EQUILIBRIUM AND STABILITY OF TWO-PHASE DEFORMATIONS WITHIN THE FRAMEWORK OF PHASE TRANSITION ZONES

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Summary Stress-induced phase transitions in nonlinear elastic materials are analyzed within the framework of phase transition zones. A procedure to examine the stability of piece-wise homogeneous two-phase deformation is developed. Spherically symmetric two-phase deformation are studied in detail to demonstrate efficiency of the approaches developed.

INTRODUCTION

If phase transitions take place in some parts of a deformable body, interfaces which bound new phase areas can be considered as surfaces across which the deformation gradient $\mathbf{F}$ is discontinuous at continuous displacements. Phase boundaries of this kind appear in the case of martensite transformations. Related phenomena are shape memory effects, localized orientation transformations in polymers (crazes, shear bands). On the whole, studies in this field direct toward the development of the theoretical basis for using and creating smart materials and structures.

The analysis of the conditions on the equilibrium interface [1, 2, 3] leads to the concept of phase transition zones (PTZ)[4]. The PTZ is formed in strain-space by all deformations which can exist on the equilibrium interface. Since the equilibrium two-phase deformations can be supported by a nonlinear elastic material only if ellipticity fails at some deformations [5], the PTZ is crossed by the non-ellipticity sub-zone. The PTZ boundary acts as a phase diagram or yield surface in strain-space. With the aid of the PTZ construction, what kind of phase boundaries will appear and when they can appear are determined entirely by properties of the strain energy function and the deformation path.

Since the PTZ results from the analysis of the local equilibrium conditions, every point of the PTZ corresponds to some equilibrium piecewise-homogeneous two-phase deformation. That is why we develop a procedure to examine the stability of such deformations. The other important reason follows from the fact that if a two-phase deformation in a non-linear elastic body is a local energy minimizer, then given any point $\mathbf{p}_0$ of the interface, the piece-wise homogeneous deformation corresponding to the two values $\mathbf{F}^{\pm}(\mathbf{p}_0)$ is a global energy minimizer [3]. Thus, instability of the latter state would imply instability of the former state.

In this paper the stability of the piece-wise homogeneous deformation is investigated with the aid of two test criteria. One is a kinetic stability criterion developed for the case of small strains in [7]: the two-phase state is unstable if superimposed perturbations grow. Our analysis draws upon recent results obtained in [8] for the half-space problem. The other criterium is the energy criterion. We compare the total energies corresponding to the two-phase piece-wise homogeneous deformation and to the perturbed inhomogeneous deformation.

Then we study some aspects of boundary value problems for bodies undergoing phase transformations. Usually the equilibrium two-phase solution is non-unique and can be meta-stable or unstable, and global minimizers of the total energy are not the only ones that are of interest from the physical point of view. Locally stable solutions can also be met in physical reality. Then the choice of the solution in a static approach can be made on the base of the stability analysis and estimations of energy changes due to phase transformations.

To discuss the problems we consider the equilibrium spherically symmetric two-phase deformation of a nonlinear elastic isotropic material. We develop a general procedure to construct the solution for a material with an arbitrary strain energy function. Then we study phase transformations in a sphere made of the Hadamard material. We describe characteristic features of the deformation fields in the equilibrium two-phase sphere for the obtained solutions in the context of the PTZ. We show that the number of solutions can be predicted if the PTZ is constructed. Then we examine the stability. Finally, we compare these results with the results obtained earlier by a small strain approach [7].

CONDITIONS ON THE EQUILIBRIUM PHASE BOUNDARY AND PHASE TRANSITION ZONES

The conditions on the equilibrium interface include the kinematic condition, traction continuity condition, and thermodynamic condition [1, 2, 3]. The last two conditions can be rewritten as

$$(\mathbf{S}(\mathbf{F}+\mathbf{f}\otimes\mathbf{m})-\mathbf{S}(\mathbf{F}))\mathbf{m}=0, \quad W(\mathbf{F}+\mathbf{f}\otimes\mathbf{m})-W(\mathbf{F})=\mathbf{f}\cdot\mathbf{S}(\mathbf{F})\mathbf{m} \tag{1}$$

where $\mathbf{F}$ is the deformation gradient on one side of the interface, $\mathbf{S}$ is the Piola stress tensor, W is the strain energy function, $\mathbf{m}$ is a unit normal to the interface. Given $\mathbf{F}$, the above equations can be considered as a system of four equations for five unknowns: $\mathbf{f}$ and $\mathbf{m}$. Those $\mathbf{F}$ for which this system of equations can be solved form the *phase transition zone* [4]. In the case of isotropic materials the PTZ can be constructed in the space of principal stretches $\lambda_i,\ i=1,2,3$ [4, 6].

If the interface is perturbed and the thermodynamic equilibrium fails then in a quasi-static approach we replace $(1)_2$ by

$$\frac{\partial\Gamma}{\partial t}=-k([W]-\mathbf{f}\cdot\mathbf{S}\widetilde{\mathbf{m}}), \quad k>0, \tag{2}$$

where $\partial\Gamma/\partial t$ is the normal speed of the interface, and $\widetilde{\mathbf{m}}$ is a normal to the perturbed interface. The evolution equation (2) is motivated by the fact that the rate of energy dissipated as the interface traverses the material must be positive [9].

STABILITY AND BIFURCATION OF PIECEWISE-HOMOGENEOUS DEFORMATIONS

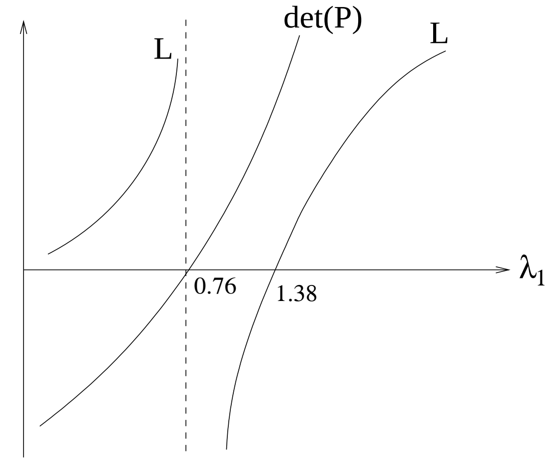

Figure 1.

The interface is given in the reference configuration by a parametric form $\mathbf{X} = \mathbf{Y}(s)$ $(-\infty < s < \infty)$. The perturbed interface is given by $\mathbf{X} = \mathbf{Y}(s) + \boldsymbol{\xi}(s,t)$. In a normal mode approach the perturbation is of the form

$$\Gamma(s,t) = \gamma e^{i\mathbf{K}\cdot\mathbf{X}} = \gamma(t)e^{is} \quad (\Gamma \equiv \boldsymbol{\xi}\cdot\mathbf{m},\ \mathbf{K}\perp\mathbf{m},\ |\boldsymbol{\xi}| \ll 1). \quad (3)$$

The incremental displacement is sinusoidal along the interface and decays away from the interface. Linearizing the right hand side of (2), we obtain a linear differential equation of the form

$$-\frac{1}{k}\frac{d\gamma}{dt} = L\gamma, \quad (4)$$

By the kinetic stability criterion, the two-phase deformation is unstable if $L < 0$. Analogously to [8], linearizing the traction continuity condition we obtain the expression for the interfacial impedance tensor P. As a result it is found that two types of bifurcations are possible, with bifurcation conditions given by $\det P = 0$ and $\det P \neq 0$ but $L = 0$. It becomes clear the necessary and sufficient conditions for the energy increment to be positive definite: (i) P is positive definite, and (ii) $L > 0$. Fig.1 shows the signs of $\det P$ and L in the case of the specified Hadamard material. The normal to the primary interface is directed as the second eigenvalue of the strain tensor. Then λ_1 is continuous across the interface. The stability conditions are satisfied only for $\lambda_1 > 1.38$, which is only a subset of the λ_1 values (namely $\lambda_1 > 0.76$) over which the pairwise homogeneous deformation is stable when viewed as joint-body problem.

EQUILIBRIUM SPHERICALLY-SYMMETRIC TWO-PHASE DEFORMATIONS

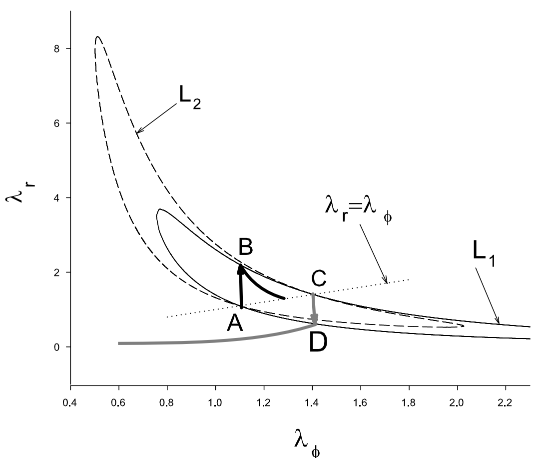

Figure 2.

Applying the general procedure to the sphere made of the Hadamard material we demonstrate that two different equilibrium two-phase states as well as a uniform one-phase state can be found. Fig. 2 shows the PTZ cross-section by the plane $\lambda_1 = \lambda_2 \equiv \lambda_\Phi$. The lines L_1 and L_2 are the PTZ-boundaries. The points A, B and C, D represent the jumps in deformations across the spherical interfaces for two solutions. The thick lines characterize strains along the radius of the sphere. We find the radius of the interface depending on the predicted external radius of the sphere, construct pressure–volume diagrams, study energy changes and analyze how the stable and unstable solutions in the cases of the Hadamard material and small strains relate with the PTZ.

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