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Maximal Domination and Maximal Total Domination in Digraphs

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ABSTRACT

Let $D=(V, A)$ be a digraph. A dominating set S of a digraph D is a maximal dominating set if $V-S$ is not a dominating set of D. The maximal domination number $\gamma_m(D)$ of D is minimum cardinality of a maximal dominating set of D. A total dominating set S of a digraph D is a *maximal* total dominating set if $V-S$ is not a total dominating set of D. The maximal total domination number $\gamma_{mt}(D)$ of D is the minimum cardinality of a maximal total dominating set of D. In this paper, we initiate a study of maximal domination and maximal total domination in digraphs and obtain some results on these two parameters.

Mathematics Subject Classification: 05*C*.

Keywords: digraph, domination, maximal domination, total domination, maximal total domination.

1. INTRODUCTION

In this paper, $D=(V, A)$ is a finite, directed graph without loops and multiple arcs (but of opposite arcs are allowed) and $G=(V,E)$ is a finite, undirected graph without loops and multiple edges. For basic terminology, we refer to Chartand and Lesniak[2] and Kulli[4].

A set S of vertices in a graph $G=(V, E)$ is a dominating set if every vertex in $V-S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(D)$. Recently many new domination parameters are given in the books by Kulli[4,5,6].

Kulli and Janakiram[8] introduced the concept of maximal domination in graphs as follows:

A set S of vertices of a graph G is a maximal dominating set if S is a dominating set and $V - S$ is not a dominating set. The minimum cardinality of a maximal dominating set of G is called the maximal domination number of G and is denoted by $\gamma_m(G)$. This concept is studied in[7, 9, 10, 11].

Let $G = (V, E)$ be a graph without isolated vertices. A dominating set S of V is a total dominating set of G if the induced subgraph $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of G is called the total domination number of G and is denoted by $\gamma_t(D)$, (see[3]).

A total dominating set S of a graph G is a maximal total dominating set if $V - S$ is not a total dominating set of G. The minimum cardinality of a maximal total dominating set of G is called the maximal total domination number of G and is denoted by $\gamma_{mt}(G)$ (see[11]).

Let $D = (V, A)$ be a digraph. For any vertex $u \in V$, the sets $I(u)= \{v/(v,u) \in A\}$ and $O(u)= \{v/(u,v) \in A\}$ are called the inset and outset of u. The indegree and outdegree of u are defined by $id(u)=|I(u)|$ and $od(u)=|O(u)|$.

A set S of vertices in a digraph $D=(V, A)$ is a dominating set if for every vertex u in $V - S$, there exists a vertex v in S such that $(v, u) \in A$. The domination number $\gamma(D)$ of D is the minimum cardinality of a dominating set of D.

Let $D=(V, A)$ be a digraph in which $id(v)+od(v) > 0$ for all $v \in V$. A subset S of V is called a *total* dominating set of D if S is a dominating set of D and the induced subgraph $\langle S \rangle$ has no isolated vertices. The total domination number $\gamma_t(D)$ of D is the minimum cardinality of a total dominating set of D,[1].

In this paper, we introduce the analog of maximal domination number and maximal total domination number in digraphs and establish some results on these parameters.

2. MAXIMAL DOMINATION IN DIGRAPHS

The concept of maximal domination can be extended to digraphs.

Definition 2.1. A dominating set S of a digraph $D=(V, A)$ is a maximal dominating set if $V - S$ is not a dominating set of D. The maximal domination number $\gamma_m(D)$ of a digraph D is the minimum cardinality of a maximal dominating set.

Definition 2.2. A dominating set S of a digraph D is minimal if for each vertex $v \in S$, $S - v$ is not a dominating set of D.

Example 2.3. For the digraph C_3 shown in Figure 1, $S=\{1, 2\}$ is a minimal dominating set and $V - S = \{3\}$ is not a dominating set. Thus S is a maximal dominating set. Hence $\gamma(C_3)= \gamma_m(C_3)=2$.

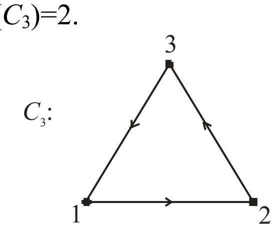

Figure 1

Theorem 2.4. If a digraph D contains an isolated vertex, then a minimal dominating set S of D is a maximal dominating set.

Proof: Let v be an isolated vertex of D. Then v is in every dominating set S of D. Thus $V - S$ is not a dominating set of D. This proves the result.

The converse of Theorem 2.4 is not true always. For example, the directed cycle C_3 (example 2.3) has a maximal dominating set, but it has no an isolated vertex.

Proposition 2.5. For any digraph D,

$$\gamma(D) \le \gamma_m(D) \tag{1}$$

Proof: Clearly, every maximal dominating set of a digraph D is a dominating set of D. Hence (1) holds.

Exact values of $\gamma_m(D)$ for some standard digraphs are given below.

Proposition 2.6. For a directed path P_p with $p \geq 2$ vertices,

$$\gamma_m(P_p) = \frac{p}{2}, \quad \text{if } p \text{ is even},$$
$$= \left\lceil \frac{p}{2} \right\rceil, \text{ if } p \text{ is odd.}$$

Remark 2.7. For any directed path P_p with $p \geq 2$ vertices, $\gamma(P_p) = \gamma_m(P_p)$.

Proposition 2.8. For a directed cycle C_p with $p \geq 3$ vertices,

$$\gamma_m(C_p) = \frac{p}{2} + 1, \text{ if } p \text{ is even},$$
$$= \left\lceil \frac{p}{2} \right\rceil, \text{ if } p \text{ is odd.}$$

Remark 2.9. For a directed cycle C_{2n+1} with $n \geq 1$ vertices, $\gamma(C_{2n+1}) = \gamma_m(C_{2n+1})$.

3. MAXIMAL TOTAL DOMINATION IN DIGRAPHS

The concept of maximal total domination can be extended to digraphs.

Definition 3.1. A total dominating set S of a digraph $D=(V, A)$ is a maximal total dominating set, if $V - S$ is not a total dominating set of D. The maximal total domination number $\gamma_{mt}(D)$ of D is the minimum cardinality of a maximal total dominating set of D.

Definition 3.2. A total dominating set S of a digraph D is minimal if for each vertex $v \in S$, $S - v$ is not a total dominating set of D.

Definition 3.3. For the directed cycle C_4 as shown in Figure 2, $S=\{1, 2, 3\}$ is a minimal total dominating set and $V - S = \{4\}$ is not a total dominating set. Thus S is a maximal total dominating set. Clearly $\gamma_t(C_4) = \gamma_{mt}(C_4) = 3$.

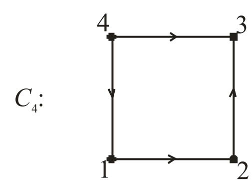

Figure 2

Proposition 3.4. For any digraph D without isolated vertices,

$$\gamma_t(D) \le \gamma_{mt}(D). \tag{2}$$

Proof: Every maximal total dominating set of a digraph D is a total dominating set of D. Thus (2) holds.

A γ_t - set is a minimum total dominating set.

Proposition 3.5. For a directed path P_p with $p \leq 2$ vertices,

$$\gamma_{mt}(P_p) = \left\lceil \frac{2p}{3} \right\rceil.$$

Proof: Let S be a γ_t-set of P_p. Then $\gamma_t(P_p) = \left\lceil \frac{2p}{3} \right\rceil$. Clearly $V - S$ does not contain a total dominating set. Thus

$$\gamma_{mt}(P_p) = \gamma_t(P_p) = \left\lceil \frac{2p}{3} \right\rceil.$$

Proposition 3.6. For a directed cycle C_p with $p \geq 3$ vertices,

$$\gamma_{mt}\left(C_p\right) = \left\lceil \frac{2p}{3} \right\rceil.$$

Proof: Let S be a γ_t-set of C_p. Then $\gamma_t(C_p) = \left\lceil \frac{2p}{3} \right\rceil$. Clearly $V - S$ does not contain a total dominating set. Hence

$$\gamma_{mt}(C_p) = \gamma_t(C_p) = \left\lceil \frac{2p}{3} \right\rceil.$$

Theorem 3.7. For any digraph D with γ_t-set and with an endvertex v such that $id(v) = 1$,

$$\gamma_t(D) = \gamma_{mt}(D). \tag{3}$$

Proof: Let v be an endvertex of a digraph D such that $id(v)=1$ and S be a γ_t-set of D. Then $v \notin S$ and is a successor of some vertex in S. Thus v is an isolated vertex in $\langle V - S\rangle$. Hence S is a γ_{mt}-set of D. Thus (3) holds.

Corollary 3.8. For any directed tree T with γ_t-set and with an endvertex v such that $id(v)=1$,

$\gamma_t(T) = \gamma_{mt}(T)$.

Proof: This follows from Theorem 3.7.

Theorem A[1]. Let $D=(V, A)$ be a weakly connected digraph with $p \geq 3$ vertices. Then $\gamma_t(D)=p$ if and only if there exists a subdigraph $D_1=(V_1, A_1)$ such that all vertices of $V - V_1$ have indegree zero and for each vertex u in V_1, there exists at least one vertex v in $V - V_1$ such that $(v, u) \in A$ and $od(v) = 1$.

Theorem 3.9. Let $D=(V, A)$ be a weakly connected digraph with $p \geq 3$ vertices. Then

$\gamma_{mt}(D) = p$

if and only if there exists a subdigraph $D_1=(V_1, A_1)$ such that all vertices of $V - V_1$ have indegree zero and for each vertex $u \in V_1$, there exists at least one vertex $v \in V - V_1$ such that $(v, u) \in A$ and $od(v)=1$.

Proof: This follows from Theorem 3.4 and Theorem A.

Theorem B[1]. If D is a digraph of order $p \geq 2$ without isolated vertices and maximum outdegree $\Delta^+(D) \geq 1$, then

$$\frac{2p}{2\Delta^+(D)+1} \leq \gamma_t(D).$$

We obtain a lower bound for $\gamma_{mt}(D)$.

Theorem 3.10. If D is a digraph of order $p \geq 2$ without isolated vertices and maximum degree $\Delta^+(D) \geq 1$, then

$$\frac{2p}{2\Delta^+(D)+1} \leq \gamma_{mt}(D). \tag{4}$$

Proof: Suppose D satisfies the condition given in the statement. Then by Theorem B, $\frac{2p}{2\Delta^{+}(D)+1} \leq \gamma_t(D)$. By Theorem 3.4, $\gamma_t(D) \leq \gamma_{mt}(D)$. Thus (4) holds.

4. OPEN PROBLEMS

In this paper, we have introduced maximal domination and maximal total domination in digraphs. Many questions are suggested by this research, among them are the following.

Problem 1. Characterize digraphs D without isolated vertices for which $\gamma(D) = \gamma_m(D)$.

Problem 2. Characterize digraphs D without isolated vertices for which $\gamma_t(D) = \gamma_{mt}(D)$.

Problem 3. Characterize digraphs D for which $\gamma_{mt}(D) = \frac{2p}{2\Delta^{+}(D)+1}$.

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