Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 185249, 16 pages
doi:10.1155/2012/185249

Research Article

Metric Subregularity for Subsmooth Generalized Constraint Equations in Banach Spaces

He Qinghai,[1] Yang Ji,[2] and Zhang Binbin[3]

[1] *Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China*
[2] *Basic Flight Training Base, Aviation University of Air Force, Changchun, Jilin 130022, China*
[3] *School of Science, Kunming University of Science And Technology, Kunming, Yunnan 650500, China*

Correspondence should be addressed to He Qinghai, heqh@ynu.edu.cn

Received 13 October 2012; Accepted 23 November 2012

Academic Editor: Jian-Wen Peng

This paper is devoted to metric subregularity of a kind of generalized constraint equations. In particular, in terms of coderivatives and normal cones, we provide some necessary and sufficient conditions for subsmooth generalized constraint equations to be metrically subregular and strongly metrically subregular in general Banach spaces and Asplund spaces, respectively.

1. Introduction

Let X be a Banach space and $f : X \to \overline{R}$ be a function. Consider the following inequality:

$$f(x) \leq 0. \tag{1.1}$$

Let $S := \{x \in X : f(x) \leq 0\}$. Recall that (1.1) has a local error bound at $a \in S$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, S) \leq \tau \left[f(x)\right]_+ \quad \forall x \in B(a, \delta), \tag{1.2}$$

where $[f(x)]_+ := \max\{f(x), 0\}$ and $B(a, \delta)$ denotes the open ball of center a and radius δ. The error bound has been studied by many authors (see [1–3] and the references therein).

Let Y be another Banach space, $b \in Y$, and let $F : X \rightrightarrows Y$ be a closed multifunction. The following generalized equation:

$$b \in F(x) \tag{GE}$$

concludes most of systems in optimization and was investigated by many researchers (see [4–9] and the references therein). Let $x \in X$ and $b \in F(a)$. Recall that (GE) is metrically subregular at (a, b) if there exist $\tau, \delta \in (0, \infty)$ such that

$$d\left(x, F^{-1}(b)\right) \leq \tau d(b, F(x)) \quad \forall x \in B(a, \delta) \tag{1.3}$$

(see [4–6] and the references therein). This property provides an estimate how far for an element x near a can be from the solution set of (GE). A stronger notion is the metric regularity: a multifunction F is metrically regular at (a, b) if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d\left(x, F^{-1}(y)\right) \leq \tau d(y, F(x)) \quad \forall (x, y) \in B((a, b), \delta). \tag{1.4}$$

There exists a wide literature on this topic. We refer the interested readers to [3, 7–11] and to the references contained therein. Let A be a closed subset of X. Consider the generalized constraint equation as follows:

$$b \in F(x) \quad \text{subject to } x \in A. \tag{GCE}$$

Let S denote the solution set of (GCE), that is, $S = \{x \in A : b \in F(x)\}$. We say that (GCE) is metrically subregular at $a \in S$ if there exist $\tau, \delta \in (0, \infty)$ such that

$$d(x, S) \leq \tau(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta). \tag{1.5}$$

When $A = X$, (GCE) reduces (GE) and (1.5) means that (GE) is metrically subregular at (a, b). When $F(x) = [f(x), +\infty), b = 0$ and $A = X$, (GCE) reduces the inequality (1.1) and (1.5) means that this inequality has a local error bound at a. Error bounds, metric subregularity and regularity have important applications in mathematical programming and have been extensively studied (see [1–12] and the references therein). The Authors [13] introduced the notions of primal smoothness and investigated the properties of primal smooth functions. Under proper conditions, the distance function is primal smooth. Differentiability of the distance function was discussed in [14]. As extension of primal smoothness and convexity, the notion of subsmoothness was introduced and some functional characterizations were provided in [15]. Recently, by variational analysis techniques (for more details, see [16–19]), Zheng and Ng [6] investigated metric subregularity of (GE) under the subsmooth assumption. In this paper, in terms of normal cones and coderivatives, we devote to metric subregularity of generalized constraint equation (GCE) under the subsmooth assumption. We will build some new necessary and sufficient conditions for (GCE) to be metrically subregular and strongly metrically subregular.

2. Preliminaries

Let X be a Banach space. We denote by B_X and X^* the closed unit ball and the dual space of X, respectively. Let A be a nonempty subset of X, $\operatorname{int}(A)$ and $\operatorname{bd}(A)$, respectively, denote the interior and the boundary of A. For $a \in X$ and $\delta > 0$, let $B(a, \delta)$ denote the open ball with center a and radius δ.

We introduce some notions of variations and derivatives needed to state our results.

For a closed subset A of X and $a \in A$, let $T_c(A, a)$ and $T(A, a)$, respectively, denote the Clarke tangent cone and contingent (Bouligand) cone of A at a defined by

$$T_c(A, a) := \liminf_{x \xrightarrow{A} a, t \to 0^+} \frac{A - x}{t}, \qquad T(A, a) := \limsup_{t \to 0^+} \frac{A - a}{t}, \tag{2.1}$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. It is easy to verify that $v \in T_c(A, a)$ if and only if for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for each natural number n; while $v \in T(A, a)$ if and only if there exists a sequence $\{v_n\}$ in X converging to v and a sequence $\{t_n\}$ in $(0, \infty)$ decreasing to 0 such that $a + t_n v_n \in A$ for all n.

We denote by $N_c(A, a)$ the Clarke normal cone of A at a, that is,

$$N_c(A, a) := \{x^* \in X^* : \langle x^*, h \rangle \le 0 \ \forall h \in T_c(A, a)\}. \tag{2.2}$$

For $\varepsilon \ge 0$ and $a \in A$, the nonempty set

$$\widehat{N}_\varepsilon(A, a) := \left\{ x^* \in X^* : \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x - a \rangle}{\|x - a\|} \le \varepsilon \right\} \tag{2.3}$$

is called the set of *Fréchet* ε-normals of A at a. When $\varepsilon = 0$, $\widehat{N}_\varepsilon(A, a)$ is a convex cone which is called the *Fréchet* normal cone of A at a and is denoted by $\widehat{N}(A, a)$. Let $N(A, a)$ denote the Mordukhovich limiting or basic normal cone of A at a, that is,

$$N(A, a) := \limsup_{x \xrightarrow{A} a, \varepsilon \to 0^+} \widehat{N}_\varepsilon(A, x), \tag{2.4}$$

that is, $x^* \in N(A, a)$ if and only if there exist sequences $\{(x_n, \varepsilon_n, x_n^*)\}$ in $A \times R_+ \times X^*$ such that $(x_n, \varepsilon_n) \to (a, 0)$, $x_n^* \xrightarrow{w^*} x^*$ and $x_n^* \in \widehat{N}_{\varepsilon_n}(A, x_n)$ for each natural number n. It is known that

$$\widehat{N}(A, a) \subseteq N(A, a) \subseteq N_c(A, a), \tag{2.5}$$

(see [4, 9, 16, 18, 19] and the references contained therein). If A is convex, then

$$\begin{gathered} T(A, a) = T_c(A, a), \\ \widehat{N}(A, a) = N(A, a) = N_c(A, a) = \{x^* \in X^* : \langle x^*, x - a \rangle \le 0 \ \forall x \in A\}. \end{gathered} \tag{2.6}$$

Recall that a Banach space X is called an Asplund space if every continuous convex function on X is *Fréchet* differentiable at each point of a dense subset of X (for other definitions and their equivalents, see [19]). It is well known that X is an Asplund space if and only if every separable subspace of X has a separable dual space. In particular, every reflexive Banach space is an Asplund space. When X is an Asplund space, it is well known that

$$N_c(A,a) = \mathrm{cl}^*(\mathrm{co}(N(A,a))), \qquad N(A,a) = \limsup_{x \xrightarrow{A} a} \widehat{N}(A,x), \tag{2.7}$$

where $\mathrm{cl}^*(\cdot)$ denotes the closure with respect to the weak* topology, see [9, 19]. Recently, Zheng and Ng [5] established an approximate projection result for a closed subset of X, which will play a key role in the proofs of our main results.

Lemma 2.1. *Let A be a closed nonempty subset of a Banach space X and let $\beta \in (0,1)$. Then for any $x \notin A$ there exist $a \in \mathrm{bd}(A)$ and $a^* \in N_c(A,a)$ with $\|a^*\| = 1$ such that*

$$\beta\|x-a\| < \min\{d(x,A), \langle a^*, x-a\rangle\}. \tag{2.8}$$

If X is an Asplund space, then $N_c(A,a)$ can be replaced by $\widehat{N}(A,a)$.

Let $F : X \rightrightarrows Y$ be a multifunction and let $\mathrm{Gr}(F)$ denote the graph of F, that is,

$$\mathrm{Gr}(F) := \{(x,y) \in X \times Y : y \in F(x)\}. \tag{2.9}$$

As usual, F is said to be closed (resp., convex) if $\mathrm{Gr}(F)$ is a closed (resp., convex) subset of $X \times Y$. Let $(x,y) \in \mathrm{Gr}(F)$. The Clarke tangent and contingent derivatives $D_cF(x,y), DF(x,y)$ of F at (x,y) are defined by

$$\begin{aligned} D_cF(x,y)(u) &:= \{v \in Y : (u,v) \in T_c(\mathrm{Gr}(F),(x,y))\} \quad \forall u \in X, \\ DF(x,y)(u) &:= \{v \in Y : (u,v) \in T(\mathrm{Gr}(F),(x,y))\} \quad \forall u \in X, \end{aligned} \tag{2.10}$$

respectively. Let $\widehat{D}^*F(x,y), D^*F(x,y)$, and $D_c^*F(x,y)$ denote the coderivatives of F at (x,y) associated with the *Fréchet*, Mordukhovich, and Clarke normal structures, respectively. They are defined by the following:

$$\begin{aligned} \widehat{D}^*F(x,y)(y^*) &:= \left\{x^* \in X^* : (x^*,-y^*) \in \widehat{N}(\mathrm{Gr}(F),(x,y))\right\} \quad \forall y^* \in Y^*, \\ D^*F(x,y)(y^*) &:= \{x^* \in X^* : (x^*,-y^*) \in N(\mathrm{Gr}(F),(x,y))\} \quad \forall y^* \in Y^*, \\ D_c^*F(x,y)(y^*) &:= \{x^* \in X^* : (x^*,-y^*) \in N_c(\mathrm{Gr}(F),(x,y))\} \quad \forall y^* \in Y^*. \end{aligned} \tag{2.11}$$

The more details of the coderivatives can be found in [9, 18, 19] and the references therein.

3. Subsmooth Generalized Constraint Equation

Let A be a closed subset of X. Recall (see [13, 14]) that A is said to be prox-regular at $a \in A$ if there exist $\tau, \delta > 0$ such that

$$\langle x^* - u^*, x - u \rangle \geq -\tau \|x - u\|^2 \tag{3.1}$$

whenever $x, u \in A \cap B(a, \delta), x^* \in N_c(A, x) \cap B_{X^*}$, and $u^* \in N_c(A, u) \cap B_{X^*}$.

As a generalization of the prox-regularity, Aussel et al. [15] introduced and studied the subsmoothness. A is said to be subsmooth at $a \in A$ if for any $\varepsilon > 0$ there exist $\tau, \delta > 0$ such that

$$\langle x^* - u^*, x - u \rangle \geq -\varepsilon \|x - u\|, \tag{3.2}$$

whenever $x, u \in A \cap B(a, \delta), x^* \in N_c(A, x) \cap B_{X^*}$, and $u^* \in N_c(A, u) \cap B_{X^*}$.

It is easy to verify that A is subsmooth at $a \in A$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u^*, x - u \rangle \leq \varepsilon \|x - u\| \tag{3.3}$$

whenever $x, u \in A \cap B(a, \delta)$ and $u^* \in N_c(A, u) \cap B_{X^*}$.

Let $F : X \rightrightarrows Y$ be a closed multifunction, $b \in Y$ and $a \in F^{-1}(b)$. Zheng and Ng [6] introduce the concept of the L-subsmoothness of F at a for b: F is called to be L-subsmooth at a for b if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle u^*, x - a \rangle + \langle v^*, y - v \rangle \leq \varepsilon (\|x - a\| + \|y - v\|), \tag{3.4}$$

whenever $v \in F(a) \cap B(b, \delta), (u^*, v^*) \in N_c(\mathrm{Gr}(F), (a, v)) \cap B_{X^* \times Y^*}$ and $(x, y) \in \mathrm{Gr}(F)$ with $\|x - a\| + \|y - b\| < \delta$. Next, we introduce the concept of the subsmoothness of generalized constraint equation (GCE) which will be useful in our discussion.

Definition 3.1. Generalized equation (GCE) is subsmooth at $a \in S$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u^*, x - u \rangle - \langle v^*, y - b \rangle \leq \varepsilon (\|x - u\| + \|y - b\|), \qquad \langle w^*, x' - u \rangle \leq \varepsilon \|x' - u\|, \tag{3.5}$$

whenever $x \in B(a, \delta), x' \in A \cap B(a, \delta), u \in S \cap B(a, \delta), y \in F(x) \cap B(b, \delta), v^* \in B_{Y^*}, u^* \in D_c^* F(u, b)(v^*) \cap B_{X^*}$, and $w^* \in N_c(A, u) \cap B_{X^*}$.

Remark 3.2. The subsmoothness of (GCE) at a means the subsmoothness of F at a for b when $A = X$, while the subsmoothness of (GCE) at a means the subsmoothness of A at a when $F(x) = b$ for all $x \in X$. If $A = X$ and $\mathrm{Gr}(F)$ is prox-regular at (a, b), then generalized equation (GCE) is subsmooth at a. If A and $\mathrm{Gr}(F)$ are convex, then F is also subsmooth at a. Finally when A is prox-regular and F is single-valued and smooth, (GCE) is subsmooth at a, too. Hence, Definition 3.1 extends notions of smoothness, convexity and prox-regularity.

Proposition 3.3. *Suppose that Generalized equation* (GCE) *is subsmooth at* $a \in S$. *Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\langle u^*, x-u\rangle \le (2+\varepsilon) d(b, F(x)) + \varepsilon\|x-u\|, \tag{3.6}$$

$$\langle w^*, x-u\rangle \le d(x, A) + \varepsilon\|x-u\|, \tag{3.7}$$

whenever $x \in B(a,\delta), u \in S\cap B(a,\delta)$, $u^* \in D_c^*F(u,b)(B_{Y^*})\cap B_{X^*}$, *and* $w^* \in N_c(A,u)\cap B_{X^*}$.

Proof. Suppose that (GCE) is subsmooth at $a \in S$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle u^*, x-u\rangle - \langle v^*, y-b\rangle \le \frac{\varepsilon}{2}(\|x-u\| + \|y-b\|), \qquad \langle w^*, x'-u\rangle \le \frac{\varepsilon}{2}\|x'-u\|, \tag{3.8}$$

whenever $x \in B(a,2\delta), x' \in A\cap B(a,2\delta), u \in S\cap B(a,2\delta), y \in F(x)\cap B(b,2\delta), v^* \in B_{Y^*}, u^* \in D_c^*F(u,b)(v^*)\cap B_{X^*}$, and $w^* \in N_c(A,u)\cap B_{X^*}$.

Let $x \in B(a,\delta), u \in S\cap B(a,\delta), v^* \in B_{Y^*}, u^* \in D_c^*F(u,b)(v^*)\cap B_{X^*}$, and $w^* \in N_c(A,u)\cap B_{X^*}$. If $F(x)\cap B(b,\delta) = \emptyset$, then

$$\langle u^*, x-u\rangle \le \|x-u\| \le \|x-a\| + \|a-u\| \le 2\delta, \qquad d(b,F(x)) \ge \delta. \tag{3.9}$$

Thus, (3.6) holds. Otherwise, one has

$$\langle u^*, x-u\rangle \le \langle v^*, y-b\rangle + \frac{\varepsilon}{2}(\|x-u\| + \|y-b\|) \le (1+\varepsilon)\|y-b\| + \varepsilon\|x-u\|, \tag{3.10}$$

whenever $y \in F(x)\cap B(b,\delta)$. Noting that $d(b,F(x)) = d(b, F(x)\cap B(b,\delta))$ (sine $F(x)\cap B(b,\delta) \neq \emptyset$), it follows that

$$\langle u^*, x-u\rangle \le (1+\varepsilon) d(b,F(x)) + \varepsilon\|x-u\|. \tag{3.11}$$

It remains to show that (3.7) holds. Since

$$\begin{aligned}\langle w^*, x-u\rangle &= \langle w^*, x-x'\rangle + \langle w^*, x'-u\rangle \le \|x-x'\| + \frac{\varepsilon}{2}\|x'-u\| \\ &\le \left(1+\frac{\varepsilon}{2}\right)\|x-x'\| + \frac{\varepsilon}{2}\|x-u\|,\end{aligned} \tag{3.12}$$

whenever $x' \in A\cap B(a,\delta)$. One has

$$\langle w^*, x-u\rangle \le d(x,A) + \frac{\varepsilon}{2}(d(x,A) + \|x-u\|). \tag{3.13}$$

Noting that $u \in A$, it follows that $d(x,A) \le \|x-u\|$ which implies that (3.7) holds and completes the proof. □

4. Main Results

This section is devoted to metric subregularity of generalized equation (GCE). We divide our discussion into two subsections addressing the necessary conditions and the sufficient conditions for metric subregularity.

4.1. Necessary Conditions for Metric Subregularity

There are two results in this subsection: one is on the Banach space setting and the other on the Asplund space setting.

Theorem 4.1. *Suppose that X, Y are Banach spaces and that generalized equation (GCE) is metrically subregular at $a \in S$. Then there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\widehat{N}(S,u) \cap B_{X^*} \subseteq \tau(D_c^* F(u,b)(B_{Y^*}) + N_c(A,u) \cap B_{X^*}) \quad \forall u \in S \cap B(a,\delta). \tag{4.1}$$

Proof. Let $\delta_{\mathrm{Gr}(F)}$ denote the indicator function of $\mathrm{Gr}(F)$ and $\delta > 0$ such that (1.5) holds. Then (1.5) can be rewritten as

$$d(x,S) \leq \delta_{\mathrm{Gr}(F)}(x,y) + \tau(\|y-b\| + d(x,A)) \quad \forall (x,y) \in B(a,\delta) \times Y. \tag{4.2}$$

Let $u \in S \cap B(a,\delta)$ and $u^* \in \widehat{N}(S,u) \cap B_{X^*}$. Noting (cf. [9, Corollary 1.96]) that $\widehat{N}(S,u) \cap B_{X^*} = \widehat{\partial} d(\cdot, S)(u)$, one gets that for any natural number n, there exists $r \in (0,\delta)$ such that $B(u,r) \subseteq B(a,\delta)$ and

$$\langle u^*, x-u \rangle \leq d(x,S) + \frac{1}{n}\|x-u\| \quad \forall x \in B(u,r). \tag{4.3}$$

Hence, by (4.2), it follows that

$$\langle u^*, x-u \rangle \leq \delta_{\mathrm{Gr}(F)}(x,y) + \tau\|y-b\| + \tau d(x,A) + \frac{1}{n}\|x-u\| \quad \forall (x,y) \in B(u,r) \times Y, \tag{4.4}$$

that is, (u,b) is a local minimizer of ϕ defined by

$$\phi(x,y) := -\langle u^*, x-u \rangle + \delta_{\mathrm{Gr}(F)}(x,y) + \tau\|y-b\| + \tau d(x,A) + \frac{1}{n}\|x-u\| \quad \forall (x,y) \in X \times Y. \tag{4.5}$$

Hence, $(0,0) \in \partial_c \phi(u,b)$. It follows from [16] that

$$(0,0) \in (-u^*,0) + N_c(\mathrm{Gr}(F),(u,b)) + \{0\} \times \tau B_{Y^*} + \tau\partial_c d(\cdot,A)(u) \times \{0\} + \frac{1}{n} B_{X^*} \times \{0\}, \tag{4.6}$$

that is,

$$\left(\frac{1}{\tau}u^* + \frac{1}{\tau n}x_n^*, -y_n^*\right) \in N_c(\mathrm{Gr}(F), (u, b)) + \partial_c d(\cdot, A)(u) \times \{0\}, \tag{4.7}$$

for some $x_n^* \in B_{X^*}$ and $y_n^* \in B_{Y^*}$. Since B_{X^*} and B_{Y^*} are weak* compact, without loss of generality (otherwise take a generalized subsequence), we can assume $x_n^* \xrightarrow{w^*} x^*, y_n^* \xrightarrow{w^*} y^*$ for some $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$ as $n \to \infty$. Noting that

$$N_c(\mathrm{Gr}(F), (u, b)) + \partial_c d(\cdot, A)(u) \times \{0\} \tag{4.8}$$

is weak* closed (since $N_c(\mathrm{Gr}(F), (u, b))$ is weak* closed and $\partial_c d(\cdot, A)(u) \times \{0\}$ is weak* compact), one has

$$\left(\frac{u^*}{\tau}, -y^*\right) \in N_c(\mathrm{Gr}(F), (u, b)) + \partial_c d(\cdot, A)(u) \times \{0\}. \tag{4.9}$$

This implies that

$$u^* \in \tau(D_c^* F(u, b)(B_{Y^*}) + N_c(A, u) \cap B_{X^*}). \tag{4.10}$$

This shows that (4.1) holds true. The proof is completed. □

When X and Y are Asplund spaces, the conclusion in Theorem 4.1 can be strengthened with $D_c^* F(u, b)(B_{Y^*})$ and $N_c(A, u) \cap B_{X^*})$ replaced by $D^* F(u, b)(B_{Y^*})$ and $N(A, u) \cap B_{X^*})$, respectively. Its proof is similar to that of Theorem 4.1.

Theorem 4.2. *Suppose that X and Y are Asplund spaces and that generalized equation (GCE) is metrically subregular at $a \in S$. Then there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\widehat{N}(S, u) \cap B_{X^*} \subseteq \tau(D^* F(u, b)(B_{Y^*}) + N(A, u) \cap B_{X^*}) \quad \forall u \in S \cap B(a, \delta). \tag{4.11}$$

4.2. Sufficient Conditions for Metric Subregularity

Under the subsmooth assumption, we will show in the next result that some conditions similar to (4.1) turns out to be sufficient conditions for metric subregularity.

Theorem 4.3. *Let X and Y be Banach spaces. Suppose that generalized constraint equation (GCE) is subsmooth at a and that there exist $\tau, \delta \in (0, +\infty)$ such that*

$$N_c(S, u) \cap B_{X^*} \subseteq \tau(D_c^* F(u, b)(B_{Y^*}) + N_c(A, u) \cap B_{X^*}), \tag{4.12}$$

whenever $u \in \mathrm{bd}(S) \cap B(a, \delta)$. *Then* (GCE) *is metrically subregular at* a *and, more precisely, for any* $\varepsilon \in (0, 1/(2(1+\tau)))$ *there exists* $\delta_\varepsilon \in (0, \delta/2)$ *such that*

$$d(x, S) \leq \frac{(1+\tau)(2+\varepsilon)}{1-2(1+\tau)\varepsilon}(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta_\varepsilon). \tag{4.13}$$

Proof. Let $\varepsilon \in (0, 1/(2(1+\tau)))$. Then, by subsmooth assmption of (GCE) at a and Proposition 3.3, there exists $\delta' \in (0, \delta/2)$ such that

$$\begin{gathered}\langle u_1^*, x-u\rangle \leq (2+\varepsilon) d(b, F(x)) + \varepsilon\|x-u\|, \\ \langle u_2^*, x-u\rangle \leq d(x, A) + \varepsilon\|x-u\|,\end{gathered} \tag{4.14}$$

whenever $x \in B(a, \delta'), u \in S \cap B(a, \delta'), u_1^* \in D_c^* F(u, b)(B_{Y^*}) \cap B_{X^*}$ and $u_2^* \in N_c(A, u) \cap B_{X^*}$.

Let $\delta_\varepsilon \in (0, \delta'/2)$ and $x \in B(a, \delta_\varepsilon) \setminus S$. Now we need only show (4.13).

(i) If $F(x) \cap B(b, \delta') = \emptyset$, then $d(x, S) \leq \|x-a\| < \delta_\varepsilon$, $d(b, F(x)) \geq \delta'$. Hence (4.13) holds.

(ii) Suppose $F(x) \cap B(b, \delta') \neq \emptyset$ and let

$$\beta \in \left(\max\left\{\frac{d(x, S)}{\delta_\varepsilon}, 2(1+\tau)\varepsilon, \frac{1}{2}\right\}, 1\right). \tag{4.15}$$

By Lemma 2.1 there exist $u_0 \in \mathrm{bd}(S)$ and $u^* \in N_c(S, u_0)$ with $\|u^*\| = 1$ such that

$$\beta\|x-u_0\| \leq \min\{\langle u^*, x-u_0\rangle, d(x, S)\}. \tag{4.16}$$

Thus, $\|x-u_0\| \leq (d(x, S)/\beta) < \delta_\varepsilon$. Hence,

$$\|u_0 - a\| \leq \|u_0 - x\| + \|x-a\| < 2\delta_\varepsilon < \delta' < \delta. \tag{4.17}$$

By (4.12) there exist $y_1^* \in B_{Y^*}, x_1^* \in D_c^* F(u_0, b)(y_1^*)$, and $x_2^* \in N_c(A, u_0) \cap B_{X^*}$ such that $u^* = \tau(x_1^* + x_2^*)$. Applying (4.14) with $((\tau/(1+\tau))x_1^*, (\tau/(1+\tau))x_2^*, u_0)$ in place of (u_1^*, u_2^*, u), it follows that

$$\begin{aligned}\langle u^*, x-u_0\rangle &= \tau\left(\langle x_1^*, x-u_0\rangle + \langle x_2^*, x-u_0\rangle\right) \\ &\leq (1+\tau)(2+\varepsilon)(d(b, F(x)) + d(x, A)) + 2(1+\tau)\varepsilon\|x-u_0\|.\end{aligned} \tag{4.18}$$

This and (4.16) imply that

$$d(x, S) \leq \|x-u_0\| \leq \frac{(1+\tau)(2+\varepsilon)}{\beta - 2(1+\tau)\varepsilon}(d(b, F(x)) + d(x, A)). \tag{4.19}$$

Letting $\beta \to 1$, it follows that (4.13) holds. The proof is completed. □

When X and Y are Asplund spaces, the assumption in Theorem 4.3 can be weakened with $N_c(S, u)$ replaced by $\widehat{N}(S, u)$.

Theorem 4.4. *Suppose X and Y are Asplund spaces. Suppose that generalized constraint equation (GCE) is subsmooth at a and that there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\widehat{N}(S, u) \cap B_{X^*} \subseteq \tau(D_c^* F(u, b)(B_{Y^*}) + N_c(A, u) \cap B_{X^*}), \tag{4.20}$$

whenever $u \in \mathrm{bd}(S) \cap B(a, \delta)$. Then for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that (4.13) *holds.*

With the Asplund space version of Lemma 2.1 applied in place of the Banach space version, similar to the proof of Theorem 4.3, it is easy to verify Theorem 4.4.

In general, (GCE) is not necessarily metrically subregular at a if (GCE) only has that $N_c(S, a) \cap B_{X^*} \subseteq \tau(D_c^* F(a, b)(B_{Y^*}) + N_c(A, a) \cap B_{X^*})$.

Finally, we end this subsection with a sufficient and necessary condition for the Clarke tangent derivative mapping $D_c F(a, b)$ to be metrically subregular at 0 for 0 over the Clerke tangent cone $T_c(A, a)$.

Let

$$\tau(F, a, b; A) := \inf\{\tau > 0 : \text{ there exists } \delta > 0 \text{ such that } (1.5) \text{ holds}\}. \tag{4.21}$$

For $u \in S$, let

$$\gamma(F, u, b; A) := \inf\{\tau > 0 : N_c(S, u) \cap B_{X^*} \subseteq \tau(D_c^* F(a, b)(B_{Y^*}) + N_c(A, u) \cap B_{X^*})\}. \tag{4.22}$$

The following lemma is known ([5, Theorem 3.2]) and useful for us in the sequel.

Lemma 4.5. *Assume that $F : X \rightrightarrows Y$ is a closed convex multifunction, A is a closed convex subset of X, and $a \in S$. And suppose that there exist a cone C and a neighborhood V of a such that $S \cap V = (a + C) \cap V$. Then,*

$$\tau(F, a, b; A) = \gamma(F, a, b; A). \tag{4.23}$$

Consequently, (GCE) is metrically subregular at a if and only if $\gamma(F, a, b; A) < +\infty$.

Theorem 4.6. *Let $a \in S$ and*

$$\tau := \inf\{\tau > 0 : d(h, T_c(S, a)) \leq \tau(d(0, D_c F(a, b)(h)) + d(h, T_c(A, a)))\ \forall h \in X\}. \tag{4.24}$$

Suppose that

$$T_c(S, a) \subseteq T_c(A, a) \cap D_c F(a, b)^{-1}(0). \tag{4.25}$$

Then,

$$\tau = \gamma(F, a, b; A). \tag{4.26}$$

If, in addition, $\tau < +\infty$*, then*

$$T_c(S,a) = T_c(A,a) \cap D_cF(a,b)^{-1}(0). \tag{4.27}$$

Consequently, $D_cF(a,b)$ *is metrically subregular at* $(0,0)$ *over* $T_c(A,a)$ *if and only if* $\gamma(F,a,b;A) < +\infty$.

Proof. First, we assume that $\tau < +\infty$. By the definition of τ, we have

$$d(x, T_c(S,a)) \leq \tau(d(0, D_cF(a,b)(x)) + d(x, T_c(A,a))) \quad \forall x \in X. \tag{4.28}$$

This implies that

$$T_c(A,a) \cap D_cF(a,b)^{-1}(0) \subseteq T_c(S,a). \tag{4.29}$$

This and (4.25) imply that (4.27) holds.

We consider the following constraint equation:

$$0 \in D_cF(a,b)(x) \quad \text{subject to } x \in T_c(A,a). \tag{GCE$'$}$$

Let S' denote the solution set of (GCE'). Then,

$$S' = T_c(A,a) \cap D_cF(a,b)^{-1}(0). \tag{4.30}$$

Noting that

$$\begin{gathered} N_c(S',0) = N_c(T_c(S,a),0) = N_c(S,a), \\ N_c(\mathrm{Gr}(F),(a,b)) = N_c(\mathrm{Gr}(D_cF(a,b)),(0,0)), \end{gathered} \tag{4.31}$$

it is straightforward to verify that

$$\tau = \tau(D_cF(a,b),0,0;T_c(A,a)), \qquad \gamma(F,a,b;A) = \gamma(D_cF(a,b),0,0;T_c(A,a)). \tag{4.32}$$

On the other hand, since $D_cF(a,b)$ is a closed convex multifunction from X to Y and $T_c(A,a)$ is a closed convex cone, Lemma 4.5 implies that

$$\tau(D_cF(a,b),0,0;T_c(A,a)) = \gamma(D_cF(a,b),0,0;T_c(A,a)). \tag{4.33}$$

This gives us $\tau = \gamma(F,a,b;A)$.

It remains to show that $\gamma(F,a,b;A) = +\infty$ when $\tau = +\infty$. Suppose that

$$\gamma(F,a,b;A) < +\infty. \tag{4.34}$$

We need only show that $\tau < +\infty$. Let $x \in X \setminus T_c(S,a)$ and $\beta \in (0,1)$. By Lemma 2.1 there exist $u \in T_c(S,a)$ and $x^* \in N_c(T_c(S,a),u)$ with $\|x^*\| = 1$ such that

$$\beta\|x-u\| \le \langle x^*, x-u\rangle. \tag{4.35}$$

Noting that $T_c(S,a)$ is a convex cone, it is easy to verify that

$$x^* \in N_c(T_c(S,a),0) = N_c(S,a), \qquad \langle x^*, u\rangle = 0. \tag{4.36}$$

Take a fixed r in $(\gamma(F,a,b;A),+\infty)$. Then there exist $y^* \in rB_{Y^*}, x_1^* \in D_c^*F(a,b)(y^*)$ and $x_2^* \in N_c(A,a) \cap rB_{X^*}$ such that

$$x^* = x_1^* + x_2^*. \tag{4.37}$$

We equip the product space $X \times Y$ with norm

$$\|(x,y)\|_r := \frac{r}{1+r}\|x\| + \|y\| \quad \forall (x,y) \in X \times Y. \tag{4.38}$$

Noting that the unit ball of the dual space of $(X \times Y, \|\cdot\|_r)$ is $(((1+r)/r)B_{X^*}) \times B_{Y^*}$, it follows from the convexity of $D_cF(x,y)$ and $T_c(A,a)$ that

$$\begin{aligned}
\frac{1}{r}(x_1^*, -y^*) &\in N_c(\mathrm{Gr}(F),(a,b)) \cap \left(\left(\frac{1+r}{r}B_{X^*}\right) \times B_{Y^*}\right) \\
&= N_c(\mathrm{Gr}(D_cF(a,b)),(0,0)) \cap \left(\left(\frac{1+r}{r}B_{X^*}\right) \times B_{Y^*}\right) \\
&= \partial_c d_{\|\cdot\|_r}(\cdot, \mathrm{Gr}(D_cF(a,b)))(0,0), \\
\frac{1}{r}x_2^* &\in N_c(A,a) \cap B_{X^*} = N_c(T_c(A,a),0) \cap B_{X^*} = \partial_c d(\cdot, T_c(A,a))(0).
\end{aligned} \tag{4.39}$$

Hence,

$$\begin{aligned}
\frac{1}{r}\langle x_1^*, x\rangle &\le d_{\|\cdot\|_r}((x,0),\mathrm{Gr}(D_cF(a,b))) \le d(0, D_cF(a,b)(x)), \\
\frac{1}{r}\langle x_2^*, x\rangle &\le d(x, T_c(A,a)),
\end{aligned} \tag{4.40}$$

whenever $x \in X$. Noting that $\langle x^*, u\rangle = 0$, it follows from (4.35) that

$$\frac{\beta\|x-u\|}{r} \le d(0, D_cF(a,b)(x)) + d(x, T_c(A,a)). \tag{4.41}$$

Therefore,

$$\frac{\beta d(x, T_c(S, a))}{r} \leq d(0, D_c F(a, b)(x)) + d(x, T_c(A, a)). \tag{4.42}$$

Letting $\beta \to 1$, one has

$$d(x, T_c(S, a)) \leq r(d(0, D_c F(a, b)(x)) + d(x, T_c(A, a))). \tag{4.43}$$

This contradicts with $\tau = +\infty$. The proof is completed. □

4.3. Strongly Metric Subregularity

Let $F : X \rightrightarrows Y$ be a multifunction and $b \in F(a)$. Recall that F is strongly subregular at a if there exist $\tau \in (0, +\infty)$, neighborhoods U of a, and V of b such that

$$\|x - a\| \leq \tau d(b, F(x) \cap V) \quad \forall x \in U. \tag{4.44}$$

It is clear that this definition is equivalent to the next one when $A = X$.

Definition 4.7. One says that generalized constraint equation (GCE) is strongly metrically subregular at a if there exists $\tau, \delta \in (0, \infty)$ such that

$$\|x - a\| \leq \tau(d(b, F(x)) + d(x, A)) \quad \forall x \in B(a, \delta). \tag{4.45}$$

It is clear that (GCE) is strongly metrically subregular at a if and only if a is an isolated point of S (i.e., $S \cap B(a, r) = \{a\}$ for some $r > 0$) and it is metrically subregular at a. Thus, if (GCE) is strongly metrically subregular at a, Then $N_c(S, a) = X^*$. We immediately have the following Corollary 4.8 from Theorem 4.1.

Corollary 4.8. *Suppose that there exists $\tau, \delta \in (0, \infty)$ such that* (4.45) *holds. Then,*

$$B_{X^*} \subseteq \tau(D_c^* F(a, b)(B_{Y^*}) + N_c(A, a) \cap B_{X^*}). \tag{4.46}$$

Applying Theorem 4.3, one obtains a sufficient condition for (GCE) to be strongly metrically subregular at a.

Corollary 4.9. *Let X, Y be Banach spaces. Suppose that generalized constraint equation* (*GCE*) *is subsmooth at a and that there exists $\tau \in (0, +\infty)$ such that*

$$B_{X^*} \subseteq \tau(D_c^* F(a, b)(B_{Y^*}) + N_c(A, a) \cap B_{X^*}). \tag{4.47}$$

Then (*GCE*) *is strongly metrically subregular at* a *and, more precisely, for any* $\varepsilon \in (0, 1/(1+2\tau))$ *there exists* $\delta_\varepsilon > 0$ *such that*

$$\|x-a\| \le \frac{(1+\tau)(2+\varepsilon)}{1-2(1+\tau)\varepsilon}(d(b,F(x))+d(x,A)) \quad \forall x \in B(a,\delta_\varepsilon). \tag{4.48}$$

Proof. From Theorem 4.3, we need only show that $S \cap B(a,\delta) = \{a\}$ for some $\delta > 0$. Since the assumption that (GCE) is subsmooth at a, by Proposition 3.3, for any $\varepsilon \in (0, 1/2(1+\tau))$, there exists $\delta > 0$ such that

$$\langle a_1^*, x-a\rangle \le (2+\varepsilon)d(b,F(x)) + \varepsilon\|x-u\|, \tag{4.49}$$

$$\langle a_2^*, x-a\rangle \le d(x,A) + \varepsilon\|x-u\|, \tag{4.50}$$

whenever $x \in B(a,\delta)$, $a_1^* \in D_c^*F(a,b)(B_{Y^*}) \cap B_{X^*}$ and $a_2^* \in N_c(A,a) \cap B_{X^*}$.

Take an arbitrary $x^* \in B_{X^*}$. By (4.47), there exist $x_1^* \in D_c^*F(a,b)(B_{Y^*})$, $x_2^* \in N_c(A,a) \cap B_{X^*}$ such that $x^* = \tau(x_1^* + x_2^*)$. Let $x \in S \cap B(a,\delta)$. Applying (4.49) with $(\tau/(1+\tau))x_1^*$ in place of a_1^*, it follows from this and (4.50) that we have

$$\tau\langle x_1^*, x-a\rangle \le (1+\tau)\varepsilon\|x-a\|, \qquad \langle x_2^*, x-a\rangle \le \varepsilon\|x-a\|. \tag{4.51}$$

Then,

$$\begin{aligned}\langle x^*, x-a\rangle &= \tau\left(\langle x_1^*, x-a\rangle + \langle x_2^*, x-a\rangle\right)\\ &\le (1+2\tau)\varepsilon\|x-a\|.\end{aligned} \tag{4.52}$$

And so,

$$\|x-a\| = \sup_{x^*\in B_{X^*}} \langle x^*, x-a\rangle \le (1+2\tau)\varepsilon\|x-a\|. \tag{4.53}$$

This shows that $S \cap B(a,\delta) = \{a\}$. The proof is completed. □

From Corollaries 4.8 and 4.9, we also have the following equivalent results.

Corollary 4.10. *Suppose that generalized constraint equation* (*GCE*) *is subsmooth at* a. *Then the following statements are equivalent:*

(i) (*GCE*) *is strongly metrically subregular at* a;

(ii) *there exists* $\tau \in (0,\infty)$ *such that* $B_{X^*} \subseteq \tau(D_c^*F(a,b)(B_{Y^*}) + N_c(A,a) \cap B_{X^*}))$;

(iii) $0 \in \operatorname{int}(D_c^*F(a,b)(Y^*) + N_c(A,a))$;

(iv) $X^* = D_c^*F(a,b)(Y^*) + N_c(A,a)$;

(v) $D_cF(a,b)$ *is strongly metrically subregular at* 0 *for* 0 *over* $T_c(A,a)$.

Proof. First, by Corollaries 4.8 and 4.9, it is clear that (i)⇔(ii). Noting that $D_c^*(D_cF(a,b))(0,0)(B_{Y^*}) = D_c^*F(a,b)(B_{Y^*})$, (ii)⇔(v) is immediate from (i)⇔(ii).

It is clear that (ii)⇔(iii). Noting that $N_c(A,a)$ and $D_c^*F(a,b)(Y^*)$ are cones, hence,

$$D_c^*F(a,b)(Y^*) + N_c(A,a) = \bigcup_{n=1}^{\infty}\left(D_c^*F(a,b)(nB_{Y^*}) + N_c(A,a) \cap nB_{X^*}\right). \tag{4.54}$$

This shows that (ii)⇒(iv).

It remains to show that (iv)⇒(ii). Suppose that (iv) holds, by the Alaogu theorem, for each n, the set $D_c^*F(a,b)(nB_{Y^*}) + N_c(A,a) \cap nB_{X^*}$ is weakly star-closed, it follows from the well-known Baire category theorem and (iv) that

$$0 \in \operatorname{int}\left(D_c^*F(a,b)(B_{Y^*}) + N_c(A,a) \cap B_{X^*}\right). \tag{4.55}$$

Hence, (ii) holds. The proof is completed. □

Acknowledgment

This paper was supported by the National Natural Science Foundations, China, (Grants nos. 11061039, 11061038, and 11261067), and IRTSTYN.

References

[1] W. Li and I. Singer, "Global error bounds for convex multifunctions and applications," *Mathematics of Operations Research*, vol. 23, no. 2, pp. 443–462, 1998.

[2] X. Y. Zheng and K. F. Ng, "Perturbation analysis of error bounds for systems of conic linear inequalities in Banach spaces," *SIAM Journal on Optimization*, vol. 15, no. 4, pp. 1026–1041, 2005.

[3] W. Li, "Abadie's constraint qualification, metric regularity, and error bounds for differentiable convex inequalities," *SIAM Journal on Optimization*, vol. 7, no. 4, pp. 966–978, 1997.

[4] B. S. Mordukhovich and Y. Shao, "Nonconvex differential calculus for infinite-dimensional multifunctions," *Set-Valued Analysis*, vol. 4, no. 3, pp. 205–236, 1996.

[5] X. Y. Zheng and K. F. Ng, "Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces," *SIAM Journal on Optimization*, vol. 18, no. 2, pp. 437–460, 2007.

[6] X. Y. Zheng and K. F. Ng, "Calmness for L-subsmooth multifunctions in Banach spaces," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1648–1673, 2008.

[7] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization*, vol. 60, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[8] B. S. Mordukhovich, "Complete characterization of openness, metric regularity, and Lipschitzian properties of multifunctions," *Transactions of the American Mathematical Society*, vol. 340, no. 1, pp. 1–35, 1993.

[9] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation. I, II*, Springer, Berlin, Germany, 2006.

[10] R. Henrion, A. Jourani, and J. Outrata, "On the calmness of a class of multifunctions," *SIAM Journal on Optimization*, vol. 13, no. 2, pp. 603–618, 2002.

[11] X. Y. Zheng and K. F. Ng, "Metric regularity and constraint qualifications for convex inequalities on Banach spaces," *SIAM Journal on Optimization*, vol. 14, no. 3, pp. 757–772, 2003.

[12] X. Y. Zheng and K. F. Ng, "Linear regularity for a collection of subsmooth sets in Banach spaces," *SIAM Journal on Optimization*, vol. 19, no. 1, pp. 62–76, 2008.

[13] F. H. Clarke, R. J. Stern, and P. R. Wolenski, "Proximal smoothness and the lower-C^2 property," *Journal of Convex Analysis*, vol. 2, no. 1-2, pp. 117–144, 1995.

[14] R. A. Poliquin, R. T. Rockafellar, and L. Thibault, "Local differentiability of distance functions," *Transactions of the American Mathematical Society*, vol. 352, no. 11, pp. 5231–5249, 2000.
[15] D. Aussel, A. Daniilidis, and L. Thibault, "Subsmooth sets: functional characterizations and related concepts," *Transactions of the American Mathematical Society*, vol. 357, no. 4, pp. 1275–1301, 2005.
[16] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, New York, NY, USA, 1983.
[17] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, vol. 1364 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1989.
[18] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, vol. 317, Springer, Berlin, Germany, 1998.
[19] B. S. Mordukhovich and Y. H. Shao, "Nonsmooth sequential analysis in Asplund spaces," *Transactions of the American Mathematical Society*, vol. 348, no. 4, pp. 1235–1280, 1996.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization