pushing up a bundle $\vec{d}$, then either the allocation does not change, or she gets $\vec{d}$.

Lemma 2 *Let f be a strategy-proof and non-bossy allocation mechanism. For any profile P, any $j \leq n$, any bundle $\vec{d}$, and any R'_j that is a pushup of $\vec{d}$ from R_j, either (1) $f(R'_j, R_{-j}) = f(P)$ or (2) $f^j(R'_j, R_{-j}) = \vec{d}$.*

The third lemma states that strategy-proofness, non-bossiness, and category-wise neutrality altogether imply *Pareto-optimality*, which means that for any profile P, there is no allocation A such that all agents prefer their bundles in A than their bundles in $f(P)$, and some of these preferences are strict.

Lemma 3 *For any basic categorized domains with $p \geq 2$, any strategy-proof, non-bossy, and category-wise neutral allocation mechanism is Pareto optimal.*

The fourth lemma says that for any strategy-proof and non-bossy allocation mechanism f, any profile P, and any pair of agents (j_1, j_2), there is no bundle $\vec{c}$ that only contains items allocated to agent j_1 and j_2, and both agents prefer $\vec{c}$ to their allocated bundles respectively.

Lemma 4 *Let f be a strategy-proof and non-bossy allocation mechanism. For any profile P and any $j_1 \neq j_2$, let $\vec{a} = f^{j_1}(P)$ and $\vec{b} = f^{j_2}(P)$, there is no $\vec{c} \in \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_p, b_p\}$ such that $\vec{c} \succ_{R_{j_1}} \vec{a}$ and $\vec{c} \succ_{R_{j_2}} \vec{b}$, where a_i is the i-th component of $\vec{a}$.*

Proof sketch: Suppose for the sake of contradiction that such a bundle $\vec{c}$ exists. Let $\vec{d}$ denote the bundle such that $\vec{c} \cup \vec{d} = \vec{a} \cup \vec{b}$. For example, if $\vec{a} = 1213$, $\vec{b} = 2431$, and $\vec{c} = 1211$, then $\vec{d} = 2433$.

We derive a contradiction in 6 steps illustrated in Table 1. In each step, we prove that the boxed bundles are allocated to agent j_1 and agent j_2 respectively, and all other agents get their top-ranked bundles. The first two steps are shown as an example.

Step 1. Let $\hat{R}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$, $\hat{R}_{j_2} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$. For any $j \neq j_1, j_2$, let $\hat{R}_j = [f^j(P) \succ \text{others}]$. By Lemma 1, $f(\hat{P}) = f(P)$.

Step 2. Let $\bar{R}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$. We will prove that $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$. Because $\bar{R}_{j_2}$ is a pushup of $\vec{a}$ from $\hat{R}_{j_2}$, by Lemma 2, $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is either $\vec{a}$ or $\vec{b}$. The former case is impossible, otherwise $f^{j_1}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ cannot be $\vec{c}$, $\vec{a}$, or $\vec{d}$ because otherwise some item will be allocated twice. This means that $f(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is Pareto dominated by the allocation where j_1 gets $\vec{d}$, j_2 gets $\vec{c}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f (Lemma 3). Hence

$\hat{R}_{j_1}: \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others $\hat{R}_{j_2}: \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others Step 1	$\mathring{R}_{j_1}: \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others $\bar{R}_{j_2}: \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others Step 2	$\bar{R}_{j_1}: \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others $\bar{R}_{j_2}: \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others Step 3
$\bar{R}_{j_1}: \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others $\mathring{R}_{j_2}: \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others Step 4	$\mathring{R}_{j_1}: \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others $\mathring{R}_{j_2}: \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others Step 5	$\mathring{R}_{j_1}: \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others $\bar{R}_{j_2}: \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others Step 6

Table 1: Proof sketch for Lemma 4. In all steps any other agents j's preferences are $f^j(P) \succ$ others.

$f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{b} = f^{j_2}(\hat{P})$. By non-bossiness we have $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$.

Contradiction. The observations in Step 5 and Step 6 (see Table 1) imply that when agents' preferences are as in Step 6, agent j_2 has incentive to report $\mathring{R}_{j_2}$ in Step 5 to improve her allocation from $\vec{b}$ to $\vec{a}$. This contradicts the strategy-proofness of f. ■

Let R^* be a linear order over $\mathfrak{D}$ that satisfies the following conditions.

- $(1, \ldots, 1) \succ (2, \ldots, 2) \succ \cdots \succ (n, \ldots, n)$.
- For any $j < n$, the bundles ranked between $(j, \ldots, j)$ and $(j+1, \ldots, j+1)$ are those that satisfy the following two conditions: (1) at least one component is j, and (2) all components are in $\{j, j+1, \ldots, n\}$. Let B_j denote these bundles.
- For any j and any $\vec{d}, \vec{e} \in B_j$, if there are more j's in $\vec{d}$ than in $\vec{e}$, then $\vec{d} \succ \vec{e}$.

Claim 1 *Let $P^* = (R^*, \ldots, R^*)$. For any $l \le n$, there exists $j_l \le n$ such that $f^{j_l}(P^*) = (l, \ldots, l)$.*

The proof of Claim 1 uses Lemma 4. W.l.o.g. let $j_1 = 1$, $j_2 = 2$, …, $j_n = n$ denote the agents in Claim 1. For any profile $P' = (R'_1, \ldots, R'_n)$, we define n bundles as follows. Let $\vec{d^1}$ denote the top-ranked bundle in R'_1, and for any $l \ge 2$, let $\vec{d^l}$ denote agent l's top-ranked available bundle after $\{\vec{d^1}, \ldots, \vec{d^{l-1}}\}$ have been allocated. Then, for any $i \le m$, we define a category-wise permutation M_i such that for all $l \le n$, $M_i(l) = [\vec{d^l}]_i$, where we recall that $[\vec{d^l}]_i$ is the item in the i-th category in $\vec{d^l}$. Let $M = (M_1, \ldots, M_m)$. It follows that for all $l \le n$, $M(l, \ldots, l) = \vec{d^l}$. By category-wise neutrality and Claim 1, $f^l(M(P^*)) = M(f^l(P^*)) = \vec{d^l}$.

Comparing $M(P^*)$ to P', we have that for all $l \le n$ and all bundle $\vec{e}$, if $\vec{d^l} \succ_{M(R^*)} \vec{e}$ then $\vec{d^l} \succ_{R'_l} \vec{e}$. This is because if there exists $\vec{e}$ such that $\vec{d^l} \succ_{M(R^*)} \vec{e}$ but $\vec{e} \succ_{R'_l} \vec{d^l}$, then $\vec{e}$ is still available after $\{\vec{d^1}, \ldots, \vec{d^{l-1}}\}$ have been allocated, and $\vec{e}$ is ranked higher than $\vec{d^l}$ in R'_l. This contradicts the selection of $\vec{d^l}$. By Lemma 1, $f(P') = f(M(P^*)) = M(f(P^*))$, which proves that f is the serial dictatorship w.r.t. the order $1 \rhd 2 \rhd \cdots \rhd n$.

Finally, we show the minimality of {strategy-proofness, non-bossiness, category-wise neutrality}.

Strategy-proofness is necessary by considering the allocation mechanism that maximizes the social welfare w.r.t. the following utility functions. For any $i \le n^p$ and $j \le n$, the bundle ranked at the i-th position in agent j's preferences gets $(n^p - i)(1 + (\frac{1}{2n^p})^j)$ points.

Non-bossiness is necessary by considering the following "conditional serial dictatorship": agent 1 chooses her favorite bundle in the first p rounds, and if the first component of agent 1's second-ranked bundle is the same as the first component of her top-ranked bundle, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise the order is $n \rhd n-1 \rhd \cdots \rhd 2$.

Category-wise neutrality is necessary by considering the following "conditional serial dictatorship": agent 1 chooses her favorite bundle in the first p rounds, and if agent 1 gets $(1, \ldots, 1)$, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise the order is $n \rhd n-1 \rhd \cdots \rhd 2$. ■

Categorical Sequential Allocation Mechanisms

Given a linear order $\mathcal{O}$ over $\{1, \ldots, n\} \times \{1, \ldots, p\}$, the *categorical sequential allocation mechanism (CSAM)* $f_{\mathcal{O}}$ allocates the items in np steps as illustrated in Protocol 1. In each step t, suppose the t-th element in $\mathcal{O}$ is (j, i), (equivalently, $t = \mathcal{O}^{-1}(j, i)$). Agent j is called the *active agent* in step t and she chooses an item $d_{j,i}$ that is still available from D_i. Then, $d_{j,i}$ is broadcast to all agents and we move on to the next step.

Protocol 1: Categorical sequential allocation mechanism (CSAM) $f_{\mathcal{O}}$.

Input: An order $\mathcal{O}$ over $\{1, \ldots, n\} \times \{1, \ldots, p\}$.
Broadcast $\mathcal{O}$ to all agents.
for $t = 1$ *to* np **do**
  Let (j, i) be the t-th element in $\mathcal{O}$.
  Agent j chooses an available item $d_{j,i} \in D_i$.
  Broadcast $d_{j,i}$ to all agents.
end

In CSAMs, in each step the active agent must choose an item from the designated category. Hence, CSAMs are different from sequential allocation protocols [6] and the draft mechanism [11], where in each step the active agent can choose any available item from any category.

Example 2 The serial dictatorship w.r.t. $\mathcal{K} = [j_1 \rhd \cdots \rhd j_n]$ is a CSAM w.r.t. $(j_1, 1) \rhd (j_1, 2) \rhd \cdots \rhd (j_1, p) \rhd \cdots \rhd (j_n, 1) \rhd (j_n, 2) \rhd \cdots \rhd (j_n, p)$.

For any even number p, given any linear order $\mathcal{K} = [j_1 \rhd \cdots \rhd j_n]$ over the agents, we define the *balanced CSAM* to be the mechanism where agents choose items in p phases, such that for each $i \leq p$, in phase i all agents choose from D_i w.r.t. $\mathcal{K}$ if i is odd, and w.r.t. inverse K if i is even.

For example, when $n = 3$, $p = 2$, and $\mathcal{K} = [1 \rhd 2 \rhd 3]$, the balanced CSAM uses the order $(1, 1) \rhd (2, 1) \rhd (3, 1) \rhd (3, 2) \rhd (2, 2) \rhd (1, 2)$. □

Similar to sequential allocations [6], CSAMs can be implemented in a distributed manner. Communication cost for CSAMs is much lower than for direct mechanisms, where agents report their preferences in full to the center, which requires $\Theta(n^p p \log n)$ bits per agent, and thus the total communication cost is $\Theta(n^{p+1} p \log n)$. For CSAMs, the total communication cost of Protocol 1 is $\Theta(n^2 p \log n + np(n \log n)) = \Theta(n^2 p \log np)$, which has a $\Theta(n^{p-2} \cdot \frac{\log n}{\log n + \log p})$ multiplicative saving. In light of this, CSAMs preserve more privacy as well.

To analyze the outcomes of CSAMs, we focus on two types of myopic agents. For any $1 \leq i \leq p$, we let $D_{i,t}$ denote the set of available items in D_i at the beginning of round t.

- *Optimistic* agents. An optimistic agent chooses the item in her top-ranked bundle that is still available, given the items she chose in previous steps.
- *Pessimistic* agents. A pessimistic agent j in round t chooses an item $d_{j,i}$ from $D_{i,t}$, such that for all $d'_i \in D_{i,t}$ with $d'_i \neq d_{j,i}$, agent j prefers the worst available bundle whose i-th component is $d_{j,i}$ to the worst available bundle whose i-th component is d'_i.

In this paper, we assume that whether an agent is optimistic or pessimistic is fixed before applying a CSAM.

Example 3 Let $n = 3$, $p = 2$. Consider the same profile as in Example 1, which can be simplified as follows.

Agent 1 (optimistic): $12 \succ 21 \succ \text{others} \succ 11$
Agent 2 (optimistic): $32 \succ \text{others} \succ 22$
Agent 3 (pessimistic): $13 \succ \text{others} \succ 33 \succ 31 \succ 23$

Let $\mathcal{O} = [(1, 1) \rhd (2, 2) \rhd (3, 1) \rhd (3, 2) \rhd (2, 1) \rhd (1, 2)]$. Suppose agent 1 and agent 2 are optimistic and agent 3 is pessimistic. When $t = 1$, agent 1 (optimistic) chooses item 1 from D_1. When $t = 2$, item 32 is the top-ranked available bundle for agent 2 (optimistic), so she chooses 2 from D_2. When $t = 3$, the available bundles are $\{2, 3\} \times \{1, 3\}$. If agent 3 chooses 2 from D_1, then the worst-case available bundle is 23, and if agent 3 chooses 3 from D_1, then the worst-case available bundle is 31. Since agent 3 prefers 31 to 23, she chooses 3 from D_1. When $t = 4$, agent 3 chooses 3 from D_2. When $t = 5$, agent 2 choses 2 from D_1 and when $t = 6$, agent 1 choses 1 from D_2. Finally, agent 1 gets 11, agent 2 gets 22, and agent 3 gets 33. □

Ordinal Efficiency of CSAMs

In this section, we focus on characterizing the *ordinal efficiency* of CSAMs measured by agents' ranks of the bundles they receive.[3] For any linear order R over $\mathfrak{D}$ and any bundle $\vec{d}$, we let $\text{Rank}(R, \vec{d})$ denote the rank of $\vec{d}$ in R, such that the highest position has rank 1 and the lowest position has rank n^p. Given a CSAM $f_{\mathcal{O}}$, we introduce the following notation for any $j \leq n$.

- Let $\mathcal{O}_j$ denote the linear order over the categories $\{1, \ldots, p\}$ according to which agent j chooses items from in $\mathcal{O}$.
- For any $i \leq p$, let $k_{j,i}$ denote the number of items in D_i that are still available right before agent j chooses from D_i. Formally, $k_{j,i} = 1 + |\{(j', i) : (j, i) \rhd_{\mathcal{O}} (j', i)\}|$.
- Let K_j denote the smallest index in $\mathcal{O}_j$ such that no agent can "interrupt" agent j from choosing all items in her top-ranked bundle that is available in round $(j, \mathcal{O}_j(K_j))$. Formally, K_j is the smallest number such that for any l with $K_j < l \leq p$, between the round when agent j chooses an item from category $\mathcal{O}_j(K_j)$ and the round when agent j chooses an item from category $\mathcal{O}_j(l)$, no agent chooses an item from category $\mathcal{O}_j(l)$. We note that K_j is defined only by $\mathcal{O}$ and is thus independent of agents' preferences.

Example 4 Let $\mathcal{O}^* = [(1, 1) \rhd (1, 2) \rhd (1, 3) \rhd (2, 1) \rhd (2, 2) \rhd (2, 3) \rhd (3, 1) \rhd (3, 2) \rhd (3, 3)]$. That is, $f_{\mathcal{O}^*}$ is a serial dictatorship. Then $\mathcal{O}^*_1 = \mathcal{O}^*_2 = \mathcal{O}^*_3 = 1 \rhd 2 \rhd 3$. $K_1 = K_2 = K_3 = 1$. $k_{1,1} = k_{1,2} = k_{1,3} = 3$, $k_{2,1} = k_{2,2} = k_{2,3} = 2$, $k_{3,1} = k_{3,2} = k_{3,3} = 1$.

Let $\mathcal{O}$ be the order in Example 3, that is, $\mathcal{O} = [(1, 1) \rhd (2, 2) \rhd (3, 1) \rhd (3, 2) \rhd (2, 1) \rhd (1, 2)]$.

$\mathcal{O}_1 = 1 \rhd 2$. $K_1 = 2$ since $(2, 2)$ is between $(1, 1)$ and $(1, 2)$ in $\mathcal{O}$. $k_{1,1} = 3$, $k_{1,2} = 1$.

$\mathcal{O}_2 = 2 \rhd 1$. $K_2 = 2$ since $(3, 1)$ is between $(2, 2)$ and $(2, 1)$. $k_{2,1} = 1$, $k_{2,2} = 3$.

$\mathcal{O}_3 = 1 \rhd 2$. $K_3 = 1$ since between $(3, 1)$ and $(3, 2)$ in $\mathcal{O}$, no agent chooses an item from D_2. $k_{3,1} = k_{3,2} = 2$. □

Proposition 1 *For any CSAM $f_{\mathcal{O}}$, any combination of optimistic and pessimistic agents, any $j \leq n$, and any profile:*

- **Upper bound for optimistic agents:** *if j is optimistic, then the rank of the bundle allocated to her is at most $n^p + 1 - \prod_{l=K_j}^{p} k_{j,O_j(l)}$.*
- **Upper bound for pessimistic agents:** *if j is pessimistic, then the rank of the bundle allocated to her is at most $n^p - \sum_{l=1}^{p} (k_{j,O_j(l)} - 1)$.*

Proof sketch: W.l.o.g. let $\mathcal{O}_j = 1 \rhd 2 \rhd \cdots \rhd p$. If j is optimistic, then we let $t_j = \mathcal{O}^{-1}(j, K_j)$ and let $(d_{j,1}, \ldots, d_{j,K_j-1}) \in D_1 \times \cdots \times D_{K_j-1}$ denote the items agent j chose in the previous rounds. It follows that at the beginning of round t_j, the following $\prod_{l=K_j}^{p} k_{j,l}$ bundles are available for agent j: $\mathfrak{D}_j = (d_{j,1}, \ldots, d_{j,K_j-1}) \times \prod_{l=K_j}^{p} D_{l,t_j}$. By the definition of K_j, no agent can

[3] This is different from the ordinal efficiency for randomized allocation mechanisms [2].

interrupt agent j from choosing the items in her top-ranked bundle in $\mathfrak{D}_j$, and $|\mathfrak{D}_j| = \prod_{l=K_j}^{p} k_{j,l}$.

If j is pessimistic, then we let $\vec{d_j} = (d_{j,1}, \ldots, d_{j,p}) = f^j_{\mathcal{O}}(P)$ denote her allocation by $f_{\mathcal{O}}$. By the definition of pessimism and the assumption that for any $1 \le l \le p$, in round $t^* = \mathcal{O}^{-1}(j,l)$ agent j chose $d_{j,l}$ from D_{l,t^*}, we must have that for all $d'_l \in D_{l,t^*}$ with $d'_l \neq d_{j,l}$, there exists an bundle $(d_{j,1}, \ldots, d_{j,l-1}, d'_l, \ldots, d'_p)$ that is ranked below $\vec{d_j}$. Such bundles are all different and the number of them is $\sum_{l=1}^{p}(k_{j,l} - 1)$, which proves the bound for pessimistic agents. ■

We note that Proposition 1 works for any combination of optimistic and pessimistic agents, which is much more general than the setting with all-optimistic agents and the setting with all-pessimistic agents. In addition, once the CSAM and the properties of the agents (that is, whether each agent is optimistic or pessimistic) is given, the bounds hold for all preference profile.

Our main theorem in this section states that, surprisingly, for all combinations of optimistic and pessimistic agents, all upper bounds described in Proposition 1 can be matched in a same profile. Even more surprisingly, for the same profile there exists an allocation where almost all agents get their top-ranked bundle, and the only agent who may not get her top-ranked bundle gets her second-ranked bundle. Therefore, the theorem not only provides a worst-case analysis in the absolute sense in that all upper bounds in Proposition 1 are matched in the same profile, but also in the comparative sense w.r.t. the optimal allocation of the profile.

Theorem 2 *For any CSAM $f_{\mathcal{O}}$ and any combination of optimistic and pessimistic agents, there exists a profile P such that for all $j \le n$:*

1. *if agent j is optimistic, then the rank of the bundle allocated to her is $n^p + 1 - \prod_{l=K_j}^{p} k_{j,O_j(l)}$;*
2. *if agent j is pessimistic, then the rank of the bundle allocated to her is $n^p - \sum_{l=1}^{p}(k_{j,O_j(l)} - 1)$;*
3. *there exists an allocation where at least $n-1$ agents get their top-ranked bundles, and the remaining agent gets her top-ranked or second-ranked bundle.*

The proof is quite involved and can be found in the supplementary material.

Example 5 The profile in Example 3 is an example of the profile guaranteed by Theorem 2: agent 1 (optimistic) gets her bottom bundle ($K_1 = 2$ and $k_{1,2} = 1$), agent 2 (optimistic) gets her bottom bundle ($K_2 = 2$ and $k_{2,1} = 1$), and agent 3 (pessimistic) gets her third bundle ($k_{3,1} = k_{3,2} = 2$). Moreover, there exists an allocation where agent 2 and agent 3 get their top bundles and agent 1 gets her second bundle. □

Theorem 2 can be used to compare various CSAMs with optimistic and pessimistic agents w.r.t. worst-case utilitarian rank and worst-case egalitarian rank.

Definition 2 *Given any CSAM $f_{\mathcal{O}}$ and any n, the* worst-case utilitarian rank *is* $\max_{P_n} \sum_{R_j \in P_n} Rank(R_j, f^j_{\mathcal{O}}(P_n))$, *and the* worst-case egalitarian rank *is* $\max_{P_n} \max_{R_j \in P_n} Rank(R_j, f^j_{\mathcal{O}}(P_n))$, *where P_n is a profile of n agents.*

In words, the worst-case utilitarian rank is the worst (largest) total rank of the bundles (w.r.t. respective agent's preferences) allocated by $f_{\mathcal{O}}$. The worst-case egalitarian rank is the worst (largest) rank of the least-satisfied agent, which is also a well-accepted measure of fairness. The worst case is taken over all profiles of n agents.

Proposition 2 *Among all CSAMs, serial dictatorships with all-optimistic agents have the best (smallest) worst-case utilitarian rank and the worst (largest) worst-case egalitarian rank.*

Proposition 3 *Any CSAM with all-optimistic agents has the worst (largest) worst-case egalitarian rank, which is n^p.*

Proposition 4 *For any even number p, the worst-case egalitarian rank of any balanced CSAM (defined in Example 2) with all-pessimistic agents is $n^p - (n-1)p/2$. These are the CSAMs with the best worst-case egalitarian rank among CSAMs with all-pessimistic agents.*

A natural question after Proposition 4 is: do the balanced CSAMs with all-pessimistic agents have optimal worst-case egalitarian rank, among all CSAMs *for any combination of optimistic and pessimistic agents*? The answer is negative.

Proposition 5 *For any even number p with $2^p > 1 + (n-1)p/2$, there exists a CSAM with both optimistic and pessimistic agents, whose worst-case egalitarian rank is strictly better (smaller) than $n^p - (n-1)p/2$.*

Summary and Future Work

In this paper we propose CDAPs to model allocation problems for indivisible and categorized items without monetary transfer, when agents have heterogenous and combinatorial preferences. We characterize serial dictatorships for basic CDAPs, propose CSAMs and characterize worst-case ordinal efficiency for CSAMs with any combination of optimistic and pessimistic agents, which leads to characterizations of utilitarian rank and egalitarian rank of various CSAMs.

There are many open questions and directions for future research, including analyzing the outcomes and ordinal efficiency for CSAMs for other types of agents, e.g. strategic agents and minimax-regret agents. We also plan to work on expected utilitarian rank and egalitarian rank (some simulation results are included in the supplementary material), and randomized allocation mechanisms. For general CDAPs, we are excited to explore generalizations of CP-nets [4], LP-trees [3], and soft constraints [24] for preference representation. Based on these new languages we can analyze fairness and computational aspects of CSAMs and other mechanisms. Mechanism design for CDAPs with sharable, non-sharable, and divisible items is also an important and promising topic for future research.

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Appendix A: Full Proofs

Proof of Theorem 1

Proof: We first prove four lemmas. The first three lemmas are standard in proving characterizations for serial dictatorships and their proofs can be found in the Appendix. The last one (Lemma 4) is new, whose proof is the most involved and heavily relies on the categorical information.

The first lemma (roughly) says that for all strategy-proof and non-bossy mechanism f and all profile P, if every agent j reports a different ranking without enlarging the set of bundles ranked above $f^j(P)$ (and she can shuffle the bundles ranked above $f^j(P)$ and she can shuffle the bundles ranked below $f^j(P)$), then the allocation to all agents does not change in the new profile. This resembles *(strong) monotonicity* in social choice.

Lemma 1 *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any pair of profiles P and P' such that for all $j \leq n$, $\{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R'_j} f^j(P)\} \subseteq \{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R_j} f^j(P)\}$, we have $f(P') = f(P)$.*

Proof: We first prove the lemma for the special case where P and P' only differ on one agent's preferences. Let j be an agent with $R'_j \neq R_j$ and $\{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R'_j} f^j(P)\} \subseteq \{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R_j} f^j(P)\}$. We will prove that $f^j(R'_j, R_{-j}) = f^j(R_j, R_{-j})$.

Suppose for the sake of contradiction $f^j(R'_j, R_{-j}) \neq f^j(R_j, R_{-j})$. If $f^j(R'_j, R_{-j}) \succ_{R_j} f^j(R_j, R_{-j})$ then it means that f is not strategy-proof since j has incentive to report R'_j when her true preferences are R_j. If $f^j(R_j, R_{-j}) \succ_{R_j} f^j(R'_j, R_{-j})$ then $f^j(R_j, R_{-j}) \succ_{R'_j} f^j(R'_j, R_{-j})$, which means that when agent j's preferences are R'_j she has incentive to report R_j,. This again contradicts the assumption that f is strategy-proof. Therefore $f^j(R_j, R_{-j}) = f^j(R'_j, R_{-j})$.

By non-bossiness, $f(R_j, R_{-j}) = f(R'_j, R_{-j})$. The lemma is proved by recursively applying this argument to $j = 1, \ldots, n$. ■

For any linear order R over $\mathfrak{D}$ and any bundle $\vec{d} \in \mathfrak{D}$, we say a linear order R' is a *pushup* of $\vec{d}$ from R, if R' can be obtained from R by raising the position of $\vec{d}$ while keeping the relative positions of other bundles unchanged. The next lemma states that for any strategy-proof and non-bossy mechanism f, if an agent reports her preferences differently by only pushing up a bundle $\vec{d}$, then either the allocation to all agents does not change, or she gets $\vec{d}$.

Lemma 2 *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any profile P, any $j \leq n$, any bundle $\vec{d}$, and any R'_j that is a pushup of $\vec{d}$ from R_j, either (1) $f(R'_j, R_{-j}) = f(R)$ or (2) $f^j(R'_j, R_{-j}) = \vec{d}$.* **Proof:** We first prove that $f^j(R'_j, R_{-j}) = f^j(R)$ or $f^j(R'_j, R_{-j}) = \vec{d}$. Suppose on the contrary that $f^j(R'_j, R_{-j})$ is neither $f^j(R)$ nor $\vec{d}$. If $f^j(R'_j, R_{-j}) \succ_{R_j} f^j(R)$, then f is not strategy-proof since when agent j's true preferences are R_j and other agents' preferences are R_{-j}, she has incentive to report R'_j to make her allocation better. If $f^j(R) \succ_{R_j} f^j(R'_j, R_{-j})$, then since $\vec{d} \neq f^j(R'_j, R_{-j})$, we have $f^j(R) \succ_{R'_j} f^j(R'_j, R_{-j})$. In this case when agent j's true preferences are R'_j and other agents' preferences are R_{-j}, she has incentive to report R_j to make her allocation better, which means that f is not strategy-proof. Therefore, $f^j(R'_j, R_{-j}) = f^j(R)$ or $f^j(R'_j, R_{-j}) = \vec{d}$. If $f^j(R'_j, R_{-j}) = f^j(R_j, R_{-j})$, then by non-bossiness $f(R'_j, R_{-j}) = f(R)$. This completes the proof. ■

We next prove that strategy-proofness, non-bossiness, and category-wise neutrality altogether imply *Pareto-optimality*, which states that for any profile P, there does not exists an allocation A such that all agents prefer their bundles in A than their bundles in $f(P)$, and some of them strictly prefer their bundles in A.

Lemma 3 *For any basic categorized domains with $p \geq 2$, any strategy-proof, non-bossy, and category-wise neutral allocation mechanism is Pareto optimal.* **Proof:** We prove the lemma by contradiction. Let f be a strategy-proof, non-bossy, category-wise neutral, but non-(Pareto optimal) allocation mechanism. Let $P = (R_1, \ldots, R_n)$ denote a profile such that $f(P)$ is Pareto dominated by an allocation A. For any $i \leq m$, let M_i denote the permutation over D_i so that for every $j \leq n$, $[f^j(P)]_i$ is permuted to $[A(j)]_i$. Let $M = (M_1, \ldots, M_m)$. It follows that for all $j \leq n$, $M(f^j(P)) = A(j)$.

Let R'_j denote an arbitrary ranking where $A(j)$ is ranked at the top place, and $f^j(P)$ is ranked at the second place if it is different from $A(j)$. Let R^*_j denote an arbitrary ranking where $f^j(P)$ is ranked at the top place, and $A(j)$ is ranked at the second place if it is different from $f^j(P)$. Let $P' = (R'_1, \ldots, R'_n)$ and $P^* = (R^*_1, \ldots, R^*_n)$. P' and P^* are illustrated as follows.

$$P' = \left\{ \begin{array}{c} R'_1 : A^1 \succ f^1(P) \succ \text{Others} \\ \vdots \\ R'_n : A^n \succ f^n(P) \succ \text{Others} \end{array} \right\}$$

$$P^* = \left\{ \begin{array}{c} R^*_1 : f^1(P) \succ A^1 \succ \text{Others} \\ \vdots \\ R^*_n : f^n(P) \succ A^n \succ \text{Others} \end{array} \right\}$$

Since A Pareto dominates $f(P)$, by Lemma 1 we have $f(P') = f(P)$, because for any $j \leq n$, in R'_j the only bundle ranked ahead of $f^j(P)$ is $A(j)$, if it is different from $f^j(P)$, and $A(j)$ is also ranked ahead of $f^j(P)$ in R_j. By Lemma 1 again we have $f(P^*) = f(P)$. Comparing $M(P')$ and P^*, we observe that the only differences are the orderings among $\mathfrak{D} \setminus \{A(j), f^j(P)\}$. Applying Lemma 1 to P^* and $M(P')$, we have that $f(M(P')) = f(P^*) = f(P)$. However, by category-wise neutrality $f(M(P')) = M(f(P')) = A$, which is a contradiction. ■

The next lemma states that for any strategy-proof and non-bossy allocation mechanism f, any profile P, and any pair of agents j_1, j_2, there is no bundle $\vec{c}$ that only contains items allocated to agent j_1 and j_2 by f, such that both j_1 and j_2 prefer $\vec{c}$ to their bundles allocated by f.

Lemma 4 *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any profile P and any $j_1 \neq j_2 \leq n$, let $\vec{a} = f^{j_1}(P)$ and $\vec{b} = f^{j_2}(P)$, there does not exist $\vec{c} \in \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_p, b_p\}$ such that $\vec{c} \succ_{R_{j_1}} \vec{a}$ and $\vec{c} \succ_{R_{j_2}} \vec{b}$, where a_i is the i-th component of $\vec{a}$.*

Proof: Suppose for the sake of contradiction that such a bundle $\vec{c}$ exists. Let $\vec{d}$ denote the bundle such that $\vec{c} \cup \vec{d} =$

$\hat{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others $\hat{R}_{j_2} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others	$\hat{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others $\bar{R}_{j_2} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others	$\bar{R}_{j_1} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others $\bar{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others
Step 1	Step 2	Step 3
$\bar{R}_{j_1} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others $\mathring{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others Other j : $f^j(P) \succ$ others	$\mathring{R}_{j_1} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others $\mathring{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others Other j : $f^j(P) \succ$ others	$\mathring{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others $\bar{R}_{j_2} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others
Step 4	Step 5	Step 6

Table 2: Proof sketch of Lemma 4.

$\vec{a} \cup \vec{b}$. More precisely, for all $i \leq m$, $\{c_i, d_i\} = \{a_i, b_i\}$. For example, if $\vec{a} = 1213$, $\vec{b} = 2431$, and $\vec{c} = 1211$, then $\vec{d} = 2433$.

The rest of the proof derives a contradiction by proving the a series of observations as illustrated in Table 2, which is the same as Table 1. In each step, we prove that the boxed bundles are allocated to agent j_1 and agent j_2 respectively, and all other agents get their top-ranked bundles.

Step 1. Let $\hat{R}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$, $\hat{R}_{j_2} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$, where "others" represents an arbitrary linear order over the remaining bundles, and for any $j \neq j_1, j_2$, let $\hat{R}_j = [f^j(P) \succ \text{others}]$. By Lemma 1, $f(\hat{P}) = f(P)$.

Step 2. Let $\bar{R}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{a}$ from $\hat{R}_{j_2}$. We will prove that $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$. Since $\bar{R}_{j_2}$ is a pushup of $\vec{a}$ from $\hat{R}_{j_2}$, by Lemma 2, $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible. Suppose for the sake of contradiction $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{a}$, then $f^{j_1}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ cannot be $\vec{c}$, $\vec{a}$, or $\vec{d}$ since otherwise some item will be allocated twice. This means that $f(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is Pareto dominated by the allocation where j_1 gets $\vec{d}$, j_2 gets $\vec{c}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f (Lemma 3). Hence $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{b} = f^{j_2}(\hat{P})$. By non-bossiness we have $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$.

Step 3. Let $\bar{R}_{j_1} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{b}$ from $\hat{R}_{j_1}$. We will prove that in $f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$, j_1 gets $\vec{b}$, j_2 gets $\vec{a}$, and all other agents get the same items as in $f(P)$. Since $\bar{R}_{j_1}$ is a pushup of $\vec{b}$ from $\hat{R}_{j_1}$, by Lemma 2, $f^{j_1}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible. Suppose for the sake of contradiction that $f^{j_1}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{a}$. By non-bossiness, $f^{j_2}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{b}$. This means that $f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is Pareto-dominated by the allocation where j_1 gets $\vec{b}$, j_2 gets $\vec{a}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f (Lemma 3).

Step 4. Let $\mathring{R}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$ be a pushup of $\vec{d}$ from $\bar{R}_{j_2}$. By Lemma 1, $f(\bar{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$.

Step 5. Let $\mathring{R}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{a}$ from $\bar{R}_{j_1}$. We will prove that $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$. Since $\mathring{R}_{j_1}$ is a pushup of $\vec{a}$ from $\bar{R}_{j_1}$, by Lemma 2, $f^{j_1}(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible. Suppose for the sake of contradiction that $f^{j_1}(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{a}$. Then in $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$, agent j_2 cannot get $\vec{c}$, $\vec{a}$, or $\vec{d}$, which means that $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is Pareto-dominated by the allocation where j_1 gets $\vec{c}$, j_2 gets $\vec{d}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f. Hence, $f^{j_1}(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{b}$. By non-bossiness $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$.

Step 6. We note that $\mathring{R}_{j_1}$ is a pushup of $\vec{b}$ from $\hat{R}_{j_1}$ (and $\vec{b}$ is still below $\vec{a}$). By Lemma 1, $f(\mathring{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\hat{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$. We note that the right hand side is the profile in Step 2.

Contradiction. Finally, the observations in Step 5 and Step 6 imply that when agents' preferences are as in Step 6, agent j_2 has incentive to report $\mathring{R}_{j_2}$ in Step 5 to improve the bundle allocated to her (from $\vec{b}$ to $\vec{a}$). This contradicts the strategy-proofness of f and completes the proof of Lemma 4. ■

It is easy to check that any serial dictatorship satisfies strategy-proofness, non-bossiness and category-wise neutrality. We now prove that any mechanism satisfying the three properties must be a serial dictatorship. Let R^* be a linear order over $\mathfrak{D}$ that satisfies the following conditions.

- $(1, \ldots, 1) \succ (2, \ldots, 2) \succ \cdots \succ (n, \ldots, n)$.
- For any $j < n$, the bundles ranked between $(j, \ldots, j)$ and $(j+1, \ldots, j+1)$ are those satisfying the following two conditions: 1) at least one component is j, and 2) all components are in $\{j, j+1, \ldots, n\}$. Let B_j denote these bundles. That is, $B_j \subseteq \mathfrak{D}$ and $B_j = \{\vec{d} : \forall l, d_l \geq j \text{ and } \exists l', d_{l'} = j\}$.
- For any j and any $\vec{d}, \vec{e} \in B_j$, if the number of j's in $\vec{d}$ is strictly larger than the number of j's in $\vec{e}$, then $\vec{d} \succ \vec{e}$.

Claim 1 *Let $P^* = (R^*, \ldots, R^*)$. For any $l \leq n$, there exists $j_l \leq n$ such that $f^{j_l}(P^*) = (l, \ldots, l)$.*

Proof: The claim is proved by induction on l. When $l = 1$.

For the sake of contradiction suppose there is no j_l with $f^{j_l}(P^*) = (1, \ldots, 1)$. Then there exist a pair of agents j and j' such that both $\vec{a} = f^j(P^*)$ and $\vec{b} = f^{j'}(P^*)$ contain 1 in at least one category.

Let $\vec{c}$ be the bundle obtained from $\vec{a}$ by replacing items in categories where $\vec{b}$ takes 1 to 1. More precisely, we let $\vec{c} = (c_1, \ldots, c_p)$, where

$$c_i = \begin{cases} 1 & \text{if } a_i = 1 \text{ or } b_i = 1 \\ a_i & \text{otherwise} \end{cases}$$

It follows that in R^*, $\vec{c} \succ_{R^*} \vec{a}$ and $\vec{c} \succ_{R^*} \vec{b}$ since the number of 1's in $\vec{c}$ is strictly larger than the number of 1's in $\vec{a}$ or $\vec{b}$. By Lemma 4, this contradicts the assumption that f is strategy-proof and non-bossy. Hence there exists $j_1 \leq n$ with $f^{j_1}(P^*) = (1, \ldots, 1)$.

Suppose the claim is true for $l \leq l'$. We next prove that there exists $j_{l'+1}$ such that $f^{j_{l'+1}}(P^*) = (l'+1, \ldots, l'+1)$. This follows after a similar reasoning to the $l = 1$ case. Formally, suppose for the sake of contradiction there does not exist such a $j_{l'+1}$. Then, there exist two agents who get $\vec{a}$ and $\vec{b}$ in $f(P^*)$ such that both $\vec{a}$ and $\vec{b}$ contain $l'+1$ in at least one category. By the induction hypothesis, items $\{1, \ldots, l'\}$ in all categories have been allocated, which means that all components of $\vec{a}$ and $\vec{b}$ are at least as large as $l'+1$. Let $\vec{c}$ be the bundle obtained from $\vec{a}$ by replacing items in all categories where $\vec{b}$ takes $l'+1$ to $l'+1$. We have $\vec{c} \succ_{R^*} \vec{a}$ and $\vec{c} \succ_{R^*} \vec{b}$, leading to a contradiction by Lemma 4. Therefore, the claim holds for $l = l'+1$. This completes the proof of Claim 1. ■

Back to the proof of Theorem 1, w.l.o.g. we let $j_1 = 1$, $j_2 = 2$, ..., $j_n = n$ denote the agents in Claim 1. For any profile $P' = (R'_1, \ldots, R'_n)$, we define n bundles as follows. Let $\vec{d^1}$ denote the top-ranked bundle in R'_1, and for any $l \geq 2$, let $\vec{d^l}$ denote agent l's top-ranked available bundle given that items in $\vec{d^1}, \ldots, \vec{d^{l-1}}$ have already been allocated. That is, $\vec{d^l}$ is the most preferred bundle in $\{\vec{d} : \forall l' < l, \vec{d} \cap \vec{d^{l'}} = \emptyset\}$ according to R'_l. Then, for any $i \leq m$, we define a category-wise permutation M_i such that for all $l \leq n$, $M_i(l) = [\vec{d^l}]_i$, where we recall that $[\vec{d^l}]_i$ is the item in the i-th category in $\vec{d^l}$. Let $M = (M_1, \ldots, M_m)$. It follows that for all $l \leq n$, $M(l, \ldots, l) = \vec{d^l}$. By category-wise neutrality and Claim 1, in $f(M(P^*))$ agent l gets $M(f^l(P^*)) = \vec{d^l}$.

Comparing $M(P^*)$ to P', we notice that for all $l \leq n$ and all bundle $\vec{e}$, if $\vec{d^l} \succ_{M(R^*)} \vec{e}$ then $\vec{d^l} \succ_{R'_l} \vec{e}$. This is because if there exists $\vec{e}$ such that $\vec{d^l} \succ_{M(R^*)} \vec{e}$ but $\vec{e} \succ_{R'_l} \vec{d^l}$, then $\vec{e}$ is still available after $\{\vec{d^1}, \ldots, \vec{d^{l-1}}\}$ have been allocated, and $\vec{e}$ is ranked higher than $\vec{d^l}$ in R'_l. This contradicts the selection of $\vec{d^l}$. By Lemma 1, $f(P') = f(M(P^*)) = M(f(P^*))$, which proves that f is the serial dictatorship w.r.t. the order $1 \rhd 2 \rhd \cdots \rhd n$.

Next, we show that strategy-proofness, non-bossiness, and category-wise neutrality are a minimal set of properties that characterize serial dictatorships.

Strategy-proofness is necessary: Consider the allocation mechanism that maximizes the social welfare w.r.t. the following utility functions. For any $i \leq n^p$ and $j \leq n$, the bundle ranked at the i-th position in agent j's preferences gets $(n^p - i)(1 + (\frac{1}{2n^p})^j)$ points.[4] It is not hard to check that for any pair of different allocations, the social welfares are different. It follows that this allocation mechanism satisfies non-bossiness. This is because if agent j's allocation is the same when only she reports differently, then the set of items left to the other agents is the same, which means that the allocation to the other agents by the mechanism is the same. Since the utility of a bundle only depends on its position in the agents' preferences rather than the name of the bundle, the allocation mechanism satisfies category-wise neutrality. This mechanism is not a serial dictatorship. To see this, consider the basic categorized domain with $p = n = 2$, $R'_1 = [11 \succ 12 \succ 22 \succ 21]$, and $R'_2 = [12 \succ 21 \succ 11 \succ 22]$. A serial dictatorship will either give 11 to agent 1 and give 22 to agent 2, or give 21 to agent 1 and give 12 to agent 2, but the allocation that maximizes social welfare w.r.t. the utility function described above is to give 12 to agent 1 and give 21 to agent 2.

non-bossiness is necessary: Consider the following "conditional serial dictatorship": agent 1 always chooses her favorite bundle in the first round, and the order over the remaining agents $\{2, \ldots, n\}$ depends on agent 1's preferences in the following way: if the first component of agent 1's second-ranked bundle is the same as the first component of her top choice, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise it is $n \rhd n-1 \rhd \cdots \rhd 2$. It is not hard to verify that this mechanism satisfies strategy-proofness and category-wise neutrality, and is not a serial dictatorship (where the order must be fixed before seeing the profile).

Category-wise neutrality is necessary: Consider the following "conditional serial dictatorship": agent 1 always chooses her favorite bundle in the first round, and the order over agents $\{2, \ldots, n\}$ depends on the allocation of agent 1 in the following way: if agent 1 gets $(1, \ldots, 1)$, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise it is $n \rhd n-1 \rhd \cdots \rhd 2$. It is not hard to verify that this mechanism satisfies strategy-proofness and non-bossiness, and is not a serial dictatorship. ■

Proof of Proposition 1

Proof: Equivalently, we need to prove that for any optimistic agent, the bundle allocated to her is ranked no lower than the $(\prod_{l=K_j}^{p} k_{j,O_j(l)})$-th position from the bottom, and for any pessimistic agent, the bundle allocated to her is ranked no lower than the $(1 + \sum_{l=1}^{p}(k_{j,O_j(l)} - 1))$-th position from the bottom.

W.l.o.g. let $\mathcal{O}_j = 1 \rhd 2 \rhd \cdots \rhd p$. That is, agent j chooses items from categories $1, \ldots, p$ in sequence in the sequential allocation. This means that in this proof, for any $l \leq p$, $\mathcal{O}_j(l) = l$. We first prove the proposition for an optimistic agent j. In the beginning of round $t_j = \mathcal{O}^{-1}(j, K_j)$ in Algorithm 1, agent j has already chosen items from $D_1, \ldots, D_{K_j-1}$, and is ready to choose an item from D_{K_j}. We recall that $D_{l,t}$ is the set of remaining items in D_l at

[4]The $(\frac{1}{2n^p})^j$ terms in the utility functions are only used to avoid ties in allocations. In fact, any utility functions where there are no ties satisfy non-bossiness and category-wise neutrality, but some of them are equivalent to serial dictatorships, which are the cases we want to avoid in our proof.

the beginning of round t. By definition, $k_{j,l} = |D_{l,t_j}|$. Let $(d_{j,1}, \ldots, d_{j,K_j-1}) \in D_1 \times \cdots \times D_{K_j-1}$ denote the items agent j has chosen in previous rounds. It follows that at the beginning of the round t_j, the following $\prod_{l=K_j}^{p} k_{j,l}$ bundles are available for agent j:

$$\mathfrak{D}_j = (d_{j,1}, \ldots, d_{j,K_j-1}) \times \prod_{l=K_j}^{p} D_{l,t_j}$$

We now show that an optimistic agent j is guaranteed to obtain her top-ranked bundle in $\mathfrak{D}_j$. Intuitively this holds because by the definition of K_j, for any $l \geq K_j$, when it is agent j's round to choose an item from D_l, the l-th component of her top-ranked bundle in $\mathfrak{D}_j$ is always available. Formally, let $\vec{d}_j = (d_{j,1}, \ldots, d_{j,p})$ denote agent j's top-ranked bundle in $\mathfrak{D}_j$. We prove that agent j will choose $d_{j,l}$ from D_l in round $\mathcal{O}^{-1}(j,l)$ by induction on l. The base case $l = K_j$ is straightforward. Suppose she has chosen $d_{K_j}, d_{K_j+1}, \ldots, d_{l'}$ for some $l' \geq K_j$. Then in round $\mathcal{O}^{-1}(j, l'+1)$ when agent j is about to choose an item from $D_{l'+1}$, the following bundles are available:

$$(d_{j,1}, \ldots, d_{j,l'}) \times \prod_{l=l'+1}^{p} D_{l,t_j}$$

This is because by the induction hypothesis, $(d_{j,1}, \ldots, d_{j,l'})$ have been chosen by agent j in previous rounds. Then, by the definition of K_j, for any $l \geq l'+1$ no agent choses an item from D_l between round $t_j = \mathcal{O}^{-1}(j, K_j)$ and round $\mathcal{O}^{-1}(j, l')$. Hence the remaining items in D_l is still the same as that in round t_j. This means that $\vec{d}_j \in (d_{j,1}, \ldots, d_{j,l'}) \times \prod_{l=l'+1}^{p} D_{l,t_j} \subseteq \mathfrak{D}_j$. Therefore, $\vec{d}_j$ is still agent j's top-ranked available bundle in the beginning of round $\mathcal{O}^{-1}(j, l')$, when she is about to choose an item from $D_{l'+1}$. Hence agent j will choose $d_{j,l'+1}$. This proves the claim for $l = l'+1$, which means that it holds for all $l \leq p$. Therefore, agent j is allocated $\vec{d}_j$ by the sequential allocation protocol. We note that $|\mathfrak{D}_j| = \prod_{l=K_j}^{p} k_{j,l}$. This proves the proposition for optimistic agents.

We next prove the proposition for an pessimistic agent j. Let $\vec{d}_j = (d_{j,1}, \ldots, d_{j,p})$ denote her allocation by the sequential allocation protocol. Since agent j is pessimistic, for any $1 \leq l \leq p$, in round $t^* = \mathcal{O}^{-1}(j,l)$ agent j chose $d_{j,l}$ from D_{l,t^*}, we must have that for all $d'_l \in D_{l,t^*}$ with $d'_l \neq d_{j,l}$, there exists an bundle $(d_{j,1}, \ldots, d_{j,l-1}, d'_l, \ldots, d'_p)$ that is ranked below $\vec{d}_j$. These bundles are all different and the number of all such bundles is $\sum_{l=1}^{p}(k_{j,l} - 1)$, which proves the proposition for pessimistic agents. ∎

Proof of Theorem 2

Proof: Given $\mathcal{O}$ and the information on whether each agent j is optimistic or pessimistic, we will construct a profile P such that in $\mathcal{O}(P)$, for all $j \leq n$, agent j obtains $(j, \ldots, j)$.

We prove the theorem in the following three steps:

- **Step 1: define bottom bundles**. We specify a set of bundles that are ranked in the bottom positions for each agent j, and require $(j, \ldots, j)$ to be ranked on the top of them.
- **Step 2: define top bundles**. We specify top-1 and sometimes also top-2 bundles for each agent.
- **Step 3: extend to full profile**. We take a profile that extends the partial orders constructed in the first two steps, and then show that it satisfies all three properties in the theorem.

The construction is summarized in Table 3 (for optimistic agents) and Table 4 (for pessimistic agents).

We first introduce some notation that will be useful to define the profile in Step 1 and Step 2. Let $\mathcal{O}(1) = (j_1, i_1)$. That is, agent j_1 is the first to choose an item in the sequential allocation, and she chooses from category D_{i_1}. Let L_{i_1} denote the order over $\{1, \ldots, n\}$ representing the order for the *agents* to choose items from D_{i_1} in $\mathcal{O}$. That is, $j \rhd_{L_{i_1}} j'$ if and only if $(j, i_1) \rhd_{\mathcal{O}} (j', i_1)$. By definition we have $j_1 = L_{i_1}(1)$. For any $j \leq n$, we let $Pred_{i_1}(j) = L_{i_1}(L_{i_1}^{-1}(j) - 1)$ denote the predecessor of agent j in L_{i_1}, that is, the latest agent who chose an item from category i_1 before agent j chooses from category i_1. If $j = 1$, then we let the last agent in L_{i_1} be her predecessor, that is, $Pred_{i_1}(1) = L_{i_1}(n)$.

Step 1: define bottom bundles. In order to match the upper bounds shown in the proof of Proposition 1, the bundles described in the proof of Proposition 1 must be the *only* bundles that are ranked below $(j, \ldots, j)$ by agent j. This is the part of the profile we will construct in the first step.

For all i and t, we first define $D^*_{i,t}$ to be the subset of $D_i = \{1, \ldots, n\}$ such that $q \in D^*_{i,t}$ if and only if agent q has not chosen an item from D_i before the t-th round. By definition, if $\mathcal{O}(t) = (j, i)$ then $j \in D^*_{i,t}$. Formally,

$$D^*_{i,t} = \{q \leq n : \mathcal{O}^{-1}(q, i) \geq t\}$$

We note that $D^*_{i,t}$ is defined solely by i, t, and $\mathcal{O}$, which means that it does not depend on agents' preferences and behavior in previous rounds. Later in this proof we will show that for our constructed profile, in each round (j, i) the active agent j will choose j from D_i, so that $D^*_{i,t}$ is the remaining items for category i at the beginning of round t of the sequential allocation.

For any $1 \leq l \leq p$, we let $t^*_{j,l} = \mathcal{O}^{-1}(j, \mathcal{O}_j(l))$. That is, $t^*_{j,l}$ is the round where agent j chooses an item from the l-th category in $\mathcal{O}_j$, which is not necessarily category l. For each agent j we specify their bottom bundles as follows.

- If agent j is optimistic, then we let the following bundles be ranked in the bottom of her preferences:

$$\text{BottomBundles}_j^{\text{Opt}} = (j_{\mathcal{O}_j(1)}, \ldots, j_{\mathcal{O}_j(K_j-1)}) \times \prod_{l=K_j}^{p} D^*_{\mathcal{O}_j(l), t^*_{j,l}}, \tag{1}$$

  where $(j, \ldots, j)$ is ranked on the top of these bundles, and the order over the remaining bundles is defined arbitrarily. It follows that $(j, \ldots, j)$ is ranked in the $(\prod_{l=K_j}^{p} k_{j,O_j(l)})$-th position from the bottom by agent j.

- If agent j is pessimistic, then we first define the following bundles:

$$\text{BottomBundles}_j^{\text{Pes}} = \bigcup_{l=1}^{p} \bigcup_{d \in D^*_{\mathcal{O}_j(l), t^*_{j,l}}} \{([d]_{\mathcal{O}_j(l)}, [j]_{-\mathcal{O}_j(l)})\}, \tag{2}$$

  where $[j]_{-\mathcal{O}_j(l)}$ means that all components except the $\mathcal{O}_j(l)$-th component is j. Bundles in $\text{BottomBundles}_j^{\text{Pes}}$ are (partially) ranked as follows: first, $(j, \ldots, j)$ is ranked

Optimistic agent		Order
$j \neq j_1$	case 1: $K_j = 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \ldots, j) \succ$ others in $\text{BottomBundles}_j^{\text{Opt}}$
	case 2: $K_j > 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ ([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ $\succ \cdots \succ (j, \ldots, j) \succ$ others in $\text{BottomBundles}_j^{\text{Opt}}$
$j = j_1$	case 1: $K_j = 1$	$(j_1, \ldots, j_1) \succ ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1}) \succ$ others
	case 2: $K_j > 1$	$([Pred_{\mathcal{O}_j(K_j)}(j_1)]_{\mathcal{O}_j(K_j)}, [j_1]_{-\mathcal{O}_j(K_j)}) \succ ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})$ $\succ \cdots \succ (j_1, \ldots, j_1) \succ$ others in $\text{BottomBundles}_{j_1}^{\text{Opt}}$

Table 3: Partial preferences for an optimistic agent j. $\text{BottomBundles}_j^{\text{Opt}}$ is defined in (1). "Others in $\text{BottomBundles}_j^{\text{Opt}}$" refers to $[\text{BottomBundles}_j^{\text{Opt}} \setminus \{(j, \ldots, j)\}]$.

Pessimistic agent	Order
$j \neq j_1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \ldots, j) \succ$ others in $\text{BottomBundles}_j^{\text{Pes}}$
$j = j_1$	$([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1}) \succ \cdots \succ (j_1, \ldots, j_1) \succ$ others in $\text{BottomBundles}_{j_1}^{\text{Pes}}$ $\succ (L_{i_1}(n), \ldots, L_{i_1}(n))$

Table 4: Partial preferences for a pessimistic agent j. $\text{BottomBundles}_j^{\text{Pes}}$ is defined in (2). For $j \neq j_1$, "others in $\text{BottomBundles}_j^{\text{Pes}}$" refers to $(\text{BottomBundles}_{j_1}^{\text{Pes}} \setminus \{(j, \ldots, j)\})$. For $j = j_1$, "others in $\text{BottomBundles}_{j_1}^{\text{Pes}}$" refers to $(\text{BottomBundles}_{j_1}^{\text{Pes}} \setminus \{(j_1, \ldots, j_1), ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})\})$.

on the top; then, for any $1 \leq l_1 < l_2 \leq p$ and any $d_1 \in D^*_{\mathcal{O}_j(l_1), t^*_{j,l_1}}$ and $d_2 \in D^*_{\mathcal{O}_j(l_2), t^*_{j,l_2}}$ with $d_1 \neq j$ and $d_2 \neq j$, we rank $([d_1]_{\mathcal{O}_j(l_1)}, [j]_{-\mathcal{O}_j(l_1)})$ below $([d_2]_{\mathcal{O}_j(l_2)}, [j]_{-\mathcal{O}_j(l_2)})$.

- If $j \neq j_1$, then we simply let $\text{BottomBundles}_j^{\text{Pes}}$ (with the partial orders specified above) be the bundles ranked in the bottom position.
- If $j = j_1$, then we move $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) = ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})$ to the bottom place and replace it by $(Pred_{i_1}(j), \ldots, Pred_{i_1}(j)) = (L_{i_1}(n), \ldots, L_{i_1}(n))$, and then let these be ranked in the bottom positions of agent j's preferences. That is, the bottom bundles are: $(j_1, \ldots, j_1) \succ (\text{BottomBundles}_j^{\text{Pes}} \setminus \{(j_1, \ldots, j_1), ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})\}) \succ (L_{i_1}(n), \ldots, L_{i_1}(n))$

In both cases $(j, \ldots, j)$ is ranked at the $(1 + \prod_{l=K_j}^{p}(k_{j,O_j(l)} - 1))$-th position from the bottom.

Step 2: define top bundles. We now specify the top two bundles (sometimes only the top bundle) for optimistic agents, and show that they are compatible with our constructions in Step 1. For any optimistic agent j:

- When $j \neq j_1$, there are following two cases:
  - case 1: $K_j = 1$. We let $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ be the top-ranked bundle of agent j.
  - case 2: $K_j > 1$. We let $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ be the top-ranked bundle of agent j. Moreover, if $i_1 \neq \mathcal{O}_j(K_j)$, then we rank $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ at the second position. We recall that $Pred_{\mathcal{O}_j(K_j)}(j)$ is the predecessor of j in $L_{\mathcal{O}_j(K_j)}$, the order for the *agents* to choose items from $D_{\mathcal{O}_j(K_j)}$. These do not conflict the preferences specified in Step 1 because item $Pred_{i_1}(j)$ in D_{i_1} is not available for agent j when she is about to choose an item in D_{i_1}, and item $Pred_{\mathcal{O}_j(K_j)}(j)$ in $D_{\mathcal{O}_j(K_j)}$ is not available for agent j when she is about to choose an item in $D_{\mathcal{O}_j(K_j)}$. Hence, none of these bundles are in $\text{BottomBundles}_j^{\text{Opt}}$.
- When $j = j_1$, there are following two cases:
  - case 1: $K_j = 1$. Since $(j_1, i_1) = \mathcal{O}(1)$, for all i, $D^*_{i,\mathcal{O}^{-1}(j,i)} = D_i$, which means that agent j is guaranteed to get her top-ranked bundle after the sequential allocation. In this case we let $(j, \ldots, j)$ be agent j's top-ranked bundle and let $([L_{i_1}(n)]_{i_1}, [j]_{-i_1})$ be ranked in agent j's second position. These do not conflict the preferences specified in Step 1 because in this case Step 1 only specifies that $(j, \ldots, j)$ be ranked in the top position.
  - case 2: $K_j > 1$. We rank $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ at the top position. We then rank $([L_{i_1}(n)]_{i_1}, [j]_{-i_1})$ at the second position. Since $i_1 = \mathcal{O}_j(1)$, we have $\mathcal{O}_j(K_j) \neq i_1$, otherwise $K_j = 1$. Hence, $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \neq ([L_{i_1}(n)]_{i_1}, [j]_{-i_1})$. These do not conflict the preferences specified in Step 1 because category i_1 is agent j_1's first category in $\mathcal{O}_{j_1}$, which means that $i_1 < K_j$, thus $([L_{i_1}(n)]_{i_1}, [j]_{-i_1}) \notin \text{BottomBundles}_{j_1}^{\text{Opt}}$; also $Pred_{\mathcal{O}_j(K_j)}(j)$ is not available when agent j_1 is about to choose an item for category $\mathcal{O}_j(K_j)$, which means that $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \notin \text{BottomBundles}_{j_1}^{\text{Opt}}$.

For any pessimistic agent j, we simply let her top-ranked bundle be $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ (we recall that

$Pred_{i_1}(j_1) = L_{i_1}(n)$). We claim that preferences specified in the second step do not conflict preferences specified in the first step for bottom bundles.

- If $j \neq j_1$, then we need to show that $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \notin \text{BottomBundles}_j^{\text{Pes}}$. When agent j is about to choose her item from D_{i_1}, agent $Pred_{i_1}(j)$ has already chosen her item from D_{i_1}, which means that $Pred_{i_1}(j)$ is unavailable for agent j. This means that $([Pred_{i_1}(j)]_{i_1}, j_{-i_1}) \notin \text{BottomBundles}_j^{\text{Pes}}$.
- If $j = j_1$, then by definition (see Table 4) $([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})$ is replaced by $(L_{i_1}(n), \ldots, L_{i_1}(n))$ in $\text{BottomBundles}_{j_1}^{\text{Pes}}$, which means that it can be ranked in the top.

Step 3: extend to full profile. For any j, let R_j be an arbitrary linear order over $\mathfrak{D}$ that satisfies all constraints defined in the previous two steps (see Table 3 and 4). Let $P = (R_1, \ldots, R_n)$.

We now show by induction on the round in the sequential allocation mechanism, denoted by t, that if we apply the sequential allocation $\mathcal{O}$ to P, then for all $j \leq n$, agent j gets $(j, \ldots, j)$.

When $t = 1$, agent j_1 chooses an item from D_{i_1}. If j_1 is optimistic, then it is not hard to check that the i_1-th component of the top-ranked bundle of R_{j_1} is j_1 (the top-ranked bundles are $(j, \ldots, j)$ and $([j']_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$, for case 1 ($K_{j_1} = 1$) and case 2 ($K_{j_1} > 1$), respectively. If agent j_1 is pessimistic, then for any $d \in D_{i_1}$ with $d \neq j_1$, there exists a bundle whose i_1th component is d and is ranked below any bundle whose i_1th component is j_1. More precisely, if $d \neq Pred_{i_1}(j_1) = L_{i_1}(n)$, then such a bundle is $([d]_{i_1}, [j]_{-i_1})$; if $d = Pred_{i_1}(j_1) = L_{i_1}(n)$, then such an bundle is $(L_{i_1}(n), \ldots, L_{i_1}(n))$. In both cases a pessimistic agent j_1 will choose item j_1 from D_{i_1}.

Suppose in every round before round t, the active agent j chose item j from the designated category. Let $\mathcal{O}(t) = (j, i)$. If j is optimistic, then we show in the following four cases that she will choose item j from D_i in round t.

- $j \neq j_1$, $K_j = 1$. In this case j is guaranteed to get her top-ranked available bundle. It is not hard to check that the available bundles are a subset of $\text{BottomBundles}_j^{\text{Opt}}$, where $(j, \ldots, j)$ is available and is ranked in the top. Therefore agent j will choose item j.
- $j \neq j_1$, $K_j > 1$. There are following cases:
  1. Agent $Pred_{i_1}(j)$ has not chosen her item from D_{i_1}. In this case the top-ranked bundle $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ is still available by the induction hypothesis.
  2. Agent $Pred_{i_1}(j)$ has chosen an item from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$. By the induction hypothesis, agent $Pred_{i_1}(j)$ chose item $Pred_{i_1}(j)$ from category D_{i_1}, which means that $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ is unavailable. The bundle $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ becomes the top-ranked available bundle due to the induction hypothesis, whose j-th component is j.
  3. $Pred_{i_1}(j)$ has chosen item $Pred_{i_1}(j)$ from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has chosen her item from $D_{\mathcal{O}_j(K_j)}$. In this case, we first claim that $\mathcal{O}_j^{-1}(i) \geq K_j$. For the sake of contradiction suppose $\mathcal{O}_j^{-1}(i) < K_j$. Then, by the definition of $Pred_{\mathcal{O}_j(K_j)}$, no agent chooses an item from $D_{\mathcal{O}_j(K_j)}$ between round $\mathcal{O}_j^{-1}(i)$ and t^*_{j,K_j}. We recall that t^*_{j,K_j} is the round when agent j chooses an item from $D_{\mathcal{O}_j(K_j)}$. However, this violates the minimality of K_j since no agent chooses an item from $D_{\mathcal{O}_j(K_j)}$ between round $t^*_{j,K_j-1} > \mathcal{O}_j^{-1}(i)$ and t^*_{j,K_j}. Hence, we must have that $\mathcal{O}_j^{-1}(i) \geq K_j$. By the induction hypothesis, the available bundles are a subset of $\text{BottomBundles}_j^{\text{Opt}}$ and $(j, \ldots, j)$ is still available and is ranked at the top, which means that agent j will choose item j from D_i.

  In all three cases above, the ith component of the top-ranked available bundle is j, which means that agent j will choose item j.
- $j = j_1$, $K_j = 1$. By the induction hypothesis, the top-ranked bundle $(j, \ldots, j)$ is still available, which means that agent j will choose item j.
- $j = j_1$, $K_j > 1$. If agent $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$, then by the induction hypothesis the top bundle $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ is still available and $i \neq \mathcal{O}_j(K_j)$. If agent $Pred_{\mathcal{O}_j(K_j)}(j)$ has chosen item $Pred_{\mathcal{O}_j(K_j)}(j)$ from $D_{\mathcal{O}_j(K_j)}$, then by the induction hypothesis the available bundles are a subset of $\text{BottomBundles}_j^{\text{Opt}}$ with $(j, \ldots, j)$ ranked at the top. In both cases the ith component of the top-ranked available bundle is j. Therefore agent j will choose item j.

If agent j is pessimistic, then by the induction hypothesis the available items in D_i are $D^*_{i,t}$, and $j \in D^*_{i,t}$. For any $d \in D^*_{i,t}$ with $d \neq j$, $([d]_i, [j]_{-i})$ is still available and is ranked lower than any *available* bundle whose i-th component is j in $\text{BottomBundles}_j^{\text{Pes}}$. Therefore, a pessimistic agent j will choose item j in this round.

It follows that after the sequential allocation, for all $j \leq n$, agent j gets $(j, \ldots, j)$. It is not hard to verify that condition 1 and 2 hold.

To show that condition 3 holds, consider the allocation where agent j gets $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$. In this allocation, all agents except j_1 get their top-ranked bundle, and j_1 gets her top-ranked bundle (if j_1 is pessimistic) or second-ranked bundle (if j_1 is optimistic). This proves the theorem. ■

Proofs of Proposition 2, 3, 4

Proposition 2. *Among all categorical sequential mechanisms, serial dictatorships with all-optimistic agents have the best (smallest) worst-case utilitarian rank and the worst (largest) worst-case egalitarian rank.*

Proof: The worst-case egalitarian rank of any serial dictatorship is n^p when all agents have the same preferences. To prove the optimality of worst-case utilitarian rank, given $f_{\mathcal{O}}$, we consider the multiset composed of the numbers of items in the designated category that the active agent can choose in each step. That is, we consider the multiset $\text{RI} = \{k_{j,l} : \forall j \leq n, l \leq p\}$. Since in each step in the execution of $f_{\mathcal{O}}$, only one item is allocated, RI is composed of p copies of $\{1, \ldots, n\}$. Since for each agent j, $(1+\sum_{l=1}^{p}(k_{j,O_j(l)}-1)) \leq \prod_{l=1}^{p} k_{j,O_j(l)}$ (we note that in the right hand side, l starts with 1 but not K_j), the best worst-case utilitarian rank is at least $n(n^p+1) - \sum_{j=1}^{n} \prod_{l=1}^{p} k_{j,O_j(l)} \geq n(n^p+1) - \sum_{j=1}^{n} j^p$. It is not hard to verify that this lower bound is achieved by any serial dictatorship with all-optimistic agents. ■

Proposition 3. *Any categorical sequential mechanisms with all-optimistic agents has the worst (largest) worst-case egalitarian rank, which is n^p.*

Proof: By Theorem 2, the proposition is equivalent to the existence of an agent j such that for all $l \geq K_j$, $k_{j,\mathcal{O}_j(l)} = 1$. For the sake of contradiction, let us assume the following condition:

Condition (*): for every agent j, there exists $l \geq K_j$ such that $k_{j,\mathcal{O}_j(l)} > 1$.

Let $\mathcal{O}(np) = (j_n, i_p)$. It follows that $k_{j_n,i_p} = 1$ because there is only one item left. By condition (*), there exists i_{p-1} with $K_n \leq \mathcal{O}_{j_n}^{-1}(i_{p-1})$ such that $k_{j_n,i_{p-1}} > 1$. Let j_{n-1} denote the agent who is the last to chose an item from category i_{p-1}. We have $j_{n-1} \neq j_n$, because agent n is not the last agent to choose an item from category i_{p-1}. By definition, we have $k_{j_{n-1},i_{p-1}} = 1$. Moreover, $(j_{n-1}, \mathcal{O}_{j_{n-1}}(K_{j_{n-1}})) \rhd_{\mathcal{O}} (j_n, i_{p-1}) \rhd_{\mathcal{O}} (j_n, \mathcal{O}_{j_n}(K_{j_n}))$, which simply states that j_{n-1} chooses an item from category $\mathcal{O}_{j_{n-1}}(K_{j_{n-1}})$ after j_n chooses an item from category i_{p-1} (the second half of the inequality is due to the way we choose i_{p-1}). This inequality holds because if agent j_{n-1} chooses an item from category $\mathcal{O}_{j_{n-1}}(K_{j_{n-1}})$ before agent j_n chooses an item from category i_{p-1}, then agent j_n "interrupts" agent j_{n-1} from choosing an item from category i_{p-1}, which contradicts the definition of $K_{j_{n-1}}$.

By condition (*), there exists i_{p-2} such that $K_{j_{n-1}} \leq \mathcal{O}_{j_{n-1}}^{-1}(i_{p-2})$ and $k_{j_{n-1},i_{p-2}} > 1$. Similarly, we can define j_{n-2}, prove that $j_{n-2} \neq j_{n-1}$ and $(j_{n-2}, \mathcal{O}_{j_{n-2}}(K_{j_{n-2}})) \rhd_{\mathcal{O}} (j_{n-1}, i_{p-2}) \rhd_{\mathcal{O}} (j_{n-1}, \mathcal{O}_{j_{n-1}}(K_{j_{n-1}}))$.

However, this process cannot continue forever, since otherwise we will obtain an infinite sequence in $\mathcal{O}$: $(j_n, \mathcal{O}_{j_n}(K_{j_n})) \lhd_{\mathcal{O}} (j_{n-1}, \mathcal{O}_{j_{n-1}}(K_{j_{n-1}})) \lhd_{\mathcal{O}} (j_{n-2}, \mathcal{O}_{j_{n-2}}(K_{j_{n-2}})) \lhd_{\mathcal{O}} \cdots$, but np is finite. This leads to a contradiction. ■

Proposition 4. *For any even number p, the worst-case egalitarian rank of any balanced CSAM with all-pessimistic agents (see Example 2) is $n^p - (n-1)p/2$. These are the mechanisms with the best worst-case egalitarian rank among categorical sequential mechanisms with all-pessimistic agents.*

Proof: For any balanced CSAM, it is not hard to see that for any agent j, $\mathcal{O}_j = 1 \rhd \cdots \rhd p$. For any $l < p/2$ and any $j \leq n$, we have $k_{j,2l-1} + k_{j,2l} = n + 1$. Since all agents are pessimistic, by part 2 of Theorem 2, their worst-case ranks are all equal to $n^p - (n-1)p/2$. The optimality of balanced CSAMs come from the fact that for any categorical sequential mechanisms $\sum_{j,l} k_{j,l} = (n+1)np/2$. Therefore, there must exists an agent j^* with $\sum_{l=1}^{p} k_{j,l} \leq (n+1)p/2$. ■

Proposition 5. *For any even number p with $2^p > 1 + (n-1)p/2$, there exists a categorical sequential mechanism with both optimistic and pessimistic agents, whose worst-case egalitarian rank is strictly better (smaller) than $n^p - (n-1)p/2$.*

Proof: We prove the proposition by explicitly constructing such a mechanism. The idea is, agents $\{1, \ldots, n-1\}$ choose the items as in a balanced CSAM for $n-1$ agents, then we let agent n "interrupt" them and choose all items in consecutive p rounds right before their last iteration, i.e. the last $(n-1)$ round. Then, we let agents 1 through $n-1$ be optimistic and let agent n be pessimistic. For example, when $n = 3$ and $p = 4$, the order is $(1,1) \rhd (2,1) \rhd (2,2) \rhd (1,2) \rhd (1,3) \rhd (2,3) \rhd (3,1) \rhd (3,2) \rhd (3,3) \rhd (3,4) \rhd (2,4) \rhd (1,4)$. Agent 1 and agent 2 are optimistic and agent 3 is pessimistic.

By part 2 of Theorem 2, for any agent $j \leq n-1$, the worst-case rank is $n^p + 1 - (1 + np/2)$. By part 1 of Theorem 2, the worst-case rank for agent n is $n^p + 1 - 2^p$. This proves the proposition. ■

Appendix B: Simulation Results

In this section, we use computer simulations to evaluate *expected* efficiency of categorical sequential mechanisms, when agents' preferences are generated i.i.d. from a well-known statistical model called the *Mallows model* [20]. Similarly to the worst-case analysis in the previous section, we evaluate two types of expected ranks: the expected utilitarian rank and the expected egalitarian rank. We first recall the definition of the Mallows model.

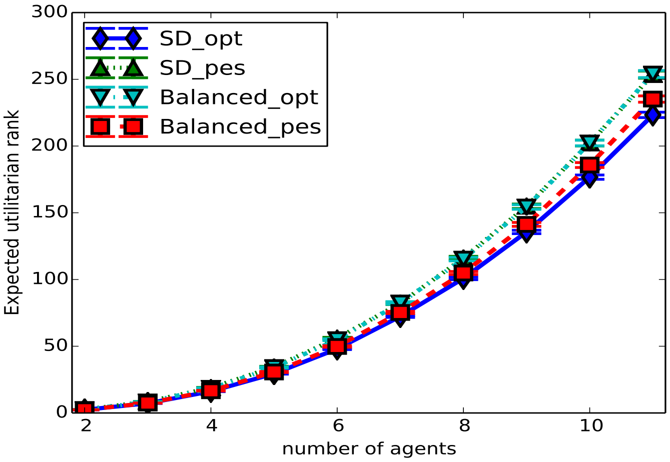
Definition 3 *Let $\mathcal{C}$ denote a set of alternatives and let $\mathcal{L}(\mathcal{C})$ denote the set of all linear orders over $\mathcal{C}$. In a* Mallows model*, each parameter consists of a ground truth linear order W over $\mathcal{C}$ and a* dispersion parameter $0 < \varphi \leq 1$. *Given (W, φ), the probability to generate a linear order V over $\mathcal{C}$ is $\Pr(V|W, \varphi) = \frac{1}{Z} \cdot \varphi^{Kendall(V,W)}$, where $Kendall(V, W)$ is the* Kendall-tau distance *between V and W, defined to be the number of different pairwise comparisons between alternatives. $Z = \sum_{V \in \mathcal{L}(\mathcal{C})} \varphi^{Kendall(V,W)}$ is the normalization factor.*

In the Mallows model, the dispersion parameter measures the centrality of the generated linear orders. The smaller φ is, the more centralized the randomly generated linear orders are (around the ground truth linear order). When $\varphi = 1$, the Mallows model degenerates to the uniform distribution for any ground truth linear order W.

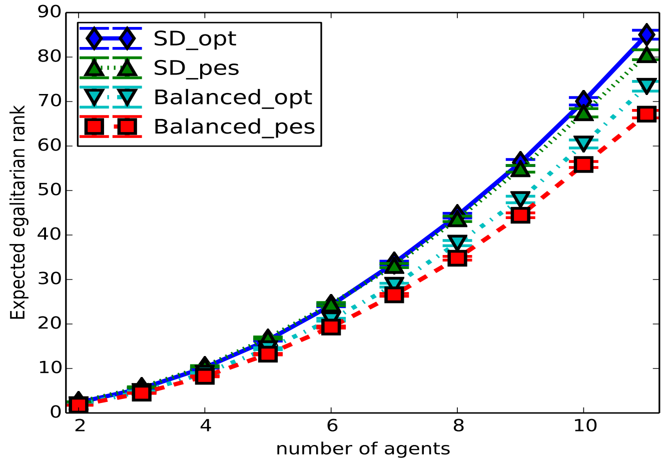
Data generation. In our experiments, we fix $p = 2$, let n range from 2 to 11, and let φ be 0.1, 0.5, and 1. For each setting, we first randomly generate a linear order W over $\mathfrak{D}$, and then use it as the ground truth linear order in the Mallows model to generate n agents' preferences. For each setting we generate 2000 datasets and use them to approximately compute the expected utilitarian rank and the expected egalitarian rank, defined by replacing $\max_{P_n}$ by E_{P_n} in Definition 2.[5] We evaluate serial dictatorships and balanced CSAMs with two configurations of agents: all-optimistic agents and all-pessimistic agents. All computations were done on a 1.8 GHz Intel Core i7 laptop with 4GB memory.

Results. Our results are summarized in Figure 1, 2, and 3. In each figure we also plot 95% confidence intervals. It can be seen from the figures that in general, serial dictatorships with all-optimistic agents have the best (smallest) expected utilitarian rank, and balanced CSAMs with all-pessimistic agents have the best (smallest) expected egalitarian rank. All these comparisons are statistically significant at the 0.05 level, except for the case of expected egalitarian rank when $\varphi = 1$ (namely, the uniform distribution), where the performance of serial dictatorships with all-optimistic agents and the performance of the balanced CSAMs with all-pessimistic agents are too close to draw informative statistical conclusions. These observations complement and are (incidentally) consistent with the worst-case results obtained in the previous section, which tell us that among the four types of mechanisms, serial dictatorships with all-optimistic agents have the best *worst-case* utilitarian rank, and the balanced

[5] The expected egalitarian rank should be distinguished from the *egalitarian expected rank*, which first computes the expected rank for every agent, then chooses the largest (expected) rank.

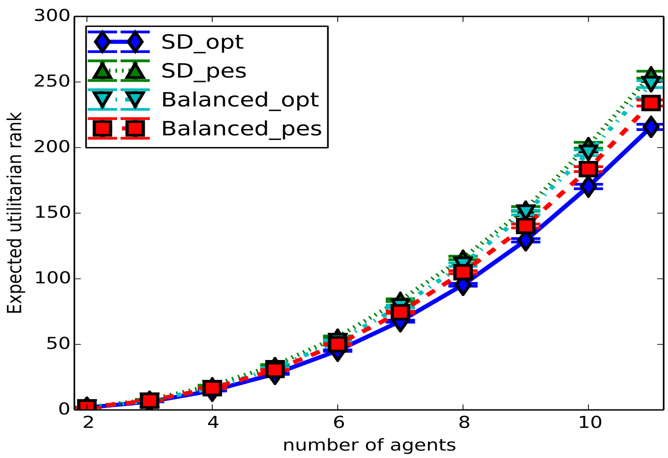


Expected utilitarian rank.

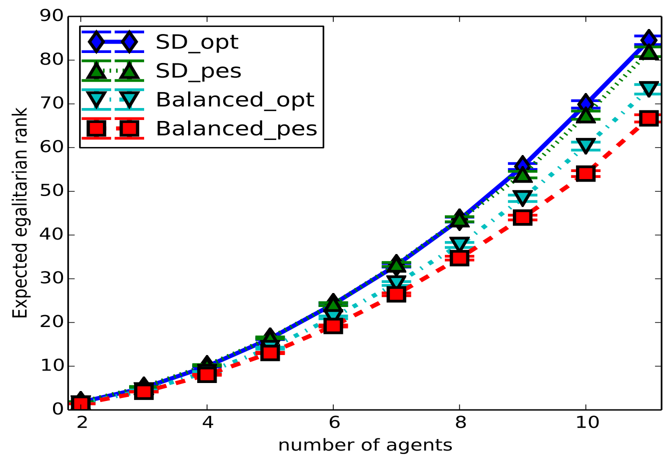


Expected egalitarian rank.

Figure 1: The data are generated from the Mallows model with $\varphi = 0.1$.

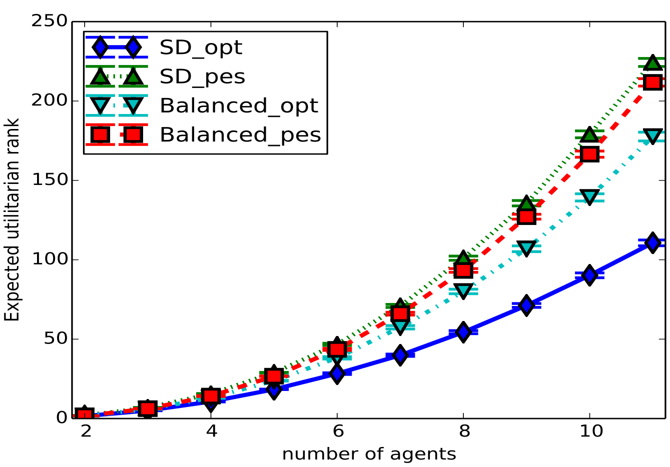


Expected utilitarian rank.

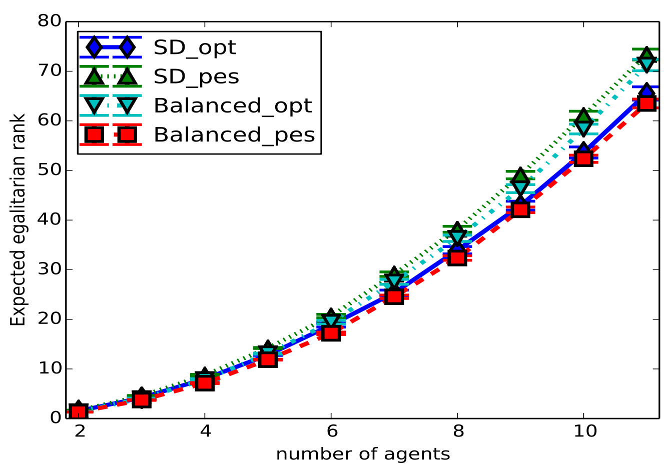


Expected egalitarian rank.

Figure 2: The data are generated from the Mallows model with $\varphi = 0.5$.



Expected utilitarian rank.



Expected egalitarian rank.

Figure 3: The data are generated from the uniform distribution (the Mallows model with $\varphi = 1$).

CSAMs with all-pessimistic agents have the best *worst-case* egalitarian rank.