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Dynamic Output Feedback Control of Discrete-Time Systems with Actuator Nonlinearities

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Abstract

This paper considers the problem of stabilization of discrete-time systems with actuator nonlinearities. Specifically, dynamic, output feedback control design for discrete-time systems with time-varying, sector-bounded, input nonlinearities is addressed. The proposed framework is based on a linear matrix inequality approach and directly accounts for robust stability and robust performance over the class of actuator nonlinearities. Furthermore, it is directly applicable to actuator saturation control and provides dynamic, output feedback controllers with guaranteed domains of attraction. The effectiveness of the approach is illustrated by a numerical example.

1. Introduction

In practical applications of feedback control, actuator nonlinearities, such as saturation, arise frequently and can severely degrade closed-loop system performance and in some cases drive the system to instability. The issue of closed-loop system stability and performance subject to actuator saturation thus carries a great deal of practical importance. For continuous-time systems, the problem of actuator saturation has been widely studied and an extensive literature is devoted to it (see, e.g., [2,6,7,12,15] and the numerous references therein). In contrast to the continuous-time, actuator saturation control problem, the problem of stabilizing discrete-time systems in the presence of control signal saturation has received scant attention. For recent exceptions see [8, 11, 13] and the references therein. Since most physical processes evolve naturally in continuous time, it is not surprising that the bulk of stability and control theory involving systems with actuator nonlinearities has been developed for continuous-time systems. Nevertheless, it is the overwhelming trend to implement controllers digitally.

For discrete-time systems, Riccati equation-based global and semi-global stabilization techniques for actuator saturation have been developed in [11,13]. In addition, the application of an anti-windup actuator saturation control framework to discrete-time systems is given in [10]. However, the research literature on controller synthesis for systems with more general time-varying, sector-bounded, input nonlinearities is rather limited. In a recent paper [8], a Riccati equation-based global and local static, output feedback control design framework for discrete-time systems with time-varying, sector-bounded, input nonlinearities was developed. Unfortunately, however, a caveat of [8] is that it can not address the dynamic, output feedback compensation problem for discrete-time systems with actuator nonlinearities. Specifically, for the aforementioned problem, the Lagrange multiplier approach of [8] leads to severe algebraic complexity in the computation of explicit closed-form expressions for the dynamic, output feedback controller.

In this paper, to overcome the limitation of [8], we focus on a tractable formulation of the dynamic, output feedback control synthesis for system with input nonlinearities using a linear matrix inequality (LMI) framework [1]. As demonstrated in this paper, dynamic, output feedback control of discrete-time systems with time-varying, sector-bounded, actuator nonlinearities leads to a matrix inequality that is nonlinear in the decision variables; hence, circumventing a direct application of the LMI theory. By a judicious over-bounding of several terms in the nonlinear matrix inequality (NMI), we provide tractable sufficient conditions, in the form of LMIs, for dynamic, output feedback control of systems with time-varying, sector-bounded, actuator nonlinearities. In addition, we detail a numerical algorithm for solving this control synthesis problem. Finally, we demonstrate the efficacy of the proposed approach via an illustrative numerical example. All proofs are omitted due to space constraint.

Nomenclature

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r$	– real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$
$I_r, 0_r$	– $r \times r$ identity matrix, $r \times r$ zero matrix
$\mathbb{P}^r$	– $r \times r$ positive-definite matrix
$\mathcal{N}$	– $\{0, 1, 2, \ldots, \}$
$n, m, p, d, n_c, \tilde{n}$	– positive integers; $1 \le n_c \le n$; $\tilde{n} = n + n_c$

2. Dynamic Output Feedback Control of Systems with Actuator Nonlinearities

In this section, we introduce the problem of dynamic, output feedback control of discrete-time, linear systems with actuators containing a set Φ of time-varying, sector-bounded nonlinearities. The goal of the problem is to determine a strictly proper, optimal, dynamic compensator (A_c, B_c, C_c) that stabilizes a given linear, dynamical system with actuator nonlinearities $\phi(u(k), k) \in \Phi$ and minimizes a quadratic performance criterion involving weighted state and control variables. The structure of Φ is specified later in the section.

Dynamic Output Feedback Stabilization Problem. Given the n^{th}-order stabilizable and detectable plant with input nonlinearities $\phi(u(k), k) \in \Phi$, $k \in \mathcal{N}$,

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$$x(k+1) = Ax(k) - B\phi(u(k),k), x(0)=x_0, k \in \mathcal{N}, \quad (1)$$
$$y(k) = Cx(k), \quad (2)$$

where $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^l$, determine an n_c^{th}-order, linear, time-invariant, dynamic compensator

$$x_c(k+1) = A_c x_c(k) + B_c y(k), \quad (3)$$
$$u(k) = C_c x_c(k), \quad (4)$$

that satisfies the following design criteria *i)* the zero solution of the closed-loop system (1)–(4) is globally asymptotically stable for all $\phi(u(k),k) \in \Phi$ and *ii)* the quadratic performance functional

$$J(A_c, B_c, C_c) \triangleq \sup_{\phi(\cdot,\cdot) \in \Phi} \sum_{k=0}^{\infty} z^T(k) z(k), \quad (5)$$

where $z(k) \triangleq E_1 x(k) + E_2 u(k)$, $z \in \mathbb{R}^p$, is minimized.

To characterize the class Φ of time-varying, sector-bounded, memoryless nonlinearities the following definitions are needed. Let $M_1, M_2 \in \mathbb{R}^{m \times m}$ be given diagonal matrices such that $M_1 = \text{diag}\,(M_{1_1}, \ldots, M_{1_m})$, $M_2 = \text{diag}\,(M_{2_1}, \ldots, M_{2_m})$, and $M \triangleq M_2 - M_1$ is positive-definite with diagonal entries M_{ii}, $i = 1, \ldots, m$. Next, we define the set of allowable nonlinearities $\phi(\cdot,\cdot)$ by

$$\Phi \triangleq \{\phi : \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}^m : M_{1_i} u_i^2 \le \phi_i(u,k) u_i \le M_{2_i} u_i^2, \; u_i \in \mathbb{R}, \; i = 1, \ldots, m, \; k \in \mathcal{N}\}. \quad (6)$$

Now, we provide a closed-loop NMI that guarantees global asymptotic stability of the closed-loop system (1)–(4) for all actuator nonlinearities $\phi(\cdot,\cdot) \in \Phi$. First, however, we decompose the nonlinearity $\phi(\cdot,\cdot)$ into linear and nonlinear parts so that $\phi(u(k),k) = \phi_s(u(k),k) + M_1 u(k)$. With the above transformation, the close-loop system (1)–(4) has a state-space representation

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) - \tilde{B}\phi_s(u(k),k), \tilde{x}(0) = \tilde{x}_0, k \in \mathcal{N}, \quad (7)$$
$$u(k) = \tilde{C}\tilde{x}(k), \quad (8)$$

where

$$\tilde{x} \triangleq \begin{bmatrix} x \\ x_c \end{bmatrix}, \qquad \tilde{A} \triangleq \begin{bmatrix} A & -BM_1C_c \\ B_cC & A_c \end{bmatrix},$$
$$\tilde{B} \triangleq \begin{bmatrix} B \\ 0_{n_c \times m} \end{bmatrix}, \qquad \tilde{C} \triangleq \begin{bmatrix} 0_{m \times n} & C_c \end{bmatrix}.$$

In addition, the performance variable $z(k)$ is given by $z(k) = \tilde{E}\tilde{x}(k)$, where $\tilde{E} \triangleq \begin{bmatrix} E_1 & E_2C_c \end{bmatrix}$. Note that the transformed nonlinearities $\phi_s(\cdot,\cdot)$ belong to the set Φ_s given by

$$\Phi_s \triangleq \{\phi_s : \mathbb{R}^m \times \mathcal{N} \to \mathbb{R}^m : 0 \le \phi_{s_i}(u,k) u_i \le M_{ii} u_i^2, \; u_i \in \mathbb{R}, \; i = 1, \ldots, m, \; k \in \mathcal{N}\}. \quad (9)$$

The following result provides the foundation for our dynamic, output feedback controller synthesis framework. For the statement of this result, let H be an $m \times m$ diagonal, positive-definite matrix and define the notation $R_0 \triangleq 2HM^{-1}$ and $\tilde{R} \triangleq \tilde{E}^T\tilde{E}$.

Theorem 2.1. Let $m \times m$ diagonal matrices M_1 and M_2 be given such that $M_2 - M_1$ is positive-definite. In addition, let (A_c, B_c, C_c) and a scalar ϵ, $0 < \epsilon < 1$, be given. Suppose there exist an $m \times m$ diagonal, positive-definite matrix H and an $\tilde{n} \times \tilde{n}$ positive-definite matrix $\tilde{P}$ satisfying

$$\begin{bmatrix} \tilde{A}^T\tilde{P}\tilde{A} - \tilde{P} + \epsilon\tilde{P} + \tilde{R} & (H\tilde{C} - \tilde{B}^T\tilde{P}\tilde{A})^T \\ H\tilde{C} - \tilde{B}^T\tilde{P}\tilde{A} & -R_0 + \tilde{B}^T\tilde{P}\tilde{B} \end{bmatrix} < 0. \quad (10)$$

Then the function $V(\tilde{x}) = \tilde{x}^T\tilde{P}\tilde{x}$ is a Lyapunov function that guarantees that the zero solution $\tilde{x}(k) \equiv 0$ of the closed-loop system (1)–(4) is globally asymptotically stable for all actuator nonlinearities $\phi(\cdot,\cdot) \in \Phi$. Furthermore, the performance functional (5) satisfies the bound $J(\tilde{x}_0, A_c, B_c, C_c) < V(\tilde{x}_0)$.

Note that $J(\tilde{x}_0, A_c, B_c, C_c) < \text{tr}\, \tilde{x}_0^T\tilde{P}\tilde{x}_0 = \text{tr}\, \tilde{P}\tilde{x}_0\tilde{x}_0^T$, which has the same form as the H_2 cost appearing in the standard Linear Quadratic Gaussian (LQG) theory. Hence, we replace $\tilde{x}_0\tilde{x}_0^T$ by $\tilde{D}\tilde{D}^T$ where $\tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}$, $D_1 \in \mathbb{R}^{n \times d}$, $D_2 \in \mathbb{R}^{l \times d}$, and $D_2D_2^T > 0$ and proceed by determining the controller gains that minimize the auxiliary cost $\mathcal{J}(\tilde{P}, A_c, B_c, C_c) \triangleq \text{tr}\, \tilde{P}\tilde{D}\tilde{D}^T$.

Theorem 2.1 provides an efficient computational approach for closed-loop stability analysis when the controller (A_c, B_c, C_c), scalar ϵ, $0 < \epsilon < 1$, and the sector-bounds M_1, M_2 for input nonlinearity $\phi(\cdot,\cdot) \in \Phi$ are given. Specifically, since (10) is an LMI in the variables H and $\tilde{P}$, one can efficiently determine the feasibility of (10) to establish the asymptotic stability of (1)–(4). In this paper, however, we focus on extending Theorem 2.1 to design stabilizing feedback controllers for systems with actuator saturation nonlinearities. Before proceeding, observe that (10) is an NMI since it contains product terms involving A_c, B_c, C_c, $\tilde{P}$, and H.

Note that multiplier theory-based robust control design problems frequently result in NMIs when simultaneous determination of the controller and multiplier matrices is attempted [4, 14]. In order to circumvent the technical difficulties arising from the numerical solution of such NMIs, in prior literature, many researchers have focused on an iterative solution of the closed-loop stability analysis and stabilizing controller synthesis sub-problems [4, 14]. Specifically, [4, 14] have shown that the multiplier theory-based robust control design can be accomplished by *i)* solving an LMI problem for closed-loop stability analysis which provides the stability multiplier for a given controller and *ii)* solving an LMI problem for controller synthesis with a given stability multiplier. Note that although sub-problems *i)* and *ii)* considered separately are convex, the problem of simultaneous multiplier and controller determination is not. In addition, no claim can be made regarding the convergence of this iterative scheme. However, the aforementioned procedure offers attractive computational advantage by exploiting the convexity of the two LMI sub-problems and has been widely used with success in practice.

Unfortunately, however, (10) does not fit the class of NMIs that arise in the standard multiplier theory-based robust control design. Specifically, in contrast to [4, 14], when the stability multiplier matrix H is assumed to be given, (10) still contains product terms involving A_c, B_c, C_c, and $\tilde{P}$. The standard controller elimination procedure discussed in [1] and widely used in prior literature is thus not directly applicable in this case. In the following section, we develop a sufficient condition for (10) to provide an efficient computational algorithm for the design of a dynamic, output feedback controller for systems with actuator nonlinearities.

3. Dynamic Output Feedback Controller Synthesis for Systems with Actuator Nonlinearities

In this section, we present the main theorem characterizing dynamic, output feedback controllers for discrete-time systems with actuator nonlinearities. In order to state this result, we assume that scalar ϵ, $0 < \epsilon < 1$, and diagonal, positive-definite matrix H are given. In addition, we assume that M_1 and M_2 are given $m \times m$ diagonal matrices such that $M_2 - M_1$ is positive-definite.

Note that (10) is a nonstandard NMI and is not directly amenable to the linearizing change of controller variables proposed in [3, 5]. However, as demonstrated in this section, by a judicious over-bounding of several terms in the NMI, we can apply the linearizing change of controller variables of [3, 5] to obtain a tractable sufficient condition, in the form of an LMI, for dynamic, output feedback control. For the remainder of this section, we set $n_c = n$ to consider the full-order, dynamic, output feedback controller synthesis. It follows from [3] that a procedure similar to Theorem 3.1 given below can be used to design reduced-order controllers.

For stating the main result of this section, without loss of generality, consider the following partitioning of $\tilde{P}$ and $\tilde{P}^{-1}$

$$\tilde{P} = \begin{bmatrix} \hat{X} & \hat{M} \\ \hat{M}^T & \hat{U} \end{bmatrix}, \quad \tilde{P}^{-1} = \begin{bmatrix} \hat{Y} & \hat{N} \\ \hat{N}^T & \hat{V} \end{bmatrix}, \quad (11)$$

where $\hat{X}, \hat{Y} \in \mathbb{P}^n$. Next, we define

$$\Pi_1 \triangleq \begin{bmatrix} \hat{Y} & I_n \\ \hat{N}^T & 0_n \end{bmatrix}, \quad \Pi_2 \triangleq \begin{bmatrix} I_n & \hat{X} \\ 0_n & \hat{M}^T \end{bmatrix}. \quad (12)$$

Using $\tilde{P}\tilde{P}^{-1} = I_{\tilde{n}}$, it now follows that $\tilde{P}\Pi_1 = \Pi_2$. With a slight modification of [3, 5], we define the change of controller variables as follows

$$A_K \triangleq \hat{M}A_c\hat{N}^T + \hat{M}B_cC\hat{Y} - \hat{X}BM_1C_c\hat{N}^T + \hat{X}A\hat{Y}, \quad (13)$$

$$B_K \triangleq \hat{M}B_c, \qquad C_K \triangleq C_c\hat{N}^T. \quad (14)$$

In addition, by defining the variables $\bar{A} \triangleq \Pi_2^T\tilde{A}\Pi_1$, $\bar{B} \triangleq \Pi_1^T\tilde{P}\tilde{B}$, $\bar{C} \triangleq \tilde{C}\Pi_1$, $\bar{E} \triangleq \tilde{E}\Pi_1$, $\bar{D} \triangleq \Pi_2^T\tilde{D}$, $\bar{P} \triangleq \Pi_1^T\tilde{P}\Pi_1$, as in [3, 5], we obtain the identities

$$\bar{A} = \begin{bmatrix} A\hat{Y} - BM_1C_K & A \\ A_K & \hat{X}A + B_KC \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ \hat{X}B \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} C_K & 0_{m\times n} \end{bmatrix}, \qquad \bar{E} = \begin{bmatrix} E_1\hat{Y} + E_2C_K & E_1 \end{bmatrix},$$

$$\bar{D} = \begin{bmatrix} D_1 \\ \hat{X}D_1 + B_KD_2 \end{bmatrix}, \bar{P} = \begin{bmatrix} \hat{Y} & I_n \\ I_n & \hat{X} \end{bmatrix}. \quad (15)$$

Before proceeding, note that the variables $\bar{A}, \bar{B}, \bar{C}, \bar{E}, \bar{D}$, and $\bar{P}$ are affine in $(\hat{X}, \hat{Y}, A_K, B_K, C_K)$.

Next, we define $\Phi_b \subset \Phi$ and consider the case such that the input nonlinearity is time-invariant, i.e., $\phi(u, k) = \phi(u)$ and $\phi(u)$ is contained in Φ for a finite range of its argument u as expressed below

$$\phi \in \Phi_b \triangleq \{\phi : \mathbb{R}^m \to \mathbb{R}^m : M_{1_i}u_i^2 \le \phi_i(u)u_i \le M_{2_i}u_i^2, \; \underline{u}_i \le u_i \le \overline{u}_i, \; i = 1, \ldots, m\}, \quad (16)$$

where $\underline{u}_i < 0$ and $\overline{u}_i > 0$, $i = 1, \ldots, m$, are given and correspond to the lower and upper limits, respectively, of u_i. Finally, for $i \in \{1, \ldots, m\}$, we define [7]

$$\begin{aligned} \chi_i &\triangleq \{\tilde{x} \in \mathbb{R}^{\tilde{n}} : \underline{u}_i \le \tilde{C}_i\tilde{x} \le \overline{u}_i\}, \quad \chi \triangleq \cap_{i=1}^m \chi_i, \\ \Psi_i(\alpha) &\triangleq \frac{\tilde{C}_i\hat{\tilde{A}}\tilde{P}^{-1}\hat{\tilde{A}}^T\tilde{C}_i^T\alpha^2}{(\tilde{C}_i\tilde{P}^{-1}\tilde{C}_i^T)(\tilde{C}_i\hat{\tilde{A}}\tilde{P}^{-1}\hat{\tilde{A}}^T\tilde{C}_i^T) - (\tilde{C}_i\hat{\tilde{A}}\tilde{P}^{-1}\tilde{C}_i^T)^2}, \\ V_i^+ &\triangleq \Psi_i(\overline{u}_i), \quad V_i^- \triangleq \Psi_i(\underline{u}_i), \\ V_S &\triangleq \min_{i=1,\ldots,m}\{\min(V_i^+, V_i^-)\}, \\ \mathcal{D}_A &\triangleq \{\tilde{x} \in \chi : V(\tilde{x}) < V_S\}, \end{aligned} \quad (17)$$

where $\tilde{C}_i$ is the i^{th} row of $\tilde{C}$, $\hat{\tilde{A}} \triangleq \tilde{A} + \tilde{B}M_1\tilde{C}$, and $\tilde{P} \in \mathbb{P}^{\tilde{n}}$ satisfies (10) for a given (A_c, B_c, C_c).

Theorem 3.1. Let $m \times m$ diagonal matrices M_1 and M_2 be given such that $M_2 - M_1$ is positive-definite. Furthermore, let $m \times m$ diagonal, positive-definite matrix H and scalar ϵ, $0 < \epsilon < 1$, be given. Suppose there exist $\hat{X}, \hat{Y} \in \mathbb{P}^n$ and (A_K, B_K, C_K) satisfying

$$\begin{bmatrix} \hat{Z}_{11} & \hat{Z}_{12} \\ \hat{Z}_{12}^T & \hat{Z}_{22} \end{bmatrix} < 0, \quad (18)$$

where

$$\hat{Z}_{11} \triangleq \begin{bmatrix} -(1-\epsilon)\bar{P} & \bar{A}^T & \bar{C}^TH \\ \bar{A} & -0.5\bar{P} & 0_{\tilde{n}\times m} \\ H\bar{C} & 0_{m\times\tilde{n}} & -R_0 \end{bmatrix},$$

$$\hat{Z}_{12} \triangleq \begin{bmatrix} 0_{\tilde{n}} & \bar{E}^T \\ 0_{\tilde{n}} & 0_{\tilde{n}\times p} \\ \bar{B}^T & 0_{m\times p} \end{bmatrix}, \quad \hat{Z}_{22} \triangleq \begin{bmatrix} -0.5\bar{P} & 0_{\tilde{n}\times p} \\ 0_{p\times\tilde{n}} & -I_p \end{bmatrix}.$$

In addition, let $\tilde{P}$ and (A_c, B_c, C_c) be given by

$$\tilde{P} = \Pi_2\Pi_1^{-1}, \quad (19)$$

$$A_c = \hat{M}^{-1}[A_K - B_KC\hat{Y} + \hat{X}BM_1C_K - \hat{X}A\hat{Y}]\hat{N}^{-T}, \quad (20)$$

$$B_c = \hat{M}^{-1}B_K, \qquad C_c = C_K\hat{N}^{-T}. \quad (21)$$

Then $\tilde{P}$ and (A_c, B_c, C_c) satisfy (10) and the zero solution $\tilde{x}(k) \equiv 0$ of the feedback interconnection of linear system with input nonlinearity $\phi(\cdot,\cdot) \in \Phi$ given by (1)–(4) is globally asymptotically stable for all $\phi(\cdot,\cdot) \in \Phi$. If $\phi(\cdot) \in \Phi_b$ then the zero solution $\tilde{x}(k) \equiv 0$ of the closed-loop system (1)–(4) is locally asymptotically stable and $\mathcal{D}_A$ defined by (17) is a subset of the domain of attraction of the closed-loop system. Finally, for all $\phi \in \Phi$, the auxiliary cost $\mathcal{J}(\tilde{P}, A_c, B_c, C_c)$ satisfies $\mathcal{J}(\tilde{P}, A_c, B_c, C_c) <$ tr $\tilde{Q}$, where $\tilde{Q} \in \mathbb{P}^d$ is such that the

LMI variables $\hat{X}, \hat{Y} \in \mathbb{P}^n$ and B_K satisfying (18) additionally satisfy

$$\begin{bmatrix} \tilde{Q} & \tilde{D}^T \\ \tilde{D} & \tilde{P} \end{bmatrix} > 0. \qquad (22)$$

Remark 3.1. LMI (18) in Theorem 3.1 is obtained by writing (10) as $(*) + \Sigma_1^T X^{-1} \Sigma_2 + \Sigma_2^T X^{-1} \Sigma_1 < 0$ and using the fact that $(\Sigma_1 - \Sigma_2)^T X^{-1} (\Sigma_1 - \Sigma_2) \geq 0$ (for $X > 0$) to yield a sufficient condition that guarantees the existence of variables satisfying (10). Finally, LMI (18) is obtained using the Schur complement [1] and the linearizing change of controller variables defined above.

Remark 3.2. It is important to note that the estimate of the domain of attraction $\mathcal{D}_{\mathrm{A}}$ given by (17) for the closed-loop system (1)–(4) is predicated on open Lyapunov surfaces. Specifically, since $\tilde{C}_i \tilde{B} \neq 0$, $i \in \{1, \ldots, m\}$, it follows from [7] that an estimate of the domain of attraction can be constructed using open Lyapunov surfaces which yields considerable improvement over domains of attraction predicated on closed Lyapunov surfaces. See [9] for a detailed discussion on the distinction between open versus closed Lyapunov surfaces for estimating domains of attraction.

Remark 3.3. A key application of Theorem 3.1 is the case in which $\phi(u(k), k)$ represents a vector of time-invariant, component-decoupled, saturation nonlinearities. Specifically, let $\phi(u(k)) = [\phi_1(u_1(k)), \ldots, \phi_m(u_m(k))]^T$, $k \in \mathcal{N}$, where $\phi_i(u_i(k))$, for $i \in \{1, \ldots, m\}$, is characterized by

$$\begin{aligned} \phi_i(u_i(k)) &= u_i(k), & |u_i(k)| \leq a_i, \\ \phi_i(u_i(k)) &= a_i \ \mathrm{sgn}(u_i(k)), & |u_i(k)| > a_i. \end{aligned} \qquad (23)$$

In this case, Theorem 3.1 can be used to guarantee asymptotic stability of the closed-loop system (1)–(4) for all $\phi(\cdot) \in \Phi_{\mathrm{b}}$ with a guaranteed domain of attraction. In particular, if $M_1 > 0$, $M_2 = I \geq M_1 > 0$, and there exist $\hat{X}, \hat{Y} \in \mathbb{P}^n$ and (A_K, B_K, C_K) satisfying (18) and thus $\tilde{P} \in \mathbb{P}^{\tilde{n}}$ satisfying (10), then take $\overline{u}_i = -\underline{u}_i = \frac{a_i}{M_{1_i}}$, $i = 1, \ldots, m$, in (16).

4. Numerical Algorithm for Dynamic Output Feedback Control of Systems with Actuator Nonlinearities

In this section, we present a numerical algorithm for the dynamic output feedback control of discrete-time systems with actuator nonlinearities. Following the approach of [4, 14], we decompose the problem of feasible stability multiplier determination (matrix H) and optimal control design (A_c, B_c, C_c) into two LMI subproblems. This enables us to exploit the computational advantage afforded by the convex formulation of the LMI-based feasibility and optimization problems. The basic structure of the numerical algorithm used is as follows.

Algorithm 4.1. To design a dynamic output feedback controller for discrete-time systems with time-varying, sector-bounded nonlinearities, carry out the following procedure:

Step 1. Obtain an initial stabilizing controller (A_c, B_c, C_c) using, e.g., the LQG scheme.

Step 2. Beginning with some initial values of M_1 and M_2 and the current controller gain, for $\phi(\cdot,\cdot) \in \Phi$, solve the feasibility problem involving LMI (10) in variables $\tilde{P} \in \mathbb{P}^{\tilde{n}}$ and $m \times m$ diagonal, positive-definite matrix H.

Step 3. With matrix H obtained in step 2, for $\phi(\cdot,\cdot) \in \Phi$, minimize tr $\tilde{Q}$ subject to LMIs (18) and (22) in variables $\hat{X}$, $\hat{Y} \in \mathbb{P}^n$, $A_K \in \mathbb{R}^{n \times n}$, $B_K \in \mathbb{R}^{n \times l}$, $C_K \in \mathbb{R}^{m \times n}$, and $\tilde{Q} \in \mathbb{P}^d$.

Step 4. Compute (A_c, B_c, C_c) using (20), (21). Now, vary M_1 and M_2 to represent larger sector nonlinearities, then repeat the above procedure (steps 2, 3) until feasible solutions are found for the target values of M_1 and M_2, or until no feasible solution is found.

Step 5. For $\phi(\cdot,\cdot) \in \Phi_{\mathrm{b}}$, compute an estimate of domain of attraction using (17).

5. Illustrative Numerical Example

In this section, we provide an illustrative numerical example to demonstrate the proposed framework for designing actuator amplitude saturation controllers.

The following state equations describe the longitudinal dynamics of the F-8 aircraft [16]

$$\dot{x}(t) = \begin{bmatrix} -0.8 & -0.0006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t) - \begin{bmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{bmatrix} \phi(u(t)), \ x(0) = x_0, \ t \geq 0, \quad (24)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} x(t). \qquad (25)$$

Discretization of the above dynamics with sampling period $T_s = 0.1$ sec yields

$$x(k+1) = \begin{bmatrix} 0.8695 & -0.0001 & -1.0485 & 0.0001 \\ -0.2315 & 0.9986 & -1.4536 & -3.2178 \\ 0.0874 & -0.0000 & 0.8084 & 0.0000 \\ 0.0943 & -0.0000 & -0.0551 & 1.0000 \end{bmatrix} x(k) - \begin{bmatrix} -1.7822 & -0.2533 \\ 0.0951 & 0.0120 \\ -0.1017 & -0.0593 \\ -0.0913 & -0.0135 \end{bmatrix} \phi(u(k)), \ x(0) = x_0, \ k \in \mathcal{N}, \ (26)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} x(k), \qquad (27)$$

where the amplitude saturation nonlinearity $\phi(u(k))$, $k \in \mathcal{N}$, is given by (16) with $i = 2$ and $a_1 = a_2 = 3.5$. The performance variable z is given by

$$z(k) = \begin{bmatrix} \mathrm{diag}(0,1,0,1) \\ 0_{2\times 4} \end{bmatrix} x(k) + \begin{bmatrix} 0_{4\times 2} \\ \mathrm{diag}(0.01, 0.01) \end{bmatrix} u(k). (28)$$

Next, we select the design variables $D_1 = [I_4 \; 0_{4\times2}]$, $D_2 = [0_{2\times4} \; I_2]$, and target M_1 and M_2 to be $0.9I_2$ and I_2, respectively. We initialize the Algorithm 4.1 with an LQG controller corresponding to the state weighting matrix $R_1 = E_1^T E_1$, the control weighting matrix $R_2 = E_2^T E_2$, with E_1 and E_2 given as in (28), the plant noise intensity $V_1 = D_1 D_1^T$, and the measurement noise intensity $V_2 = D_2 D_2^T$. For this design data, we computed an optimal controller using Algorithm 4.1. Finally, we computed an LQG controller for the state and control weighting matrices given by (28) and the plant and measurement noise intensities given by $V_1 = D_1 D_1^T$ and $V_2 = D_2 D_2^T$, respectively.

To illustrate the closed-loop behavior of the system let $x_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$. For the controller designed using Algorithm 4.1, the guaranteed domain of attraction is computed via the stability analysis subproblem and is given by $\mathcal{D}_{\rm A} = \{x : V(x) < 7.1663 \times 10^7\}$. Note that $V(x_0) = 3.6868 \times 10^7$ so that $x_0 \in \mathcal{D}_{\rm A}$. It can be seen from Figure 1 that the controller designed by Algorithm 4.1 results in an asymptotically stable system while the LQG controller in the presence of an input saturation nonlinearity yields unstable response.

6. Conclusion

In this paper, we developed a dynamic, output feedback control design framework for discrete-time systems with time-varying, sector-bounded, actuator nonlinearities via weighted circle criterion. Our results are directly applicable to systems with saturating actuators and provide guaranteed domains of attraction. The technical difficulties associated with the NMIs involving nonlinear terms in the decision variables were nullified by developing an LMI-based sufficient condition for actuator saturation control. A numerical algorithm was developed to exploit the computational advantage afforded by the convex formulation of the LMI conditions. Finally, the effectiveness of the design approach was demonstrated via a numerical example.

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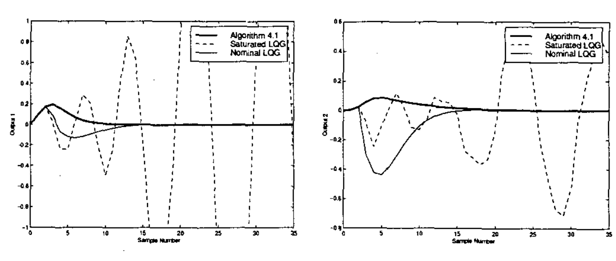

Figure 1: Comparison of LQG, Saturated LQG, and Algorithm 4.1 Designs