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REGRESSIONS FOR SUMS OF SQUARES OF SPACINGS

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Abstract. Starting with a new formula for the regression of sum of squares of spacings (SSS) with respect to the maximum we present a characterization of a family of beta type mixtures in terms of the constancy of regression of normalized SSS of order statistics. Related characterization for records describes a family of minima of independent Weibull distributions.

Key words and phrases: Sum of squares of spacings, regression characterizations, mixtures, beta distribution, Weibull distribution, order statistics, record values.

1. Introduction

If $X_{1:n}, \dots, X_{n:n}$ are the order statistics of a random sample $X_1, \dots, X_n$ from a continuous distribution then $V_i = X_{i:n} - X_{i-1:n}$, $i = 2, \dots, n$, are called spacings of the order statistics. The spacings play an important role in statistics and applied probability. Sukhatme (1936) and Greenwood (1946) are two of the earliest references in this regard. We refer to Misra and van der Meulen (2003) and the references therein for some recent work on the distribution theory and reliability-theoretic aspects of spacings. If the distribution sampled has bounded support $[a, b]$ then an important statistic is Greenwood's $G = \sum_{i=1}^{n+1} V_i^2$; $X_{0:n} = a$, $X_{n+1:n} = b$. This statistic is particularly useful in testing goodness of fit when primary interest is in detecting discrepancies between density functions; see Kirmani and Alam (1974). If F is the distribution function (df) sampled, F_0 a specified continuous df and $H_0 : F = F_0$ the goodness-of-fit hypothesis, let $U_{i:n} = F_0(X_{i:n})$, $i = 0, 1, \dots, n+1$, with $X_{0:n} = -\infty$ and $X_{n+1:n} = \infty$. If H_0 is true and $D_i = U_{i:n} - U_{i-1:n}$, then $E(D_i) = 1/(n+1)$, so that

$$\sum_{i=1}^{n+1} [D_i - E(D_i)]^2 = G - \frac{1}{n+1},$$

which shows the relevance of G in testing H_0. This, of course, is one of many situations where the sum of squares of spacings (SSS) enters in a natural manner. It does, however, suggest the question of predicting SSS. Another situation, where distributional properties of the SSS are very useful, arises for instance in investigations of Poisson driven sequences of observations as noted in Kirmani and Wesołowski (2003).

We show in Section 2 that the best mean square error predictor of SSS through the largest order statistic has a rather interesting form in terms of the distribution function sampled. This form leads to an intriguing result (Proposition 2.1) for the beta $B(p, 1)$ distribution. On a more general note, our expression for the best predictor of SSS opens new possibilities for exploring the extent to which the form of this predictor determines the distribution sampled. We study the case $n = 2$ in detail.

It is shown in Section 3 that, for $n = 2$, constancy of regression of normalized SSS on the maximum characterizes a family of beta type mixtures. The case $n > 2$ is out of our reach at present. Related investigations for records are presented in Section 4. Rather unexpectedly they lead to characterizations of a family of distributions of minima of two independent Weibull rv's.

The results we present here fall in the wide area of characterizations of probability distributions via regression properties of ordered statistics and records. The area develops rapidly in recent years. To have a wider perspective one can consult some of recent papers as for example: Ahsanullah (2000), Ahsanullah and Nevzorov (2000), Asadi *et al.* (2001), Dembińska (2001), Dembińska and Wesołowski (2000, 2003), Ferguson (2002), Franco and Ruiz (2001), Gupta and Wesołowski (2001), López-Blázquez and Wesołowski (2001, 2004), Raquab (2002), Wesołowski and Ahsanullah (2001), Wu (2000, 2001*a*, 2001*b*), Wu and Lee (2001).

2. Regression for sum of squares of spacings (on $X_{n:n}$) and the beta $B(p, 1)$ distribution

While studying distributional properties of Poisson driven sequences of observations, we (Kirmani and Wesołowski (2003)) obtained the following concise formula for independent identically distributed (iid) random variables (rv's) $U_1, \ldots, U_n$ assuming values in $[0, a]$ and having a continuous df H:

$$(2.1)\qquad E\left[U_{1:n}^2 + \sum_{k=1}^{n-1}(U_{k+1:n} - U_{k:n})^2 + (a - U_{n:n})^2\right] = 2\iint_{0<x<y<a} [H(x) + 1 - H(y)]^n dx dy.$$

Let $X_1, X_2, \ldots, X_n$ be iid square integrable positive rv's with a continuous df F. Then, according to the general formula (2.1) and the fact that for any $x > 0$ the conditional distribution of $(X_{1:n}, \ldots, X_{n-1:n})$ given $X_{n:n} = x$ is the same as the distribution of $(Y_{1:n-1}, \ldots, Y_{n-1:n-1})$ for iid observations $Y_1, \ldots, Y_{n-1}$ having the df

$$F_x(y) = \begin{cases} 0, & y < 0, \\ \frac{F(y)}{F(x)}, & y \in [0, F(x)), \\ 1, & y \geq F(x), \end{cases}$$

see for instance Nevzorov (2001), it follows that

$$(2.2)\qquad E[X_{1:n}^2 + (X_{2:n} - X_{1:n})^2 + \cdots + (X_{n:n} - X_{n-1:n})^2 \mid X_{n:n}] = \frac{2}{[F(X_{n:n})]^{n-1}} \iint_{0<x<y<X_{n:n}} [F(x) + F(X_{n:n}) - F(y)]^{n-1} dx dy.$$

The formula is a starting point of our investigation of families of distributions with constant regression of the normalized SSS. We show below that any beta distribution $B(p, 1)$ with the density $f(x) = px^{p-1}I_{(0,1)}(x)$ has this property.

PROPOSITION 2.1. *If $X_1, \ldots, X_n$ are iid beta $B(p, 1)$ rv's then*

$$(2.3)\qquad E[X_{1:n}^2 + (X_{2:n} - X_{1:n})^2 + \cdots + (X_{n:n} - X_{n-1:n})^2 \mid X_{n:n}] = \alpha(p, n)X_{n:n}^2,$$

where

$$\alpha(p,n) = \frac{2}{p}\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{B\left(k+\frac{2}{p}, n-k\right)}{kp+1}.$$

PROOF. Using (2.2) we have for any $z \in (0,1)$

$$\begin{aligned} E[X_{1:n}^2 + (X_{2:n} - X_{1:n})^2 + \cdots + (X_{n:n} - X_{n-1:n})^2 \mid X_{n:n} = z] \\ = \frac{2}{z^{(n-1)p}} \int_0^z \int_0^y [x^p + z^p - y^p]^{n-1} dx dy. \end{aligned} \tag{2.4}$$

To compute the last integral we use the Newton formula:

$$\begin{aligned} \int_0^z \int_0^y [x^p + z^p - y^p]^{n-1} dx dy &= \int_0^z \sum_{k=0}^{n-1} \binom{n-1}{k} \int_0^y x^{kp}(z^p - y^p)^{n-1-k} dx dy \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{1}{kp+1} \int_0^z y^{kp+1}(z^p - y^p)^{n-1-k} dy. \end{aligned}$$

Now, introducing in the last integral a new variable by $y^p = z^p t$ we get

$$\begin{aligned} \int_0^z y^{kp+1}(z^p - y^p)^{n-1-k} dy &= \int_0^1 (zt^{1/p})^{kp+1} z^{p(n-1-k)} (1-t)^{n-1-k} \frac{z}{p} t^{1/p-1} dt \\ &= \frac{z^{2+(n-1)p}}{p} \int_0^1 t^{k+2/p-1}(1-t)^{n-1-k} dt \\ &= B\left(k + \frac{2}{p}, n-k\right) \frac{z^{2+(n-1)p}}{p}. \end{aligned}$$

Plugging it back into (2.4) we arrive at (2.3). □

If we divide both sides of (2.3) by $X_{n:n}^2$ we see that the regression of the normalized (by $X_{n:n}^2$) SSS in the case of the beta $B(p,1)$ distribution is constant.

In particular for $n = 2$ we get

$$E\left[\frac{X_{1:2}^2 + (X_{2:2} - X_{1:2})^2}{X_{2:2}^2} \,\middle|\, X_{2:2}\right] = 1 - \frac{2p}{(p+1)(p+2)}. \tag{2.5}$$

Since the function

$$h(p) = \frac{2p}{(p+1)(p+2)}, \qquad p > 0,$$

attains its maximum value $2(3 - 2\sqrt{2})$ at $p = \sqrt{2}$ it follows that $\alpha(p,2) \in [4\sqrt{2} - 5, 1)$.

Observe also that for $p = 1$, i.e. for the uniform distribution one gets

$$E\left[\frac{X_{1:2}^2 + (X_{2:2} - X_{1:2})^2}{X_{2:2}^2} \,\middle|\, X_{2:2}\right] = \frac{2}{3}. \tag{2.6}$$

It might be worthy to notice that $\alpha(1,2) = 2/3 \approx 0.66667$ is very close to the lower bound $\alpha(\sqrt{2},2) = 4\sqrt{2} - 5 \approx 0.65685$.

3. Characterizations of mixtures of beta type distributions

In this section we will identify the family of distributions on the positive half axis having the property of constancy of regression of normalized SSS. It appears that the family is much wider that just beta $B(p,1)$ distributions. However this type of the beta distribution is the main building element of the family.

THEOREM 3.1. *Let X_1, X_2 be iid rv's with a continuous df F and the support $[0, r_F]$, where $r_F = \inf\{x > 0 : F(x) = 1\} \leq \infty$. Assume that*

$$E\left(\frac{X_{1:2}^2 + (X_{2:2} - X_{1:2})^2}{X_{2:2}^2} \,\middle|\, X_{2:2}\right) = c, \tag{3.1}$$

where c is a constant.

Then up to a positive scale the df F is a mixture of the beta $B(b,1)$ df F_1 and a df F_2, i.e.

$$F = \alpha F_1 + (1-\alpha)F_2$$

for some $\alpha \in [0,1]$ and only the following two cases are possible: **either** $c = 4\sqrt{2} - 5$ *(i.e. $a = b$, see below) and then the df F_2 is of the form*

$$F_2(z) = \begin{cases} 0, & z < 0, \\ z^{\sqrt{2}}(1 - \sqrt{2}\log(z)), & z \in [0,1], \\ 1, & z > 1, \end{cases}$$

or $c \in (4\sqrt{2} - 5, 1)$ *(i.e. $a < b$, see below) and then the df F_2 is of the form*

$$F_2(z) = \begin{cases} 0, & z < 0, \\ \frac{bz^a - az^b}{b-a}, & z \in [0,1], \\ 1, & z > 1, \end{cases}$$

where

$$b = \frac{3c - 1 + \sqrt{c^2 + 10c - 7}}{2(1-c)}, \qquad a = \frac{3c - 1 - \sqrt{c^2 + 10c - 7}}{2(1-c)} > 0. \tag{3.2}$$

PROOF. Since

$$\frac{1}{2} \leq \frac{x^2 + (y-x)^2}{y^2} < 1, \qquad 0 < x \leq y,$$

then it follows that $c \in [0.5, 1)$.

Since $F(z) = 0$ for $z \leq 0$ and $F(z) = 1$ for $z \geq r_F$, where r_F is the right end of the support of the distribution, we consider below only the arguments $z \in (0, r_F)$. From (2.4) with $n = 2$ it follows that (3.1) is equivalent to

$$2\int_0^z \left(\int_0^y [F(x) + F(z) - F(y)]dx\right) dy = cz^2 F(z). \tag{3.3}$$

Thus F is differentiable in $(0, r_F)$ and differentiating the above formula we get

$$2\int_0^z F(x)dx = 2czF(z) + (c-1)z^2 F'(z).$$

Hence F is twice differentiable. Differentiating again we obtain the equation

$$z^2F''(z) + \frac{2(2c-1)}{c-1}zF'(z) + 2F(z) = 0. \tag{3.4}$$

Introduce a new function $u = u(z)$ by $F(z) = \exp(u(z))$. Then the differential equation (3.4) assumes the form

$$z^2u''(z) + [zu'(z)]^2 + \frac{2(2c-1)}{c-1}zu'(z) + 2 = 0.$$

Define $v(z) = zu'(z)$. Then $z^2u''(z) = zv'(z) - v$ and the above equation can be written as

$$-zv'(z) = [v(z)]^2 + \frac{3c-1}{c-1}v(z) + 2. \tag{3.5}$$

Since the discriminant of the quadratic at the rhs of (3.5) is

$$\Delta = \frac{c^2 + 10c - 7}{(1-c)^2},$$

then $\Delta < 0$ iff $c \in [0.5, 4\sqrt{2} - 5)$ and $\Delta \geq 0$ iff $c \in [4\sqrt{2} - 5, 1)$ since, as observed earlier, $c \in [0.5, 1)$.

In the first case solving the differential equation (3.5) and returning to the density $f = F'$ we obtain

$$f(z) = F(z)z[\beta + \alpha \tan(D - \alpha \log(z))],$$

where

$$\alpha = -\Delta = \frac{7 - 10c - c^2}{(1-c)^2} > 0, \qquad \beta = \frac{3c-1}{1-c} > 0,$$

and D is a real constant. Note that f satisfying the above equation cannot be a density function since the rhs assumes negative values in the set of positive Lebesgue measure in the right neighborhood of zero.

Thus only the case $c \in [4\sqrt{2} - 5, 1)$ is possible.

Consider first the case $c = 4\sqrt{2} - 5$. Then solving (3.5) we arrive at the equation

$$\frac{F'(z)}{F(z)} = \frac{\sqrt{2}}{z} + \frac{1}{z(D_1 + \log(z))},$$

where $D_1 \in [-\infty, \infty]$ is a constant. Hence for $|D_1| < \infty$

$$F(z) = D_2 z^{\sqrt{2}} |D_1 + \log(z)|,$$

where D_2 is a constant. Note that since F is a df then necessarily $r_F < \infty$. Without any loss of generality we assume that $r_F = 1$. Then, since $F(1) = 1$, it follows that $D_2 = 1/|D_1|$. Thus necessarily $D_1 < 0$ and writing $d = -1/D_1$ we have

$$F(z) = z^{\sqrt{2}}(1 - d\log(z)).$$

Note that $0 < d \leq \sqrt{2}$, since otherwise the function F is not non-decreasing. The case $|D_1| = \infty$ allows as to write $0 \leq d \leq \sqrt{2}$. It is an elementary exercise to check that any df of the above form satisfies (3.3), i.e. proceeding via differentiation procedure we remained within the set of solutions of our original problem.

Denoting $\alpha = 1 - d/\sqrt{2} \in [0,1]$, the above formula can be written as

$$F(z) = \alpha z^{\sqrt{2}} + (1-\alpha) z^{\sqrt{2}} (1 - \sqrt{2}\log(z)),$$

where $\alpha \in [0,1]$, which proves the first part of the assertion since $a = b = \sqrt{2}$ (see (3.2)).

If $c \in (4\sqrt{2} - 5, 1)$ the solution of (3.5) written in terms of the density f has the form

$$f(z) = F(z) \frac{bz^{b-1} - D_1 a z^{a-1}}{z^b - D_1 z^a},$$

where a and b are defined in (3.2) and $D_1 \in [-\infty, \infty]$ is a constant. Therefore for $|D_1| < \infty$

$$F(z) = D_2 z^a |z^{b-a} - D_1|,$$

where D_2 is a constant. Similarly as in the previous case, we have to have $r_F < \infty$ and again, without loosing generality, we take $r_F = 1$. Obviously, then we have to have $D_1 \neq 1$ and $D_2 = 1/|1 - D_1| > 0$ and thus

$$F(z) = \frac{z^a |z^{b-a} - D_1|}{|1 - D_1|}.$$

Consider first the case $D_1 \leq 0$. Then for $d = -D_1 \geq 0$ we have

$$F(z) = \frac{z^b + dz^a}{1+d} = \alpha_1 z^b + (1-\alpha_1) z^a$$

with $\alpha_1 = 1/(1+d) \in (0,1]$. The case $|D_1| = d = \infty$ allows to write $\alpha_1 \in [0,1]$, and thus F is a df of a mixture of beta distributions $B(a,1)$ and $B(b,1)$. Note that defining α by $(1-\alpha)b = (1-\alpha_1)(b-a)$ we have $\alpha \in [0,1]$ and

$$F(z) = \alpha z^b + (1-\alpha)\frac{bz^a - az^b}{b-a} = \alpha F_1(z) + (1-\alpha) F_2(z).$$

If $D_1 > 0$ then, since F is non-decreasing, we have to have $D_1 > 1$. Then

$$F(z) = \frac{D_1 z^a - z^b}{D_1 - 1} \quad \text{and} \quad f(z) = \frac{D_1 a z^{a-1} - bz^{b-1}}{D_1 - 1},$$

which is nonnegative iff $D_1 \geq b/a > 1$. Thus

$$F(z) = \frac{D_1 z^a - z^b}{D_1 - 1},$$

which can be written as

$$F(z) = \alpha F_1(z) + (1-\alpha) F_2(z),$$

for

$$\alpha = 1 - \frac{D_1(b-a)}{(D_1 - 1)b} \in \left[0, \frac{a}{b}\right],$$

since $D_1 = \infty$ is also allowed.

Again, on noting that for a and b defined by (3.2) one has

$$\frac{a}{(a+1)(a+2)} = \frac{b}{(b+1)(b+2)},$$

it is elementary to check that the above df satisfies (3.3), and thus this is the general form of the solution of our problem in the second case. □

Observe that the case of uniform distribution is included in Theorem 3.1 by specifying $c = 2/3$ (then $a = 1$, $b = 2$) and $\alpha = 1/2$. Also if $\alpha > a/b$ then the mixture characterized in Theorem 3.1 can be represented as a mixture of beta $B(a,1)$ and beta $B(b,1)$ distributions, with new mixing coefficients $(1-\alpha)b/(b-a)$ and $(\alpha b - a)/(b-a)$, respectively.

4. Characterization of minima of Weibull distributions

For a sequence (X_i) of iid non-negative rv's with a continuous df F and $R(x) = -\log(1-F(x))$ for $x \in (0, r_F)$ let (R_n) denote the record sequence starting with $R_1 = X_1$. Since the conditional distribution of $(R_1, \ldots, R_{n-1})$ given $R_n = z > 0$ is the same as the joint distribution of order statistics from a sample of size n from the df defined by

$$F_z(x) = \begin{cases} 0, & x < 0, \\ \frac{R(x)}{R(z)}, & x \in [0, z), \\ 1, & x \geq z, \end{cases}$$

(see for instance Lemma 4.3.3 in Arnold *et al.* (1998)) then it follows from (2.1) that the regression of SSS of first n records on the n-th record has the form

$$E\left[R_1^2 + \sum_{k=1}^{n-1}(R_{k+1} - R_k)^2 \;\middle|\; R_n\right] = \frac{2}{[R(R_n)]^{n-1}} \int_{0<x<y<R_n} [R(x) + R(R_n) - R(y)]^{n-1} dx dy.$$

Consider now a special case of a sequence (X_i) of iid Weibull $W(\alpha, p)$ rv's with the df $F(x) = [1 - \exp(-\alpha x^p)]I_{(0,\infty)}(x)$, with the parameters $\alpha, p > 0$. Then $R(x) = \alpha x^p I_{(0,\infty)}(x)$ and thus, since α plays a role of a scale only, immediately from the above formula and considerations of Section 2 we find out that in this case the following analogue of (2.5) holds

$$E[R_1^2 + (R_2 - R_1)^2 + \cdots + (R_n - R_{n-1})^2 \mid R_n] = \alpha(p,n) R_n^2, \tag{4.1}$$

where $\alpha(p,n)$ is defined in Proposition 2.1. Taking $n = 2$ we have, as in Section 2, that

$$E\left[\frac{R_1^2 + (R_2 - R_1)^2}{R_2^2} \;\middle|\; R_2\right] = 1 - \frac{2p}{(p+1)(p+2)}. \tag{4.2}$$

Note that for $p = 1$, i.e. for the exponential distribution the constant at the right hand side of (4.2) is 2/3—it was the case of uniform distribution for the SSS of the first two order statistics.

Similarly as in Section 3 we are unable to study the converse of (4.1) in full generality. Instead we are concerned only with the first two records i.e. with the question, if (4.2) describes a property which is characteristic for the Weibull (exponential) distributions. It appears that a wider family of minima of Weibull (exponential) distributions is identified by this property.

THEOREM 4.1. *Let* (X_i) *be a sequence of iid rv's with a continuous df and the support* $[0, r_X]$, $r_X \le \infty$. *Assume that*

$$E\left[\frac{R_1^2 + (R_2 - R_1)^2}{R_2^2} \;\middle|\; R_2\right] = c,$$

where c *is a constant.*

Then $c \in [4\sqrt{2} - 5, 1)$, $r_X = \infty$ *and*

$$X \stackrel{d}{=} \min\{W_1, W_2\},$$

where W_1 *and* W_2 *are independent Weibull rv's* $W(\alpha, a)$ *and* $W(\beta, b)$, *respectively, where* a *and* b *are defined in* (3.2) *and* α *and* β *are some nonnegative constants,* $\alpha + \beta > 0$.

PROOF. Exchanging the roles of R and F in the previous proof we arrive at the equation (3.5) with $R(z) = e^{u(z)}$, $v(z) = zu'(z)$. Then, similarly as earlier, we observe that necessarily $c \in [4\sqrt{2} - 5, 1)$.

Considering first the case $c = 4\sqrt{2} - 5$, as in the previous proof, we see that, since $\lim_{z\to\infty} R(z) = \infty$ and R is non-decreasing in $[0, r_F)$, we have to have $R(z) = Dz^{\sqrt{2}}$, $D > 0$ (only $|D_1| = \infty$ is admissible) and $r_F = \infty$. Consequently, X_i's are Weibull with the df

$$F(z) = (1 - e^{-Dz^{\sqrt{2}}})I_{(0,\infty)}(z)$$

and $a = b = \sqrt{2}$, $\alpha = \beta = D$.

If $c \in (4\sqrt{2} - 5, 1)$ then again following the previous proof we find out that

$$R(z) = D_2 z^a |z^{b-a} - D_1|, \qquad z \in (0, r_F).$$

Since R is nondecreasing, necessarily $D_1 \le 0$ (this time the case $D_1 > 0$ is impossible) and thus $r_F = \infty$. Finally, we conclude that the df has the form

$$F(z) = (1 - e^{-\alpha z^a - \beta z^b})I_{(0,\infty)}(z),$$

which is a df of minimum of two independent Weibull $W(\alpha, a)$ and $W(\beta, b)$ random variables, where $\alpha, \beta \ge 0$ and are not zero together.

For the above F on returning to R one can easily check that the analogue of (3.3) is satisfied and thus F gives the complete solution of our problem. □

Similarly as in the previous section we observe easily that the exponential distribution is included in the statement of Theorem 4.1 by specifying $c = 2/3$ (then $a = 1$, $b = 2$) and $\beta = 0$.

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