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A New Study on Reliability-Based Design Optimization

This paper presents a general approach for probabilistic constraint evaluation in the reliability-based design optimization (RBDO). Different perspectives of the general approach are consistent in prescribing the probabilistic constraint, where the conventional reliability index approach (RIA) and the proposed performance measure approach (PMA) are identified as two special cases. PMA is shown to be inherently robust and more efficient in evaluating inactive probabilistic constraints, while RIA is more efficient for violated probabilistic constraints. Moreover, RBDO often yields a higher rate of convergence by using PMA, while RIA yields singularity in some cases.

1 Introduction

In engineering design, the traditional deterministic design optimization model (Arora, 1989; Haftka and Gurdal, 1991) has been successfully applied to systematically reduce the cost and improve quality. However, the existence of uncertainties in either engineering simulations or manufacturing processes calls for a reliability-based design optimization (RBDO) model for robust and cost-effective designs.

In the RBDO model for robust system parameter design, mean values of random system parameters are usually used as design variables, and the cost is optimized subject to prescribed probabilistic constraints by solving a mathematical nonlinear programming problem. Therefore, the solution from RBDO provides not only an improved design but also a higher level of confidence in the design.

To date, almost all researchers on RBDO (Enevoldsen, 1994; Enevoldsen and Sorensen, 1994; Chandu and Grandi, 1995; Frangopol and Corotis, 1996; Choi et al., 1996; Yu et al., 1997, 1998; Wu and Wang, 1996; Grandhi and Wang, 1998) have used the reliability index evaluated in the traditional reliability analysis to prescribe the probabilistic constraint. In this paper, the probabilistic constraint evaluation in RBDO is studied from a broader perspective. It is shown that the target probabilistic performance measure of the proposed performance measure approach (PMA) evaluated in an inverse reliability analysis is consistent with the conventional reliability index approach (RIA) in prescribing the probabilistic constraint for RBDO. Moreover, it is illustrated that the probabilistic constraint can be effectively evaluated from different perspectives in a general approach where RIA and PMA are two special cases.

Thus, different perspectives of the general approach are consistent in prescribing the probabilistic constraint because any of them can sufficiently identify the exact status of the probabilistic constraint. However, they are not equivalent in solving the RBDO problem. It is shown in this paper that PMA is inherently robust for RBDO and is more efficient in evaluating the inactive probabilistic constraint. In contrast, RIA may yield singularity in many RBDO applications though it is more efficient in evaluating the violated probabilistic constraint. Thus, efficiency and robustness in solving RBDO problems can be achieved by using PMA and RIA adaptively depending on the estimated marginal status of the probabilistic constraint in the RBDO iterations.

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2 Probability Analysis of the System Performance Function

The uncertainties of an engineering system are identified by the variations of the random system parameter $\mathbf{X} = [X_i]^T$ $(i = 1, 2, \ldots, n)$. The probability distribution of X_i is described by its cumulative distribution function (CDF) $F_{X_i}(x_i)$ or probability density function (PDF) $f_{X_i}(x_i)$, and is often bounded by the tolerance limits of the system parameter (Dai and Wang, 1992; Ayyub and McCuen, 1997).

The system performance criteria are described by system performance functions. Consider a system performance function $G(\mathbf{x})$, where the system fails if $G(\mathbf{x}) < 0$. The statistic description of $G(\mathbf{x})$ is characterized by its CDF $F_G(g)$ as

$$F_G(g) = P(G(\mathbf{x}) < g) = \int_{G(\mathbf{x})<g} \cdots \int f_x(\mathbf{x}) dx_1 \ldots dx_n, \quad \mathbf{x}^L \le \mathbf{x} \le \mathbf{x}^U \tag{1}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (JPDF) of all random system parameters and g is named the probabilistic performance measure. The probability analysis of the system performance function is to evaluate the non-decreasing $F_G(g) \sim g$ relationship, which is performed in the probability integration domain bounded by the system parameter tolerance limits given in Eq. (1).

A generalized probability index β_G, which is a non-increasing function of g, is introduced (Madsen et al., 1986) as

$$F_G(g) = \Phi(-\beta_G) \tag{2}$$

which can be expressed in two ways using the following inverse transformations (Rubinstein, 1981), respectively, as

$$\beta_G(g) = -\Phi^{-1}(F_G(g)) \tag{3a}$$

$$g(\beta_G) = F_G^{-1}(\Phi(-\beta_G)) \tag{3b}$$

Thus, the non-increasing $\beta_G \sim g$ relationship represents a one-to-one mapping of $F_G(g) \sim g$ and also completely describes the probability distribution of the performance function. Since the system performance is often non-normal distribution, the $\beta_G \sim g$ relationship is generally nonlinear.

3 General Definition of the RBDO Model

In the robust system parameter design, the RBDO model (Enevoldsen and Sorensen, 1994; Chandu and Grandi, 1995; Choi et al., 1996; Wu and Wang, 1996; Yu et al., 1997, 1998; Grandhi and Wang, 1998) can generally be defined as

$$\text{minimize} \quad \text{Cost}(\mathbf{d}) \tag{4a}$$

subject to $P_{fj} = P(G_j(\mathbf{x}) < 0) \le \bar{P}_{fj}$,

$$j = 1, 2, \ldots, np \quad (4b)$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \quad (4c)$$

where the cost can be any function of the design variable $\mathbf{d} = [d_i]^T \equiv [\mu_i]^T$ $(i = 1, 2, \ldots, n)$, and each prescribed failure probability limit $\bar{P}_f$ is often represented by the reliability target index as $\beta_t = -\Phi^{-1}(\bar{P}_f)$. Hence, any probabilistic constraint in Eq. (4b) can be rewritten using Eq. (1) as

$$F_G(0) \le \Phi(-\beta_t) \quad (5)$$

which can also be expressed in two ways through inverse transformations as

$$\beta_s = -\Phi^{-1}(F_G(0)) \ge \beta_t \quad (6a)$$

$$g^* = F_G^{-1}(\Phi(-\beta_t)) \ge 0 \quad (6b)$$

where β_s is traditionally called the reliability index and g^* is named the target probabilistic performance measure in this paper.

To date, most researchers have used the reliability index approach (RIA) of Eq. (6a) to directly prescribe the probabilistic constraint as

$$\beta_s(\mathbf{d}) \ge \beta_t \quad (7a)$$

At a given design $\mathbf{d}^k = [d_i^k]^T \equiv [\mu_i^k]^T$, the evaluation of reliability index $\beta_s(\mathbf{d}^k)$ for RIA is performed using the well-developed reliability analysis (Madsen et al., 1986) as

$$\beta_s(\mathbf{d}^k) = -\Phi^{-1}\left(\int_{G(\mathbf{x})<0} \cdots \int f_X(\mathbf{x}) dx_1 \ldots dx_n\right),$$

$$x_i^L \le x_i \le x_i^U \quad (7b)$$

It is clear that Eq. (6b) can also be used to prescribe the probabilistic constraint and it is called the performance measure approach (PMA) as

$$g^*(\mathbf{d}) \ge 0 \quad (8a)$$

and the evaluation of target probabilistic performance measure $g^*(\mathbf{d}^k)$ in PMA is called the inverse reliability analysis (Tu and Choi, 1997) as

$$g^*(\mathbf{d}^k) = F_G^{-1}\left(\int_{G(\mathbf{x})<g^*} \cdots \int f_X(\mathbf{x}) dx_1 \ldots dx_n\right),$$

$$x_i^L \le x_i \le x_i^U \quad (8)$$

4 Broader Perspective of the Probabilistic Constraint Evaluation

RIA and PMA are directly derived from the general definition of the probabilistic constraint and they are consistent in prescribing the probabilistic constraint in RBDO. In fact, the probabilistic constraint can be understood from an even broader perspective, where RIA and PMA are just two special cases.

4.1 Example. Consider a system described by two independent, uniformly distributed random system parameters, $X_i \sim$ Uniform$[a_i, b_i]$ $(i = 1, 2)$, and their PDFs are expressed as

$$f_{X_i}(x_i) = 1/(b_i - a_i), \quad a_i \le x_i \le b_i, \quad i = 1, 2 \quad (9a)$$

where the mean values and variances of system parameters are expressed, respectively, as

$$\mu_i = \int_{a_i}^{b_i} x_i f_{X_i}(x_i) dx_i = (a_1 - b_1)/2, \quad i = 1, 2 \quad (9b)$$

$$\sigma_i^2 = \int_{a_i}^{b_i} (x_i - \mu_i)^2 f_{X_i}(x_i) dx_i = (b_i - a_i)^2/12, \quad i = 1, 2 \quad (9c)$$

In the system parameter design, both μ_1 and μ_2 are chosen as design variables, $\mathbf{d} = [d_1, d_2]^T \equiv [\mu_1, \mu_2]^T$, and their variances are constants as $\sigma_1^2 = \sigma_2^2 = \frac{1}{3}$. Thus, the PDFs of system parameters can be expressed in terms of design variables as

$$f_{X_i}(x_i) = \tfrac{1}{2}, \quad d_i - 1 \le x_i \le d_i + 1, \quad i = 1, 2 \quad (9d)$$

Since X_1 and X_2 are mutually independent, their JPDF can be explicitly expressed as

$$f_X(\mathbf{x}) = f_{X_1}(x_1) f_{X_2}(x_2) = \tfrac{1}{4}, \quad d_i - 1 \le x_i \le d_i + 1, \quad i = 1, 2 \quad (9e)$$

Consider a probabilistic constraint in the RBDO model that is defined as

$$P(G(\mathbf{x}) < 0) \le \bar{P}_f = 2.275\% = \Phi(-\beta_t) \quad (9f)$$

where $\beta_t = -\Phi^{-1}(0.02275) = 2$. The system performance function $G(\mathbf{x})$ and its CDF are

$$G(\mathbf{x}) = x_1 + 2x_2 - 10 \quad (9g)$$

$$F_G(g) = \frac{1}{4} \int_{G(\mathbf{x})<g} \cdots \int dx_1 \ldots dx_n,$$

$$d_i - 1 \le x_i \le d_i + 1, \quad i = 1, 2 \quad (9h)$$

At three different designs, $\mathbf{d}^1 = [3.7, 3.7]^T$, $\mathbf{d}^2 = [4.2, 4.2]^T$, and $\mathbf{d}^3 = [4.5, 4.5]^T$, the $F_G(g) \sim g$ relationship can be obtained by performing the probability integration in Eq. (9h) repeatedly

Nomenclature

- $\mathbf{X}$ = Random system parameter; $\mathbf{X} = [X_i]^T$ $(i = 1, 2, \ldots, n)$
- $\mathbf{x}$ = Outcomes of the random system parameter; $\mathbf{x} = [x_i]^T$ $(i = 1, 2, \ldots, n)$
- $\mathbf{x}^L$, $\mathbf{x}^U$ = Lower and upper tolerance limits of the system parameter; $\mathbf{x}^L \le \mathbf{x} \le \mathbf{x}^U$
- $\Phi(\cdot)$ = Standard normal cumulative distribution function (CDF)
- $G(\mathbf{x})$ = System performance function; system fails if $G(\mathbf{x}) < 0$
- $F_G(g)$ = CDF of the system performance function $G(\mathbf{x})$; $F_G(g) = P(G(\mathbf{x}) < g)$
- np = Total number of the probabilistic constraints in the RBDO model
- P_f = Failure probability; $P_f = F_G(0) = P(G(\mathbf{x}) < 0)$
- $\bar{P}_f$ = Prescribed failure probability limit
- β_s = Reliability index; $\beta_s = -\Phi^{-1}(P_f) = -\Phi^{-1}(F_G(0))$
- β_t = Reliability target index; $\beta_t = -\Phi^{-1}(\bar{P}_f)$
- g^* = Target probabilistic performance measure; $P(G(\mathbf{x}) < g^*) = \bar{P}_f$
- $\mathbf{u}^*_{g=0}$ = MPP of RIA corresponding to $G(\mathbf{u}) = 0$ in the $\mathbf{u}$-space; $\beta_s = \|\mathbf{u}^*_{g=0}\|$
- $\mathbf{u}^*_{\beta=\beta_t}$ = MPP of PMA corresponding to $G(\mathbf{u}) = g^*$ in the $\mathbf{u}$-space; $g^* = G(\mathbf{u}^*_{\beta=\beta_t})$

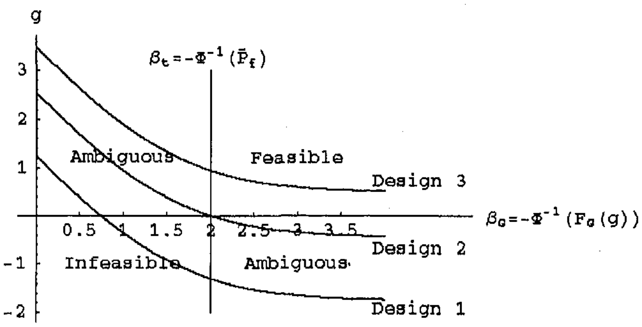

Fig. 1 General Interpretation of Probabilistic Constraint

with different values of g. Then, the corresponding non-increasing $\beta_G \sim g$ curves are obtained using Eq. (3a), which are illustrated in Fig. 1.

4.2 General Interpretation of the Probabilistic Constraint. Note that the comprehensive definition of the probabilistic constraint in Eq. (4b) includes two inequality relations. Conceptually, any probabilistic constraint in Eq. (4b) (or Eq. (5)) can be represented by a set of three simple constraints, where two inequality constraints are related to each other through an equality constraint by Eq. (2), as

$$\beta_G \geq \beta_t \quad (10a)$$

$$g \geq 0 \quad (10b)$$

$$F_G(g) = \Phi(-\beta_G) \quad (10c)$$

The limit-state of Eq. (10a) is represented in Fig. 1 by the vertical line at $\beta_t = -\Phi^{-1}(\bar{P}_f)$, the limit-state of Eq. (10b) is represented by the β_G-axis, and the limit-state of Eq. (10c) is represented by the non-increasing $\beta_G \sim g$ curve. Thus, the $\beta_G - g$ space is naturally divided into four regions as

$$\text{Active Point:} \quad \beta_G = \beta_t \quad \text{and} \quad g = 0 \quad (11a)$$

$$\text{Infeasible Region:} \quad \beta_G \leq \beta_t \quad \text{and} \quad g \leq 0 \quad (11b)$$

$$\text{Feasible Region:} \quad \beta_G \geq \beta_t \quad \text{and} \quad g \geq 0 \quad (11c)$$

$$\text{Ambiguous Regions:} \quad (\beta_G - \beta_t) \cdot g < 0 \quad (11d)$$

The probabilistic constraint is violated for design $\mathbf{d}^1$ as the corresponding $\beta_G \sim g$ curve passes through the infeasible region. It is active for design $\mathbf{d}^2$ as its $\beta_G \sim g$ curve passes through the active point. And it is inactive for design $\mathbf{d}^3$ as its $\beta_G \sim g$ curve passes through the feasible region. In other words, a given design is infeasible if the non-increasing $\beta_G \sim g$ curve passes through the infeasible region, while the design is feasible if the curve passes through the feasible region. For the active probabilistic constraint at design $\mathbf{d}^2$, the only point outside the ambiguous regions is the active point $(\beta_t, 0)$ because $\beta_s(\mathbf{d}^2) = \beta_t$ and $g^*(\mathbf{d}^2) = 0$. That is, the probabilistic constraint can be evaluated by finding any point on the $\beta_G \sim g$ curve that is outside the ambiguous regions.

Thus, a single inequality relation can be used to represent the probabilistic constraint, such as Eq. (7a) in the conventional RIA or Eq. (8a) in the proposed PMA. On the $\beta_G \sim g$ curve for design $\mathbf{d}^k = [4, 4]^T$, as shown in Fig. 2, the point $(\beta_s, 0)$ is identified in RIA by performing reliability analysis of Eq. (7b), and the point (β_t, g^*) is identified in PMA by performing inverse reliability analysis of Eq. (8b). The probabilistic constraint is violated in RIA because $\beta_s = 1.512 < \beta_t = 2$ as well as in PMA because $g^* = -0.452 < 0$.

4.3 Singularity of RIA in the Probabilistic Constraint Evaluation. Note that any point on the $\beta_G \sim g$ curve that is outside the ambiguous regions, such as the point (β_a, g_a) in Fig. 2, can sufficiently identify the status of the probabilistic constraint, while RIA and PMA are two extreme cases.

However, RIA can yield singularity in the probabilistic constraint evaluation. For this, consider design $\mathbf{d}^3 = [4.5, 4.5]^T$, whose non-increasing $\beta_G \sim g$ curve is shown in Fig. 1. Note that the point $(\beta_s, 0)$ does not exist because the failure probability of the design is zero. Numerically, the reliability index $\beta_s(\mathbf{d}^3)$ approaches infinity and thus RIA yields singularity. This happens because the system performance function $G(\mathbf{x})$ is positive everywhere in the corresponding probability integration domain of the design. If $G(\mathbf{x})$ is negative everywhere, the failure probability of the design is one hundred percent and RIA yields singularity again as the reliability index approaches negative infinity. In contrast, PMA is inherently robust because the point (β_t, g^*) always exists.

5 A General Approach for the Probabilistic Constraint Evaluation

A general approach for the probabilistic constraint evaluation can be established by finding the point (β_a, g_a) between $(\beta_s, 0)$ and (β_t, g^*) so that a single inequality relation can be used to represent the probabilistic constraint.

For general evaluation of the probabilistic constraint given in Eqs. (10a) to (10c), the Taylor series expansion of Eq. (10c) at the point (β_a, g_a) can be obtained in two ways by using its equivalent forms in Eq. (3a) and Eq. (3b), respectively, as

$$\beta_G(g) = \beta_a + \sum_{n=1}^{\infty} \frac{d^n\beta_a}{dg^n} \frac{(g - g_a)^n}{n!} \quad (12a)$$

$$g(\beta_G) = g_a + \sum_{n=1}^{\infty} \frac{d^n g_a}{d\beta_G^n} \frac{(\beta_G - \beta_a)^n}{n!} \quad (12b)$$

By assuming $g = 0$ (RIA) in Eq. (10b) and substituting Eq. (12a) into Eq. (10a), an inequality relation can be obtained to represent the probabilistic constraint as

$$\beta_G(0) = \beta_a + \sum_{n=1}^{\infty} \frac{d^n\beta_a}{dg^n} \frac{(-g_a)^n}{n!} \geq \beta_t \quad (13a)$$

Similarly, by assuming $\beta_G = \beta_t$ (PMA) in Eq. (10a) and substituting Eq. (12b) into Eq. (10b), another inequality relation is obtained to represent the probabilistic constraint as

$$g(\beta_t) = g_a + \sum_{n=1}^{\infty} \frac{d^n g_a}{d\beta_G^n} \frac{(\beta_t - \beta_a)^n}{n!} \geq 0 \quad (13b)$$

Because high order derivatives in Eqs. (13a) and (13b) are generally difficult to obtain in practical applications, the mth-order approximation of the probabilistic constraint is instead used in two ways depending on whether the point (β_a, g_a) is exemplified by reliability or inverse reliability analysis, respectively, i.e.,

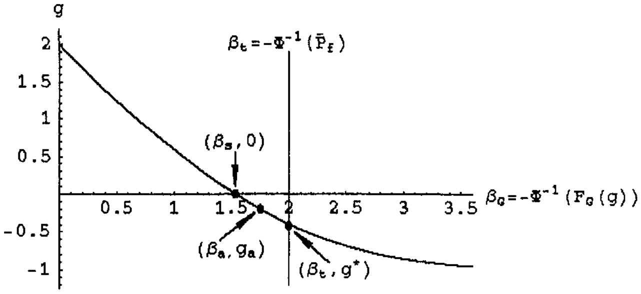

Fig. 2 Illustration of Probabilistic Constraint Evaluation at $\mathbf{d}^k = [4, 4]^T$

$$\beta_a(\mathbf{d}) + \sum_{n=1}^{m} \frac{d^n\beta_a}{dg^n}\frac{(-g_a)^n}{n!} \geq \beta_t, \quad \text{for given} \quad g_a = \alpha g^* \quad (14a)$$

$$g_a(\mathbf{d}) + \sum_{n=1}^{m} \frac{d^n g_a}{d\beta_G^n}\frac{(\beta_t - \beta_a)^n}{n!} \geq 0, \quad \text{for given}$$

$$\beta_a = (1-\alpha)\beta_s + \alpha\beta_t \quad (14b)$$

where the adaptive factor is in $0 \leq \alpha \leq 1$, and m depends on the specific approximate probability integration method. For example, $m = 1$ if the first-order reliability method (FORM) is used and $m = 2$ if the second-order reliability method (SORM) is used. It is clear that Eq. (14*a*) becomes the conventional RIA of Eq. (7*a*) if $\alpha = 0$, and Eq. (14*b*) becomes the proposed PMA of Eq. (8*a*) if $\alpha = 1$.

The consistency of various perspectives in the general approach is maintained by using a point (β_a, g_a) that can sufficiently identify the limit-state of the probabilistic constraint, which is ensured by the adaptive factor so that the point is in between $(\beta_s, 0)$ and (β_t, g^*). If the probabilistic constraint is active at design $\mathbf{d}^k$, then the unique point is $(\beta_t, 0)$ for arbitrary adaptive factor $0 \leq \alpha \leq 1$ because $\beta_a(\mathbf{d}^k) = \beta_s(\mathbf{d}^k) = \beta_t$ and $g_a(\mathbf{d}^k) = g^*(\mathbf{d}^k) = 0$. Thus, various perspectives of the general approach are consistent in prescribing the probabilistic constraint and they are exchangeable in RBDO iterations.

6 FORM for Approximate Probability Integration

Either Eq. (14*a*) or Eq. (14*b*) can be used to prescribe a probabilistic constraint in the RBDO model. At design $\mathbf{d}^k$ in the RBDO iterations, the evaluation of Eq. (14*a*) requires reliability analysis and the evaluation of Eq. (14*b*) requires inverse reliability analysis. In either case, the multiple integration is involved and the exact probability integration is in general extremely complicated to compute, i.e.,

$$\beta_a(\mathbf{d}^k) = -\Phi^{-1}(F_G(g_a)) = -\Phi^{-1}\left(\int_{G(\mathbf{x})<g_a} \cdots \int f_X(\mathbf{x})dx_1 \ldots dx_n\right), \quad x_i^L \leq x_i \leq x_i^U \quad (15a)$$

$$g_a(\mathbf{d}^k) = F_G^{-1}(\Phi(-\beta_a)) = F_G^{-1}\left(\int_{G(\mathbf{x})<g_a} \cdots \int f_X(\mathbf{x})dx_1 \ldots dx_n\right), \quad x_i^L \leq x_i \leq x_i^U \quad (15b)$$

The Monte Carlo simulation (MCS) (Rubinstein, 1981) provides a convenient approximation for both reliability analysis and inverse reliability analysis because it directly approximates the $\beta_G \sim g$ relationship. The minimum MCS sample size for finding the point (β_a, g_a) is usually suggested as

$$L = 10/P(G(\mathbf{x}) \leq g_a) = 10/F_G(g_a) = 10/\Phi(-\beta_a) \quad (16)$$

where L increases exponentially in terms of β_a and becomes very large if the reliability target is high, e.g., $L = 7692$ for $g_a = 3$. Thus, MCS becomes prohibitively expensive for many engineering applications.

Some approximate probability integration methods has been developed to provide efficient solutions (Breitung, 1984; Madsen et al., 1986; Kiureghian et al., 1987; Wu and Wirsching, 1987; Tvedt, 1990), such as FORM or the asymptotic SORM. FORM often provides adequate accuracy and is widely accepted for RBDO applications. The RIA and PMA can be used effectively with FORM in the probabilistic constraint evaluation. If the more accurate (and also more expensive) SORM is necessary in some engineering applications, the intermediate perspective of the general approach becomes attractive. This paper focuses on RBDO using FORM for approximate probability integration. Thus, RIA and PMA, the two extreme cases of the general approach, are analyzed next and compared in RBDO applications.

6.1 General Interpretation of FORM. In FORM, the transformation (Hohenbichler and Rackwitz, 1981; Madsen et al., 1986) from the nonnormal random system parameter $\mathbf{X}$ ($\mathbf{x}$-space) to the independent and standard normal variable $\mathbf{U}$ ($\mathbf{u}$-space) is required. If all system parameters are mutually independent, the transformations can be simplified as

$$u_i = \Phi^{-1}(F_{X_i}(x_i)), \quad i = 1, 2, \ldots, n \quad (17a)$$

$$x_i = F_{X_i}^{-1}(\Phi(u_i)), \quad i = 1, 2, \ldots, n \quad (17b)$$

The performance function $G(\mathbf{x})$ can then be represented as $G_U(\mathbf{u})$ in the $\mathbf{u}$-space. The point on the hypersurface $G_U(\mathbf{u}) = g_a$ with the maximum joint probability density is the point with the minimum distance from the origin and is named the most probable point (MPP) $\mathbf{u}^*_{g=g_a}$. The minimum distance, named the first-order reliability index $\beta_{a,\text{FORM}}$, is an approximation of the generalized probability index corresponding to g_a as

$$\beta_{a,\text{FORM}} \approx \beta_a = \beta_G(g_a) \quad (18)$$

Inversely, the performance function value at the MPP $\mathbf{u}^*_{\beta=\beta_a}$ with *the distance* β_a *from the origin is an approximation of the* probabilistic performance measure g_a as

$$g_{a,\text{FORM}} = G_U(\mathbf{u}^*_{\beta=\beta_a}) \approx g_a = g(\beta_a) \quad (19)$$

Thus, the first-order reliability analysis is to find the MPP on the hypersurface $G_U(\mathbf{u}) = g_a$ in the $\mathbf{u}$-space, and first-order inverse reliability analysis is to find the MPP that renders the minimum distance β_a from the origin. In two special cases, the MPP $\mathbf{u}^*_{g=0}$ is found by performing first-order reliability analysis in RIA, while the MPP $\mathbf{u}^*_{\beta=\beta_t}$ is found by performing first-order inverse reliability analysis in PMA.

6.2 First-Order Reliability Analysis. In traditional first-order reliability analysis (Madsen et al., 1986), the first-order reliability index $\beta_{a,\text{FORM}}$ is the solution of a nonlinear optimization problem

$$\text{minimize} \quad \|\mathbf{u}\| \quad (20a)$$

$$\text{subject to} \quad G_U(\mathbf{u}) = g_a \quad (20b)$$

where the optimum is the MPP $\mathbf{u}^*_{g=g_a}$ and thus $\beta_{a,\text{FORM}} = \|\mathbf{u}^*_{g=g_a}\|$. Many MPP search algorithms (such as HL-RF, Modified HL-RF, AMVFO) and general optimization algorithms (such as SLP, SQP, MFD, augmented Lagrangian method, etc.) can be used to find the MPP (Wu and Wirsching, 1987; Wu et al., 1990; Liu and Kiureghian, 1991; Wang and Grandhi, 1994; Wu, 1994; Choi et al., 1996; Yu et al., 1997, 1998).

6.3 First-Order Inverse Reliability Analysis. In first-order inverse reliability analysis, the first-order target probabilistic performance measure $g_{a,\text{FORM}}$ is the solution of a sphere-constrained nonlinear optimization problem (Tu and Choi, 1997)

$$\text{minimize} \quad G_U(\mathbf{u}) \quad (21a)$$

$$\text{subject to} \quad \|\mathbf{u}\| = \beta_a \quad (21b)$$

where the optimum is the MPP $\mathbf{u}^*_{\beta=\beta_a}$ and thus $g_{a,\text{FORM}}(\beta_a) = G_U(\mathbf{u}^*_{\beta=\beta_a})$.

General optimization algorithms (such as SLP, SQP, and MFD) can be used to solve this sphere-constrained optimization problem, which is generally easier to solve than the optimization problem in

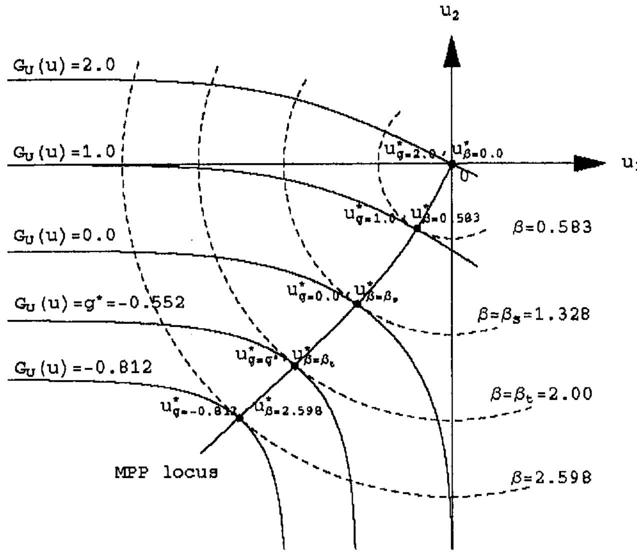

Fig. 3 Illustration of MPP locus in the *u*-space

Eqs. (20*a*) and (20*b*) due to the regular sphere constraint of Eq. (21*b*). In particular, the advanced mean-value first-order method (AMVFO) (Wu et al., 1990; Wu, 1994) can also be used effectively in PMA for many engineering applications.

6.4 Example. Consider the same probabilistic constraint defined in Section 4.1, where the CDFs of the uniformly distributed system parameters are

$$F_{X_i}(x_i) = \int_{a_i}^{x_i} f_{X_i}(x_i)dx_i = (x_i - a_i)/(b_i - a_i),$$

$$a_i \leq x_i \leq b_i, \quad i = 1, 2 \quad (22a)$$

Since mean values are chosen as design variables, $\mathbf{d} = [d_1, d_2]^T \equiv [\mu_1, \mu_2]^T$, and variances are constants as $\sigma_1^2 = \sigma_2^2 = \frac{1}{3}$, the CDFs of system parameters can be rewritten in terms of design variables as

$$F_{X_i}(x_i) = (x_i - d_i + 1)/2, \quad d_i - 1 \leq x_i \leq d_i + 1,$$

$$i = 1, 2 \quad (22b)$$

The transformations between the **x**-space and the **u**-space at design $\mathbf{d}^k$ for two independent system parameters can be expressed as

$$u_i = \Phi^{-1}(F_{X_i}(x_i)) = \Phi^{-1}((x_i - d_i^k + 1)/2), \quad i = 1, 2 \quad (22c)$$

$$x_i = 2\Phi(u_1) + d_i^k - 1, \quad i = 1, 2 \quad (22d)$$

and the performance function is then transformed into the **u**-space as

$$G_U(\mathbf{u}) = 2\Phi(u_1) + 4\Phi(u_2) + d_1^k + 2d_2^k - 13 \quad (22e)$$

At design $\mathbf{d}^k = [4.0, 4.0]^T$, the contours of the performance function, i.e., $G_U(\mathbf{u}) = g$ for different g values, and the MPP locus in the **u**-space are illustrated in Fig. 3, where the MPP $\mathbf{u}^*_{g=0}$ (or $\mathbf{u}^*_{\beta=\beta_s}$) is found using first-order reliability analysis of RIA, and the MPP $\mathbf{u}^*_{\beta=\beta_t}$ (or $\mathbf{u}^*_{g=g^*}$) is found using first-order inverse reliability analysis of PMA. The corresponding $(\beta_G \sim g)_{\text{FORM}}$ curve is then compared with the exact $\beta_G \sim g$ curve in Fig. 4, where RIA identifies the point $(\beta_{s,\text{FORM}}, 0)$ and PMA identifies the point $(\beta_t, g^*_{\text{FORM}})$.

As discussed in Section 4.3, at design $\mathbf{d}^3 = [4.5, 4.5]^T$, the performance function of Eq. (22*e*) is positive everywhere in the probability integration domain as

$$G(\mathbf{x}) = G_U(\mathbf{u}) = 2\Phi(u_1) + 4\Phi(u_2) + 0.5 > 0,$$

$$d_i^3 - 1 \leq x_i \leq d_i^3 + 1, \quad i = 1, 2 \quad (23)$$

and thus first-order reliability analysis by Eqs. (20*a*) and (20*b*) of RIA yields no solution. In contrast, the first-order inverse reliability analysis of PMA can always be performed.

7 Computational Efficiency in Probabilistic Constraint Evaluation

If the Monte Carlo simulation (MCS) is used for probability analysis, the computational efforts required to find point $(\beta_s, 0)$ in RIA and point (β_t, g^*) in PMA can be quantified by the minimum MCS sample size L suggested in Eq. (16) as $L_{\text{RIA}} = 10/\Phi(-\beta_s)$ and $L_{\text{PMA}} = 10/\Phi(-\beta_t)$. That is,

a. if the probabilistic constraint is inactive, then $\beta_s > \beta_t$ and $L_{\text{PMA}} < L_{\text{RIA}}$;
b. if the probabilistic constraint is active, then $\beta_s = \beta_t$ and $L_{\text{PMA}} = L_{\text{RIA}}$;
c. if the probabilistic constraint is violated, then $\beta_s < \beta_t$ and $L_{\text{PMA}} > L_{\text{RIA}}$.

It is pointed out that inverse reliability analysis is easier to solve than reliability analysis since the spherical constraint in Eq. (21*b*) is more regular compared to the general nonlinear constraint in Eq. (20*b*). In practical applications, the computational efforts associated with RIA (using first-order reliability analysis) and PMA (using first-order inverse reliability analysis) cannot be easily quantified, since RIA and PMA are searching for different MPPs. However, it is generally easier to find the MPP that is closer to the origin of the **u**-space (which means searching an MPP in a more restrictive solution space as shown in Fig. 3). Thus, the estimations of the computational efforts associated with RIA and PMA can also be established for three different scenarios so that

a. if $\beta_s > \beta_t$, then the MPP of PMA $\mathbf{u}^*_{\beta=\beta_t}$ is closer to the origin than the MPP of RIA $\mathbf{u}^*_{g=0}$ (i.e., $\mathbf{u}^*_{\beta=\beta_s}$);
b. if $\beta_s = \beta_t$, then PMA and RIA search the same MPP as $\mathbf{u}^*_{\beta=\beta_t} = \mathbf{u}^*_{\beta=\beta_s}$;
c. if $\beta_s < \beta_t$, then the MPP of RIA is closer to the origin than the MPP of PMA.

Therefore, PMA is not only inherently robust but is also more efficient for evaluating inactive probabilistic constraints, while RIA is more efficient for violated probabilistic constraints. Note that the computational difference between RIA and PMA becomes significant if $\mathbf{u}^*_{\beta=\beta_t}$ and $\mathbf{u}^*_{g=0}$ are far apart in the **u**-space, while the exact status of the probabilistic constraint is unknown until either $\mathbf{u}^*_{\beta=\beta_t}$ or $\mathbf{u}^*_{g=0}$ is finally found. Hence, it is desired to adaptively select RIA or PMA in the RBDO iterations depending on the marginally estimated status of the probabilistic constraint at the beginning of the MPP search.

8 Difference of PMA and RIA in RBDO

In previous sections, it has been illustrated that the probabilistic constraint in RBDO can be interpreted from a broader perspective

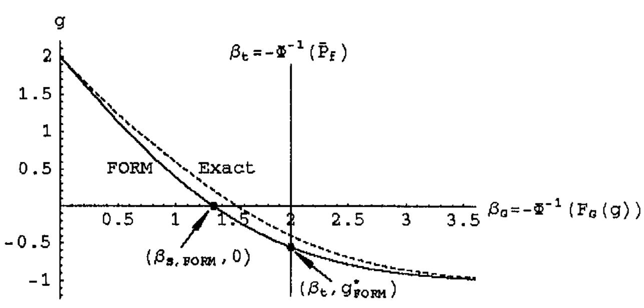

Fig. 4 Probabilistic Constraint Evaluation by FROM

Table 1 RBDO History using RIA ($\mathbf{d}^0 = [4.000, 4.000]^T$)

Iteration k	Cost	d_1^k	d_2^k	$j = 1$		$j = 2$	
				$\beta_{s,1}^k$	$\beta_{s,1}^k - \beta_{t,1}$	$\beta_{s,2}^k$	$\beta_{s,2}^k - \beta_{t,2}$
0	8.000	4.000	4.000	1.327	−0.727	1.327	−0.554
1	8.455	4.135	4.320	2.481	0.427	2.066	0.185
2	8.371	4.116	4.255	2.135	0.081	1.897	0.016
3	8.354	4.123	4.231	2.057	0.003	1.881	0.000
4	8.353	4.123	4.230	2.054	0.000	1.881	0.000
5	8.353	4.123	4.230	2.054	active	1.881	active

in the general approach. These different perspectives of the general approach are consistent in prescribing the probabilistic constraint, but they are different in terms of robustness and computational efficiency in probabilistic constraint evaluation. Furthermore, using different perspectives of the general approach in prescribing the probabilistic constraint actually yields different rates of convergence in solving the RBDO problem.

The RBDO problem is usually solved by search methods for constrained nonlinear optimization, such as SLP, SQP, and MFD. The search method starts with an initial design and iteratively improves it with the design change obtained by solving an approximate subproblem defined by the linearized probabilistic constraints. The difference is that the linearized probabilistic constraints from different perspectives are not equivalent in predicting the design change. The RIA and PMA are compared here to illustrate their differences in solving the RBDO problem.

In RIA, the probabilistic constraint of Eq. (7*a*) is linearized at design $\mathbf{d}^k$ in defining the search direction determination subproblem as

$$\beta_s(\mathbf{d}^k) + \nabla_d^T\beta_s(\mathbf{d}^k)(\mathbf{d} - \mathbf{d}^k) \geq \beta_t \quad (24a)$$

where $\beta_s(\mathbf{d}^k)$ and $\nabla_d^T\beta_s(\mathbf{d}^k)$ are obtained in the first-order reliability analysis, i.e.,

$$\beta_s(\mathbf{d}^k) = \|\mathbf{u}^*_{g=0}\| \quad (24b)$$

$$\nabla_\mathbf{d}^T\beta_s(\mathbf{d}^k) = \frac{\nabla_\mathbf{d}^T G_U(\mathbf{u}^*_{g=0})}{\|\nabla_\mathbf{u} G_U(\mathbf{u}^*_{g=0})\|} \quad (24c)$$

Similarly, the probabilistic constraint of Eq. (8*a*) in PMA is linearized as

$$g^*(\mathbf{d}^k) + \nabla_\mathbf{d}^T g^*(\mathbf{d}^k)(\mathbf{d} - \mathbf{d}^k) \geq 0 \quad (25a)$$

where $g^*(\mathbf{d}^k)$ and $\nabla_\mathbf{d}^T g^*(\mathbf{d}^k)$ are obtained in the first-order inverse reliability analysis as

$$g^*(\mathbf{d}^k) = G_U(\mathbf{u}^*_{\beta=\beta_t}) \quad (25b)$$

$$\nabla_\mathbf{d}^T g^*(\mathbf{d}^k) = \nabla_\mathbf{d}^T G_U(\mathbf{u}^*_{\beta=\beta_t}) \quad (25c)$$

For comparison, the linearized probabilistic constraints from Eq. (24*a*) of RIA and Eq. (25*a*) of PMA are rearranged, respectively, as

$$\nabla_d^T \mathrm{G_U}(\mathbf{u}^*_{g=0})(\mathbf{d} - \mathbf{d}^k) \geq -\|\nabla_\mathbf{u} G_U(\mathbf{u}^*_{g=0})\|(\beta_s(\mathbf{d}^k) - \beta_t) \quad (26a)$$

$$\nabla_\mathbf{d}^T G_U(\mathbf{u}^*_{\beta=\beta_t})(\mathbf{d} - \mathbf{d}^k) \geq -G_U(\mathbf{u}^*_{\beta=\beta_t}) \quad (26b)$$

If the probabilistic constraint is active at a given design $\mathbf{d}^k$, then $\mathbf{u}^*_{\beta=\beta_t} = \mathbf{u}^*_{g=0}$, $\beta_s(\mathbf{d}^k) = \beta_t$, and $G_U(\mathbf{u}^*_{\beta=\beta_t}) = 0$. Thus, Eqs. (26*a*) and (26*b*) become identical, which means RIA and PMA are the same in identifying the limit-state of the probabilistic constraint in the design space. On the other hand, Eqs. (26*a*) and (26*b*) are rather different if the constraint is violated or inactive. As a result, the design changes computed from them are different and therefore the RBDO convergence rates are different, too.

8.1 Example. Consider the same system described in Section 4.1, where the design variable $\mathbf{d} = [d_1, d_2]^T \equiv [\mu_1, \mu_2]^T$, and variances are constants as $\sigma_1^2 = \sigma_2^2 = \frac{1}{3}$. The RBDO problem is to

$$\text{minimize} \quad \mathrm{Cost}(\mathbf{d}) = d_1 + d_2 \quad (27a)$$

$$\text{subject to} \quad P(G_j(\mathbf{x}) < 0) \leq \bar{P}_{f,j}, \quad j = 1, 2 \quad (27b)$$

$$1 \leq d_1 \leq 10 \quad \& \quad 1 \leq d_2 \leq 10 \quad (27c)$$

where the two system performance functions are defined as

$$G_1(\mathbf{x}) = x_1 + 2x_2 - 10 \quad (28a)$$

$$G_2(\mathbf{x}) = 2x_1 + x_2 - 10 \quad (28b)$$

and the prescribed failure probability limits are $\bar{P}_{f,1} = 2.00\%$ & $\bar{P}_{f,2} = 3.00\%$ (i.e., $\beta_{t,1} = -\Phi^{-1}(0.02) = 2.054$ & $\beta_{t,2} = -\Phi^{-1}(0.03) = 1.881$).

At design $\mathbf{d}^k$, the transformations for the two system parameters are defined in Eqs. (22*c*) and (22*d*). Thus, the performance functions can be represented in the $\mathbf{u}$-space as

$$G_{U,1}(\mathbf{u}) = 2\Phi(u_1) + 4\Phi(u_2) + (d_1^k + 2d_2^k - 13) \quad (28c)$$

$$G_{U,2}(\mathbf{u}) = 4\Phi(u_1) + 2\Phi(u_2) + (2d_1^k + d_2^k - 13) \quad (28d)$$

8.1.1 In RIA, the RBDO problem is to

$$\text{minimize} \quad \mathrm{Cost}(\mathbf{d}) = d_1 + d_2 \quad (29a)$$

$$\text{subject to} \quad \beta_{s,1}(\mathbf{d}) \geq \beta_{t,1} = 2.054 \quad (29b)$$

$$\beta_{s,2}(\mathbf{d}) \geq \beta_{t,2} = 1.881 \quad (29c)$$

$$1 \leq d_1 \leq 10 \quad \& \quad 1 \leq d_2 \leq 10 \quad (29d)$$

At design $\mathbf{d}^k = [d_1^k, d_2^k]^T$, two reliability indexes are computed by performing two first-order reliability analyses to

$$\text{minimize} \quad u_1^2 + u_2^2 \quad (30a)$$

$$\text{subject to} \quad G_{U,1}(\mathbf{u}) = 2\Phi(u_1) + 4\Phi(u_2) + (d_1^k + 2d_2^k - 13) = 0 \quad (30b)$$

and

$$\text{minimize} \quad u_1^2 + u_2^2 \quad (30c)$$

$$\text{subject to} \quad G_{U,2}(\mathbf{u}) = 4\Phi(u_1) + 2\Phi(u_2) + (2d_1^k + d_2^k - 13) = 0 \quad (30d)$$

In this example, any general nonlinear programming algorithm can easily solve the optimization problems in Eqs. (30*a*) to (30*d*). The SLP can be used to solve the overall RBDO problem. Starting from an initial design $\mathbf{d}^0 = [4, 4]^T$, the SLP converges after five iterations and ten reliability analyses. The RBDO history is listed in Table 1.

8.1.2 In PMA, the RBDO problem is to

$$\text{minimize} \quad \text{Cost}(\mathbf{d}) = d_1 + d_2 \tag{31a}$$

$$\text{subject to} \quad g_j^*(\mathbf{d}) \geq 0, \quad j = 1, 2 \tag{31b}$$

$$1 \leq d_1 \leq 10 \quad \& \quad 1 \leq d_2 \leq 10 \tag{31c}$$

At initial design $\mathbf{d}^0 = [d_1^0, d_2^0]^T$, two target probabilistic performance measures are computed by performing two first-order inverse reliability analyses to

$$\text{minimize} \quad G_{U,1}(\mathbf{u}) = 2\Phi(u_1) + 4\Phi(u_2) + (d_1^0 + 2d_2^0 - 13) \tag{32a}$$

$$\text{subject to} \quad u_1^2 + u_2^2 = \beta_{t,1}^2 = (2.054)^2 \tag{32b}$$

and

$$\text{minimize} \quad G_{U,2}(\mathbf{u}) = 4\Phi(u_1) + 2\Phi(u_2) + (2d_1^0 + d_2^0 - 13) \tag{32c}$$

$$\text{subject to} \quad u_1^2 + u_2^2 = \beta_{t,2}^2 = (1.881)^2 \tag{32d}$$

Note that the last terms in Eqs. (32*a*) and (32*c*) are not functions of **u** but are related only to the initial design. Thus, the MPPs for these two inverse reliability analyses are the same for any initial design, even though the values of the corresponding target probabilistic performance measures are different for using various $\mathbf{d}^0$.

The solutions of the nonlinear optimization problem of Eqs. (32*a*) and (32*b*) can be obtained using SLP, SQP, or MFD as

$$\mathbf{u}^{*}_{\beta=\beta_t}{}^{j=1} = [u_1^{*j=1}, u_2^{*j=1}]^T \equiv [-1.278, -1.608]^T \tag{33a}$$

$$g_1^*(\mathbf{d}^0) = 2\Phi(u_j^{*j=1}) + 4\Phi(u_2^{*j=1}) + (d_1^0 + 2d_2^0 - 13) \tag{33b}$$

and the solutions of the nonlinear optimization problem of Eqs. (32*c*) and (32*d*) are

$$\mathbf{u}^{*}_{\beta=\beta_t}{}^{j=2} = [u_1^{*j=2}, u_2^{*j=2}]^T \equiv [-1.478, -1.163]^T \tag{33c}$$

$$g_2^*(\mathbf{d}^0) = 4\Phi(u_1^{*j=2}) + 2\Phi(u_2^{*j=2}) + (2d_2^0 + d_2^0 - 13) \tag{33c}$$

Thus, the probabilistic constraints by PMA is linear in terms of design variables as

$$g_1^*(\mathbf{d}) = g_1^*(\mathbf{d}^0) + (d_1 - d_1^0) + 2(d_2 - d_2^0) \equiv d_1 + 2d_2 - 12.583 \geq 0 \tag{34a}$$

$$g_2^*(\mathbf{d}) = g_2^*(\mathbf{d}^0) + 2(d_1 - d_1^0) + (d_2 - d_2^0) \equiv 2d_1 + d_2 - 12.476 \geq 0 \tag{34b}$$

Consequently, this RBDO problem can be solved as a linear programming problem. The optimum $\mathbf{d}^{\text{opt}} = [4.123, 4.230]^T$ can be obtained in one iteration from an arbitrary initial design and only two first-order inverse reliability analyses of Eqs. (32*a*) to (32*d*) are required to compute $g_1^*(\mathbf{d}^0)$ and $g_2^*(\mathbf{d}^0)$ in Eqs. (34*a*) and (34*b*). The RBDO results by PMA and RIA are compared in Table 2.

8.2 Singularity of RIA in RBDO. In the previous section, it is shown that the convergence of RBDO is independent of the initial design if PMA is used for the probabilistic constraint evaluation. In this section, it will be shown that RBDO using RIA is very sensitive with respect to the initial design. For the given initial design $\mathbf{d}^0 = [3.5, 3.5]^T$, the RBDO history using RIA by SLP is listed in Table 3. The SLP fails in the first iteration due to the singularity of RIA at design $\mathbf{d}^1 = [4.325, 4.625]^T$, where the first-order reliability analysis for both probabilistic constraints have no solution. This is because the last terms in Eqs. (30*b*) and (30*d*) are positive at design $\mathbf{d}^1$, i.e.,

$$2\mu_1^1 + \mu_2^1 - 13 = 0.375 > 0 \tag{36a}$$

$$\mu_1^1 + 2\mu_2^1 - 13 = 0.575 > 0 \tag{36b}$$

Thus, the performance functions are positive everywhere in the probability integration domains of the design, and the corresponding failure probabilities are zeros.

Table 2 RBDO using PMA and RIA

RBDO	Cost	d_1^{opt}	d_2^{opt}	Total Number of RBDO Iteration	Total Number of Reliability or Inverse Reliability Analyses
PMA	8.353	4.123	4.230	1	2
RIA	8.353	4.123	4.230	5	10

Table 3 RBDO History using RIA ($\mathbf{d}^0 = [3.500, 3.500]^T$)

Iteration k	Cost	d_1^k	d_2^k	$\beta_{s,1}^k$	$\beta_{s,2}^k$
0	7.000	3.500	3.500	0.283	0.283
1	8.950	4.625	4.325	∞	∞

8.3 Discussion. The comprehensive probabilistic constraint is represented by Eq. (8*a*) in PMA and is directly measured by the target probabilistic performance measure. In RIA, on the other hand, the probabilistic constraint is measured by the reliability index, which is often a nonlinear transformation of the corresponding probabilistic performance measure. In a case where the system has non-normally distributed random system parameters and the probabilistic constraints are for linear performance functions, PMA yields linear constraints of design variables while RIA yields nonlinear constraints. It is expected that, for the general nonlinear performance functions in practical applications, PMA yields a higher rate of convergence for RBDO than the conventional RIA.

9 Summary

It is clearly shown in this paper that the well-accepted RIA represents only one perspective of the probabilistic constraint. From a broader perspective, the general approach for the probabilistic constraint evaluation is developed, where RIA and the proposed PMA are two extreme cases. Although various perspectives of the general approach are consistent in prescribing the probabilistic constraint, their significant differences in solving the RBDO problem is illustrated. The PMA is inherently robust and more efficient in evaluating inactive probabilistic constraints, and it yields a higher overall RBDO rate of convergence. On the other hand, RIA is more efficient for violated probabilistic constraints, but the singularity behavior of RIA restricts its applications in broader engineering design practices.

The overall efficiency of solving the RBDO problem depends on the balance between the total number of overall iterations and the computational efforts in each iteration. In practical applications, the RBDO problem can be solved robustly and more efficiently by adaptively choosing RIA and PMA depending on the estimated marginal status of the probabilistic constraint in the RBDO iterations.

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