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Attenuation Factors in Multivariate Fourier Analysis*

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Summary. W. Gautschi's theory of attenuation factors for families of periodic functions in one variable is extended to families of functions in several variables. Again, the linearity and the translation invariance of the operator which maps the data space onto the family are crucial. Special results are obtained for tensor product families and for the interpolation by translates of one generating function. Interesting examples are provided by box splines, which include certain finite elements.

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1. Introduction

It has been known for more than 50 years that the Fourier coefficients of an odd-degree periodic spline interpolant to data given at equidistant abscissas, which are also the knots (breakpoints) of the spline function, can be computed easily from the discrete Fourier transform (DFT) of the data, namely by multiplying the periodically extended sequence of DFT coefficients by certain factors, called attenuation factors, which are independent of the data. In 1972 Gautschi [11] presented a general theory for attenuation factors, where he showed that such factors arise under much more general circumstances, namely, whenever the approximation process that maps the data into a family of periodic approximants is linear and (in a natural sense) translation invariant. Gautschi [11] gave also a number of new examples, e.g., attenuation factors for defective spline functions, and new formulas based on a polynomial recurrence for the factors for splines interpolants of odd degree. (The respective formulas for even degree splines are listed in [12].) Moreover, Gautschi's article contains a brief historical survey of the earlier work on attenuation factors for periodic splines, notably

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by Eagle (1928), Quade and Collatz (1938), Bauer and Stetter (1959), Ehlich (1966), and Golomb (1968) (see [11] for the complete references). In a more recent contribution Locher [16] studied the attenuation factors of interpolants from the linear space spanned by the translates $h(\cdot - j/N)$, $j \in \mathbb{Z}$, of a periodic generating function h; he also gave a number of examples.

Here we extend the basic results of both Gautschi's and Locher's paper to the multivariate case. It is not surprising that the linearity and the translation invariance of the approximation process are again crucial, and that in the tensor product case the attenuation factors are products of univariate ones. Yet in some points our results give improvements even in the univariate case. For example, in the basic characterization theorem our assumptions are weaker than Gautschi's. In the case of interpolation by translates of a periodic generating function h interesting relations hold if h is itself the sum of translates of a nonperiodic germ function H. Interesting examples for the latter are provided by box splines, which are treated in the last section.

In the multivariate setting, suitable notation is important. There are several reasonable normalizations; for example, one might assume that the functions are 2π-periodic [11, 16] in each coordinate direction, or that the data points have integer coordinates [3–5, 8, 9, 13]. Since reducing the case of a rectangular domain to the one with square domain is trivial, we decided for 1-periodicity in each coordinate direction, but allowed for a different meshsize in each direction.

Among the possible applications for attenuation factors in several variables we mention the high-order fast Poisson solvers outlined in [12] for the case of periodic boundary conditions.

2. Notation

In $R := \mathbb{R}^D$ the following abbreviations are used for coordinatewise multiplication and division, for the inner product, and for a particular weighted inner product:

$$\begin{aligned} \mathbf{x}\cdot\mathbf{y} &:= (x_d\, y_d)_{d=1}^{D}, & \mathbf{x}/\mathbf{y} &:= (x_d/y_d)_{d=1}^{D}, \\ (\mathbf{x},\mathbf{y}) &:= \sum_{d=1}^{D} x_d\, y_d, & \langle \mathbf{x},\mathbf{y}\rangle &:= \sum_{d=1}^{D} x_d\, y_d/N_d. \end{aligned} \tag{2.1}$$

N_d denotes the number of grid points per unit length in the coordinate direction d, and we set $\mathbf{N} := (N_d)_{d=1}^{D}$. The following subsets of $Z := \mathbb{Z}^D$ and of R play a role:

$$\begin{aligned} &K := \{\mathbf{k} \in Z;\, 0 \leqq k_d < N_d (d=1,\ldots,D)\}, \\ &Q := \{\mathbf{x} \in R;\, 0 \leqq x_d \leqq 1 (d=1,\ldots,D)\}, \\ &\Xi := \{\mathbf{k}/\mathbf{N};\, \mathbf{k} \in Z\}. \end{aligned}$$

We mainly consider complex-valued functions f defined on R that are 1-periodic in each coordinate direction: $f(\mathbf{x}) = f(\mathbf{x}+\mathbf{k}) (\forall \mathbf{k} \in Z)$. The space of such periodic functions f whose restriction to the fundamental square Q is in $L^p(Q)$ is denoted by $\mathscr{L}^p (p \geqq 1)$. (In the univariate case we use the standard notation $L^p(T)$ instead.) The restriction of any such f to the grid Ξ is thought of as a periodic sequence $\mathbf{f} = (f_{\mathbf{k}})_{\mathbf{k} \in Z}$ with multi-index $\mathbf{k}$, i.e., $\mathbf{f} \in \Pi_{\mathbf{N}}$, where

$$\Pi_{\mathbf{N}} := \{\mathbf{f}\colon\ Z \to \mathbb{C};\, f_{\mathbf{k}+\mathbf{l}\cdot\mathbf{N}} = f_{\mathbf{k}} (\mathbf{k}, \mathbf{l} \in Z)\}$$

is the space of $\mathbf{N}$-periodic sequences. As usual, we set

$$\ell^p := \{\varphi\colon\ Z \to \mathbb{C};\ \sum_{\mathbf{k} \in Z} |\varphi_{\mathbf{k}}|^p < \infty\},$$

and denote the corresponding space of bounded sequences by ℓ^∞. The notation in (2.1) for coordinatewise multiplication and division is also used for elements of $\Pi_{\mathbf{N}}$ and ℓ^p; e.g., $(\mathbf{f} \cdot \varphi)_{\mathbf{k}} = f_{\mathbf{k}}\, \varphi_{\mathbf{k}}$.

Fourier analysis defines an invertible linear mapping $\mathscr{F}$ from $\mathscr{L}^1$ onto some subset of ℓ^∞; the restriction of $\mathscr{F}$ to $\mathscr{L}^2$ is an isomorphism of $\mathscr{L}^2$ onto ℓ^2:

$$\mathscr{F}: f \in \mathscr{L}^1 \mapsto \varphi \in \ell^\infty, \qquad \varphi_{\mathbf{n}} := \int_Q f(\mathbf{x})\, e^{-2\pi i (\mathbf{n}, \mathbf{x})} d\mathbf{x} \qquad (\mathbf{n} \in Z). \tag{2.2}$$

The discrete Fourier transform (DFT) $\mathscr{F}_{\mathbf{N}}$ is an automorphism of $\Pi_{\mathbf{N}}$ [13]:

$$\mathscr{F}_{\mathbf{N}}\colon\ \mathbf{f} \in \Pi_{\mathbf{N}} \mapsto \hat{\mathbf{f}} \in \Pi_{\mathbf{N}}, \qquad \hat{f}_{\mathbf{n}} := \frac{1}{N_0} \sum_{\mathbf{k} \in K} f_{\mathbf{k}}\, e^{-2\pi i \langle \mathbf{n}, \mathbf{k} \rangle} \qquad (\mathbf{n} \in Z), \tag{2.3}$$

where $N_0 := \prod_{d=1}^{D} N_d$. Conversely,

$$f(\mathbf{x}) \sim \sum_{\mathbf{n} \in Z} \varphi_{\mathbf{n}}\, e^{2\pi i (\mathbf{n}, \mathbf{x})} \qquad (\mathbf{x} \in R), \tag{2.4}$$

$$f_{\mathbf{k}} = \sum_{\mathbf{n} \in K} \hat{f}_{\mathbf{n}}\, e^{2\pi i \langle \mathbf{n}, \mathbf{k} \rangle} \qquad (\mathbf{k} \in Z). \tag{2.5}$$

(In the univariate case we write $\mathscr{F}_T$ and $\mathscr{F}_N$ instead of $\mathscr{F}$ and $\mathscr{F}_{\mathbf{N}}$, respectively.)

If f has an absolutely convergent Fourier series, i.e., if f lies in

$$\mathscr{A} := \{f \in \mathscr{L}^2; \mathscr{F} f \in \ell^1\},$$

we always assume that f is continuous. Then (2.4) holds with the equality sign. If $f \in \mathscr{A}$, it is well known and easy to prove (e.g., see [11], p. 381, or [13], p. 489, for the one-dimensional case) that $\varphi := \mathscr{F} f$ and $\hat{\mathbf{f}} := \mathscr{F}_{\mathbf{N}} \mathbf{f}$, where $f_{\mathbf{k}} := f(\mathbf{k}/\mathbf{N})$ is obtained by evaluating f on the grid, are related by *aliasing*:

$$\hat{f}_{\mathbf{n}} = \sum_{\mathbf{k} \in Z} \varphi_{\mathbf{n}+\mathbf{k}\cdot\mathbf{N}} \qquad (\mathbf{n} \in Z). \tag{2.6}$$

In Sect. 5 and 6 certain nonperiodic functions $F \in L^1(R)$ will also play a role. The Fourier transform $\mathscr{F}_R$ associates with each such function a uniformly continuous function $\hat{F} = \mathscr{F}_R F$ defined on R [15, p. 121]:

$$\mathscr{F}_R\colon F \in L^1(R) \mapsto \hat{F}, \qquad \hat{F}(\mathbf{u}) := \int_R F(\mathbf{x})\, e^{-2\pi i (\mathbf{u}, \mathbf{x})} d\mathbf{x} \qquad (\mathbf{u} \in R). \tag{2.7}$$

(The factor 2π in the exponent is *not* standard, but in our application some formulas are simplified.)

On $\mathscr{L}^1$, $L^1(R)$, and $\Pi_{\mathbf{N}}$ we define shift operators $E_{\mathbf{y}}$, $E_{\mathbf{y}}$ and $\mathbf{E}_{\mathbf{l}}$, respectively, by

$$(E_{\mathbf{y}} f)(\mathbf{x}) := f(\mathbf{x}-\mathbf{y}) \qquad (f \in \mathscr{L}^1 \text{ or } L^1(R), \mathbf{y} \in R),$$
$$(\mathbf{E}_{\mathbf{l}} \mathbf{f})_{\mathbf{k}} := f_{\mathbf{k}-\mathbf{l}} \qquad (\mathbf{f} \in \Pi_{\mathbf{N}}, \mathbf{l} \in Z).$$

The effects of shifts on Fourier transforms are described by the relations [13, 15]:

$$(\mathscr{F} E_{\mathbf{y}} f)_{\mathbf{n}} = e^{-2\pi i(\mathbf{n},\mathbf{y})} (\mathscr{F} f)_{\mathbf{n}} \qquad (f \in \mathscr{L}^1, \mathbf{y} \in R), \tag{2.8}$$

$$(\mathscr{F}_R E_{\mathbf{y}} F)(\mathbf{u}) = e^{-2\pi i(\mathbf{u},\mathbf{y})} (\mathscr{F}_R F)(\mathbf{u}) \qquad (F \in \mathscr{L}^1(R), \mathbf{y} \in R), \tag{2.9}$$

$$(\mathscr{F}_{\mathbf{N}} \mathbf{E}_{\mathbf{l}} \mathbf{f})_{\mathbf{n}} = e^{-2\pi i\langle \mathbf{n},\mathbf{l}\rangle} (\mathscr{F}_{\mathbf{N}} \mathbf{f})_{\mathbf{n}} \qquad (\mathbf{f} \in \Pi_{\mathbf{N}}, \mathbf{l} \in Z). \tag{2.10}$$

We also make use of the Kronecker symbol and of various modifications of it:

$$\delta_{m,n} := \begin{cases} 0 & \text{if} \quad m \neq n, \\ 1 & \text{if} \quad m = n, \end{cases}$$

$$\delta_{m,n}^{M} := \begin{cases} 0 & \text{if} \quad m \not\equiv 0 \pmod{M}, \\ 1 & \text{if} \quad m \equiv n \pmod{M}, \end{cases}$$

$$\delta_{\mathbf{m},\mathbf{n}} := \prod_{d=1}^{D} \delta_{m_d,n_d}, \qquad \delta_{\mathbf{m},\mathbf{n}}^{\mathbf{N}} := \prod_{d=1}^{D} \delta_{m_d,n_d}^{N_d}.$$

Finally,

$$\mathbf{e} := (e_{\mathbf{k}})_{\mathbf{k} \in Z} \quad \text{with} \quad e_{\mathbf{k}} := \delta_{\mathbf{k},\mathbf{0}}^{\mathbf{N}} = \begin{cases} 1 & \text{if} \quad \mathbf{k}/\mathbf{N} \in Z \\ 0 & \text{otherwise,} \end{cases}$$

is the $\mathbf{N}$-periodic unit pulse.

3. Attenuation Factors

The following treatment of attenuation factors for periodic functions of several variables is an extension of Gautschi's theory in one variable [11]. Briefly, attenuation factors exist if and only if the approximation process $P\colon \Pi_{\mathbf{N}} \to \mathscr{L}^1$ is linear and translation invariant. (Of course, linearity implies continuity since $\Pi_{\mathbf{N}}$ is finite-dimensional.)

Definition. $P\colon \Pi_{\mathbf{N}} \to \mathscr{L}^1$ is *translation invariant* if for all $\mathbf{f} \in \Pi_{\mathbf{N}}$ and all $\mathbf{l} \in Z$

$$P \mathbf{E}_{\mathbf{l}} \mathbf{f} = E_{\mathbf{l}/\mathbf{N}} P \mathbf{f}. \tag{3.1}$$

Theorem 1. *$P\colon \Pi_{\mathbf{N}} \to \mathscr{L}^1$ is linear and translation invariant if and only if there exists a sequence of attenuation factors, $\tau = (\tau_{\mathbf{n}})_{\mathbf{n} \in Z}$, such that for every $\mathbf{f} \in \Pi_{\mathbf{N}}$ the Fourier coefficients $\gamma := \mathscr{F} P \mathbf{f}$ of $P\mathbf{f}$ and the DFT coefficients $\hat{\mathbf{f}} := \mathscr{F}_{\mathbf{N}} \mathbf{f}$ of $\mathbf{f}$ are related by*

$$\gamma = \tau \cdot \hat{\mathbf{f}}, \quad \text{i.e.,} \quad \gamma_{\mathbf{n}} = \tau_{\mathbf{n}} \hat{f}_{\mathbf{n}} \qquad (\mathbf{n} \in Z). \tag{3.2}$$

Proof. Since $\mathbf{f}=\sum_{\mathbf{k}\in K} f_{\mathbf{k}} \mathbf{E}_{\mathbf{k}} \mathbf{e}$, we obtain, assuming the linearity and translation invariance of P,

$$P\mathbf{f}=\sum_{\mathbf{k}\in K} f_{\mathbf{k}} E_{\mathbf{k}/\mathbf{N}} P\mathbf{e}. \tag{3.3}$$

Due to the shifting relation (2.8), and in view of $(\mathbf{n},\mathbf{k}/\mathbf{N})=\langle\mathbf{n},\mathbf{k}\rangle$, we get

$$(\mathscr{F} E_{\mathbf{k}/\mathbf{N}} P\mathbf{e})_{\mathbf{n}}=e^{-2\pi i\langle\mathbf{n},\mathbf{k}\rangle}(\mathscr{F} P\mathbf{e})_{\mathbf{n}},$$

and then, using the linearity of $\mathscr{F}$ and (2.3),

$$\begin{aligned}\gamma_{\mathbf{n}}=(\mathscr{F} P\mathbf{f})_{\mathbf{n}}&=\sum_{\mathbf{k}\in K} f_{\mathbf{k}}(\mathscr{F} E_{\mathbf{k}/\mathbf{N}} P\mathbf{e})_{\mathbf{n}}=\sum_{\mathbf{k}\in K} f_{\mathbf{k}}\, e^{-2\pi i\langle\mathbf{n},\mathbf{k}\rangle}(\mathscr{F} P\mathbf{e})_{\mathbf{n}}\\&=N_0\hat{f}_{\mathbf{n}}(\mathscr{F} P\mathbf{e})_{\mathbf{n}}=\tau_{\mathbf{n}}\hat{f}_{\mathbf{n}}\end{aligned}$$

if we let

$$\tau_{\mathbf{n}}:=N_0(\mathscr{F} P\mathbf{e})_{\mathbf{n}}. \tag{3.4}$$

Conversely, if (3.2) holds, we conclude from the injectivity of $\mathscr{F}$ [15, p. 13] that

$$P\mathbf{f}=\mathscr{F}^{-1}\mathscr{F} P\mathbf{f}=\mathscr{F}^{-1}\gamma=\mathscr{F}^{-1}(\tau\cdot\hat{\mathbf{f}})=\mathscr{F}^{-1}(\tau\cdot\mathscr{F}_{\mathbf{N}}\mathbf{f}).$$

Hence, P is a composition of linear maps, thus linear itself. Moreover, the shifting relations (2.8) and (2.10) now yield

$$(\mathscr{F} E_{\mathbf{l}/\mathbf{N}} P\mathbf{f})_{\mathbf{n}}=e^{-2\pi i(\mathbf{n},\mathbf{l}/\mathbf{N})}(\mathscr{F} P\mathbf{f})_{\mathbf{n}}=e^{-2\pi i\langle\mathbf{n},\mathbf{l}\rangle}\tau_{\mathbf{n}}\hat{f}_{\mathbf{n}}=\tau_{\mathbf{n}}(\mathscr{F}_{\mathbf{N}}\mathbf{E}_{\mathbf{l}}\mathbf{f})_{\mathbf{n}}=(\mathscr{F} P\mathbf{E}_{\mathbf{l}}\mathbf{f})_{\mathbf{n}},$$

which, by applying $\mathscr{F}^{-1}$, gives (3.1). □

Note that even in the univariate case Theorem 1 generalizes Gautschi's Theorems 3.1 and 3.2 [11], where $P\mathbf{f}$ is assumed to belong to a certain subset of $L^2(T)\cong L^2[0,1]$.

From now on we always assume that $P\colon \Pi_{\mathbf{N}}\to\mathscr{L}^1$ is linear and translation invariant, and not the null operator, and that $\tau=(\tau_{\mathbf{n}})_{\mathbf{n}\in Z}$ is the sequence of attenuation factors of P. From (3.4) and the fact that $\mathscr{F}(\mathscr{L}^p)\subseteq\ell^q$ if $1\leq p\leq 2$, $1/q=1-1/p$ [15, p. 25] we have

Corollary. (i) $\tau=N_0\mathscr{F} P\mathbf{e}$.
(ii) *If* $P_{\mathbf{e}}\in\mathscr{L}^p$, $1\leq p\leq 2$, *then* $\tau\in\ell^q$, *where* $1/q=1-1/p$.
(iii) $\tau\in\ell^1$ *if and only if* $P\mathbf{e}\in\mathscr{A}$.
In most applications, P has additional natural properties which are reflected by properties of τ:

Theorem 2. (i) $\tau_{\mathbf{n}}\in\mathbb{R}\,(\forall\mathbf{n}\in Z)$ *if and only if P preserves symmetry in the sense that*

$$\bar{f}_{-\mathbf{k}}=f_{\mathbf{k}}\quad(\forall\mathbf{k}\in K)\ \Rightarrow\ \overline{(P\mathbf{f})(-\mathbf{x})}=(P\mathbf{f})(\mathbf{x})\qquad(a.e.\ \mathbf{x}\in R). \tag{3.5}$$

(ii) $\bar{\tau}_{-\mathbf{n}}=\tau_{\mathbf{n}}(\forall\mathbf{n}\in Z)$ *if and only if P preserves the reality of the data,*

$$f_{\mathbf{k}}\in\mathbb{R}\quad(\forall\mathbf{k}\in K)\ \Rightarrow\ (P\mathbf{f})(\mathbf{x})\in\mathbb{R}\qquad(a.e.\ \mathbf{x}\in R). \tag{3.6}$$

(iii) $\tau_{\mathbf{0}}=1$ *and* $\tau_{\mathbf{k}\cdot\mathbf{N}}=0$ $(\forall\mathbf{k}\neq\mathbf{0})$ *if and only if P preserves unity in the sense of*

$$f_{\mathbf{k}}=1 \quad (\forall\mathbf{k}\in K) \Rightarrow (P\mathbf{f})(\mathbf{x})\equiv 1 \qquad (a.e.\ \mathbf{x}\in R). \tag{3.7}$$

Proof. (i) Since $(P\mathbf{f})(\mathbf{x})=(\mathscr{F}^{-1}(\tau\cdot\hat{\mathbf{f}}))(\mathbf{x})$ and

$$\overline{(P\mathbf{f})(-\mathbf{x})}\sim\sum_{\mathbf{n}\in Z}\overline{\tau_{\mathbf{n}}\hat{f}_{\mathbf{n}}}\,e^{2\pi i(\mathbf{n},\mathbf{x})}\sim(\mathscr{F}^{-1}(\bar{\tau}\cdot\bar{\hat{\mathbf{f}}}))(\mathbf{x}),$$

$(P\mathbf{f})(\mathbf{x})=\overline{(P\mathbf{f})(-\mathbf{x})}$ if and only if $\tau\cdot\hat{\mathbf{f}}=\bar{\tau}\cdot\bar{\hat{\mathbf{f}}}$, i.e., $\tau_{\mathbf{n}}\hat{f}_{\mathbf{n}}\in\mathbb{R}(\forall\mathbf{n})$. Similarly, $\bar{f}_{-\mathbf{k}}=f_{\mathbf{k}}(\forall\mathbf{k})$ if and only if $\hat{f}_{\mathbf{n}}\in\mathbb{R}(\forall\mathbf{n})$. Hence, (3.5) implies $\tau_{\mathbf{n}}\in\mathbb{R}(\forall\mathbf{n})$ and vice versa.

(ii) $\bar{\gamma}_{-\mathbf{n}}=\gamma_{\mathbf{n}}(\forall\mathbf{n})$ if and only if $g:=\mathscr{F}^{-1}\gamma$ is a real function. Hence $\bar{\tau}_{-\mathbf{n}}=\tau_{\mathbf{n}}(\forall\mathbf{n})$ if and only if $P\mathbf{e}$ is a real function, cf. (3.4). In view of (3.3) this is seen to be equivalent to the implication (3.6).

(iii) $f_{\mathbf{k}}=1(\forall\mathbf{k})$ if and only if $\hat{\mathbf{f}}=\mathbf{e}$. Also, $g(\mathbf{x})\equiv 1$ a.e. if and only if $\gamma_{\mathbf{n}}=\delta_{\mathbf{n},\mathbf{0}}$. Therefore, the implication (3.7) holds if and only if $\delta_{\mathbf{n},\mathbf{0}}=\gamma_{\mathbf{n}}:=\mathscr{F}P\mathscr{F}_{\mathbf{N}}^{-1}\mathbf{e}=\tau_{\mathbf{n}}\mathbf{e}$ (here, (3.2) was used). For $\mathbf{n}\neq\mathbf{k}\cdot\mathbf{N}$ the equality $\delta_{\mathbf{n},\mathbf{0}}=\tau_{\mathbf{n}}\mathbf{e}$ holds trivially. Hence, it holds for all $\mathbf{n}$ if and only if $\tau_{\mathbf{0}}=1$ and $\tau_{\mathbf{k}\cdot\mathbf{N}}=0(\forall\mathbf{k}\neq\mathbf{0})$. □

Of particular interest are operators P that interpolate any given data $\mathbf{f}\in\Pi_{\mathbf{N}}$, i.e., satisfy $P\mathbf{f}(\mathbf{k}/\mathbf{N})=f_{\mathbf{k}}(\forall\mathbf{k}\in Z)$. In view of (3.3) it is sufficient to require that P interpolate $\mathbf{e}$:

Definition. P is an *interpolation operator* if $P\mathbf{e}\in\mathscr{A}$ and $P\mathbf{e}(\mathbf{k}/\mathbf{N})=e_{\mathbf{k}}(\mathbf{k}\in Z)$.

For an interpolation operator the aliasing formula (2.6) implies

$$\hat{f}_{\mathbf{n}}=\sum_{\mathbf{k}\in Z}\gamma_{\mathbf{n}+\mathbf{l}\cdot\mathbf{N}}(\mathbf{n}\in Z) \quad \text{with} \quad \gamma:=\mathscr{F}P\mathbf{f}. \tag{3.8}$$

From (3.2) and Corollary 1 we therefore readily conclude

Theorem 3. *P is an interpolation operator if and only if $\tau\in\ell^1$* (i.e., $P\mathbf{e}\in\mathscr{A}$) *and*

$$\sum_{\mathbf{k}\in Z}\tau_{\mathbf{n}+\mathbf{k}\cdot\mathbf{N}}=1 \quad (\forall\mathbf{n}\in Z). \tag{3.9}$$

Gautschi's Theorem 3.3 [11] is a special means to compute attenuation factors of an interpolation operator. Its straightforward generalization to the multivariate case is left to the reader. In Sect. 5 we will discuss another method which, however, is restricted to a particular class of interpolation operators generated by translation of one (periodic or nonperiodic) function. The following trivial but very useful remark will be applied there: If for some data $\mathbf{f}$ we *know* the Fourier (series) coefficients γ of $P\mathbf{f}$ and *compute* the DFT coefficients $\hat{\mathbf{f}}$ of $\mathbf{f}$, then

$$\tau_{\mathbf{n}}=\frac{\gamma_{\mathbf{n}}}{\hat{f}_{\mathbf{n}}} \quad \text{if} \quad \hat{f}_{\mathbf{n}}\neq 0. \tag{3.10}$$

In particular, if it is known that $\hat{f}_{\mathbf{n}}=0\Rightarrow\tau_{\mathbf{n}}=0$, then all attenuation factors can be computed via the one FFT to determine $\hat{f}_{\mathbf{n}}$.

4. Tensor Product Operators

Many of the most widely used families of approximants for data given on rectangular domains are tensor product spaces of well-known univariant families. In fact, whenever the data can be approximated well by such a tensor product, this approach is very efficient. It is not surprising that the attenuation factors have product form in this case.

Definition. P is a *tensor product operator* if there exist linear operators $P_d\colon \Pi_{N_d} \to L^1(T)$ such that

$$f_{\mathbf{k}} = \prod_{d=1}^{D} f_{k_d}^{[d]} (\forall \mathbf{k} \in Z) \Rightarrow (P\mathbf{f})(\mathbf{x}) = \prod_{d=1}^{D} (P_d (f_k^{[d]})_{k \in \mathbb{Z}})(x_d) \quad \text{a.e.} \tag{4.1}$$

The usual notation is $P = \otimes_{1 \leq d \leq D} P_d$. It is easy to verify that the translation invariance of P is inherited to every P_d. Also, $P_d \neq 0$ as a consequence of our assumption $P \neq 0$.

Theorem 4. *The following statements are equivalent* (*for a linear and translation invariant operator* P):

(i) P *is a tensor product operator.*

(ii) *There are functions* $p_d \in L^1(T)$, $p_d \neq 0 (d = 1, \ldots, D)$ *such that*

$$(P\mathbf{e})(\mathbf{x}) = \prod_{d=1}^{D} p_d(x_d) \qquad a.e. \tag{4.2}$$

(iii) *There are sequences* $\tau^{[d]} \in \mathscr{F}_T(L^1(T))$, $\tau^{[d]} \neq 0 (d = 1, \ldots, D)$ *such that*

$$\tau_{\mathbf{n}} = \prod_{d=1}^{D} \tau_{n_d}^{[d]}. \tag{4.3}$$

In view of (3.3), (4.2) generalizes for arbitrary $\mathbf{f} \in \Pi_{\mathbf{N}}$ to

$$P\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{k} \in K} f_{\mathbf{k}} \prod_{d=1}^{D} p_d(x_d - k_d/N_d) \qquad a.e. \tag{4.4}$$

Proof. (i)$\Rightarrow$(ii): Since $e_{\mathbf{k}} = \Pi\, \delta_{0,k_d}^{N_d}$, (4.1) implies that $P\mathbf{e}$ has the asserted form.

(ii)$\Rightarrow$(iii): Applying $\mathscr{F}$ to both sides of (4.2) we see that in view of (3.4)

$$\tau_{\mathbf{n}} = N_0 \prod_{d=1}^{D} (\mathscr{F}_T\, p_d)_{n_d} = \prod_{d=1}^{D} N_d (\mathscr{F}_T\, p_d)_{n_d}, \tag{4.5}$$

where $\mathscr{F}_T\, p_d$ is now the sequence of Fourier coefficients of the univariate function p_d. Hence (4.3) is satisfied with

$$\tau_n^{[d]} = N_d (\mathscr{F}_T\, p_d)_n. \tag{4.6}$$

(iii)⇒(i): We define P_d by letting, for any univariate N_d-periodic sequence $\mathbf{f}^{[d]} \in \Pi_{N_d}$,

$$P_d \mathbf{f}^{[d]} := \mathscr{F}_T^{-1}(\tau^{[d]} \cdot \mathscr{F}_{N_d} \mathbf{f}^{[d]}),$$

$d=1, \ldots, D$. Since $\tau^{[d]} = \mathscr{F}_T \tilde{p}_d$ for some $\tilde{p}_d \in L^1(T)$, $P_d(\delta_{k,N_d}^{N_d})_{k \in \mathbf{Z}} = N_d^{-1} \mathscr{F}_T^{-1} \mathscr{F}_T \tilde{p}_d = N_d^{-1} \tilde{p}_d \in L^1(T)$, and by the obvious linearity of P_d, we have P_d: $\Pi_{N_d} \to L^1(T)$. By Theorem 1 (with $D=1$, $\mathbf{N}=N_d$), P_d is translation invariant. Finally, it is easy to check that the tensor product operator defined now by (4.1) for data $\mathbf{f}=(f_{\mathbf{k}})_{\mathbf{k} \in Z}$ of product form and extended to $\Pi_{\mathbf{N}}$ by linearity has exactly the attenuation factors (4.3). □

5. Generation by Translation

As a consequence of the linearity and translation invariance of P, $P\mathbf{f}$ can always be written as a linear combination of finitely many shifted versions of $P\mathbf{e}$, cf. Formula (3.3). In this section we investigate two similar representations of P – one more special, the other more general – and give appropriate formulas for the attenuation factors. In the first case, we assume that $P\mathbf{e}$ is itself a superposition of infinitely many shifted versions of a nonperiodic function which is the image of a nonperiodic unit pulse $(\delta_{\mathbf{k},0})_{\mathbf{k} \in Z}$. In the other case, $P\mathbf{f}$ (and hence $P\mathbf{e}$) is some finite linear combination of translates of a given periodic generating function (as in [8]), and $P\mathbf{f}$ interpolates $\mathbf{f}$ (as in [16]).

Let $H \in L^1(R)$ and define h by

$$h(\mathbf{x}) := (\sum_{\mathbf{j} \in Z} E_{\mathbf{j}} H)(\mathbf{x}) = \sum_{\mathbf{j} \in Z} H(\mathbf{x}-\mathbf{j}) \qquad (\mathbf{x} \in R). \tag{5.1}$$

Clearly, h is periodic; we say that it is *generated by the germ function* H. Since $\int_Q |h(\mathbf{x})| d\mathbf{x} \leqq \int_R |H(\mathbf{x})| d\mathbf{x}$ (by the Monotone Convergence Theorem), $h \in \mathscr{L}^1$. Using Lebesgue's Convergence Theorem we get (cf. [15], p. 128):

$$\begin{aligned} \eta_{\mathbf{n}} := (\mathscr{F} h)_{\mathbf{n}} &= \int_Q \sum_{\mathbf{j} \in Z} H(\mathbf{x}-\mathbf{j})\, e^{-2\pi i(\mathbf{n},\mathbf{x})} d\mathbf{x} \\ &= \sum_{\mathbf{j} \in Z} \int_Q H(\mathbf{x}-\mathbf{j})\, e^{-2\pi i(\mathbf{n},\mathbf{x}-\mathbf{j})} d\mathbf{x} \\ &= \int_R H(\mathbf{x})\, e^{-2\pi i(\mathbf{n},\mathbf{x})} d\mathbf{x} \\ &= \hat{H}(\mathbf{n}), \quad \text{where} \quad \hat{H} = \mathscr{F}_R H. \end{aligned} \tag{5.2}$$

We may now think of H as the image of the nonperiodic unit pulse $(\delta_{\mathbf{k},0})_{\mathbf{k} \in Z}$ under some linear and translation invariant operator, called P again, defined on the space $\mathbb{C}^Z$ of (in general nonperiodic) sequences. On the subspace $\Pi_{\mathbf{N}} \subset \mathbb{C}^Z$ this operator then satisfies

$$(P\mathbf{f})(\mathbf{x}) = \sum_{\mathbf{j} \in Z} f_{\mathbf{j}} H(\mathbf{x}-\mathbf{j}/\mathbf{N}) = \sum_{\mathbf{k} \in K} f_{\mathbf{k}} h(\mathbf{x}-\mathbf{k}/\mathbf{N}) \qquad (\mathbf{f} \in \Pi_{\mathbf{N}}, \mathbf{x} \in R), \tag{5.3}$$

i.e.,

$$\mathbf{P}\mathbf{f} = \sum_{\mathbf{j}\in Z} f_{\mathbf{j}}\, E_{\mathbf{j}/\mathbf{N}} H = \sum_{\mathbf{k}\in K} f_{\mathbf{k}}\, E_{\mathbf{k}/\mathbf{N}} h. \tag{5.3'}$$

Obviously, $h = P\mathbf{e}$, so by (3.4) and (5.2) the attenuation factors of P are

$$\tau_{\mathbf{n}} = N_0\, \hat{H}(\mathbf{n}). \tag{5.4}$$

Let us now turn to *interpolation* by translates of a given *periodic* generating function $h \in \mathscr{A}$. (Trivially, a corresponding function $H \in L^1(R)$ satisfying (5.1) always exists, but first we do not assume that such a function is given.) For given data $\mathbf{f} \in \Pi_{\mathbf{N}}$ we want to determine coefficients $c_{\mathbf{k}} = c_{\mathbf{k}}(\mathbf{f})(\mathbf{k} \in K)$ such that

$$f_{\mathbf{j}} = \sum_{\mathbf{k}\in K} c_{\mathbf{k}}(\mathbf{f})\, h((\mathbf{j}-\mathbf{k})/\mathbf{N}) \qquad (\mathbf{j} \in Z), \tag{5.5}$$

if this interpolation problem has a solution for all. As noted by Locher [16], this linear systems becomes diagonal after applying the DFT since the right-hand side of (5.5) is a convolution: i.e., if we extend $c_{\mathbf{k}}$ periodically and let

$$\begin{aligned} &\mathbf{c} := (c_{\mathbf{k}})_{\mathbf{k}\in Z}, \quad \mathbf{h} := (h_{\mathbf{k}})_{\mathbf{k}\in Z} := (h(\mathbf{k}/\mathbf{N}))_{\mathbf{k}\in Z}, \\ &\hat{\mathbf{f}} := \mathscr{F}_{\mathbf{N}} f, \quad \hat{\mathbf{c}} := \mathscr{F}_{\mathbf{N}} \mathbf{c}, \quad \hat{\mathbf{h}} := \mathscr{F}_{\mathbf{N}} \mathbf{h}, \end{aligned} \tag{5.6}$$

(5.5) is equivalent to

$$\hat{\mathbf{f}} = N_0\, \hat{\mathbf{c}} \cdot \hat{\mathbf{h}}. \tag{5.7}$$

Obviously, this system can be solved for the unknown $\hat{\mathbf{c}}$, and therefore (5.5) can be solved for $\mathbf{c}$ if and only if $\hat{h}_{\mathbf{k}} = 0$ implies $\hat{f}_{\mathbf{k}} = 0$. In particular, both systems have for arbitrary $\mathbf{f} \in \Pi_{\mathbf{N}}$ a unique solution if and only if $\hat{h}_{\mathbf{k}} \neq 0$ for all $\mathbf{k}$ [16]. In this case, P defined by

$$(P\mathbf{f})(\mathbf{x}) := \Big(\sum_{\mathbf{k}\in K} c_{\mathbf{k}}\, E_{\mathbf{k}/\mathbf{N}} h\Big)(\mathbf{x}) = \sum_{\mathbf{k}\in K} c_{\mathbf{k}}\, h(\mathbf{x} - \mathbf{k}/\mathbf{N}) \tag{5.8}$$

is an interpolation operator as defined in Sect. 3. In particular, $P\mathbf{h} = h$, i.e., $c_{\mathbf{k}}(\mathbf{h}) = e_{\mathbf{k}} = \delta^{\mathbf{N}}_{\mathbf{k},\mathbf{0}}$. According to the basic relation (3.2) and the aliasing formula (2.6), the Fourier coefficients $\eta_{\mathbf{n}}$ of h and the attenuation factors $\tau_{\mathbf{n}}$ of P satisfy

$$\tau_{\mathbf{n}} = \frac{\eta_{\mathbf{n}}}{\hat{h}_{\mathbf{n}}} = \frac{\eta_{\mathbf{n}}}{\sum_{\mathbf{j}\in Z} \eta_{\mathbf{n}+\mathbf{j}\cdot\mathbf{N}}} \qquad (\mathbf{n} \in Z). \tag{5.9}$$

(The left-hand side equality is an example for (3.10). Note also that (3.9) holds.) If h itself is generated by a nonperiodic germ function H in the sense of (5.1), then by (5.2),

$$\tau_{\mathbf{n}} = \frac{\hat{H}(\mathbf{n})}{\hat{h}_{\mathbf{h}}} = \frac{\hat{H}(\mathbf{n})}{\sum_{\mathbf{j}\in Z} \hat{H}(\mathbf{n} + \mathbf{j}\cdot\mathbf{N})} \qquad (\mathbf{n} \in Z). \tag{5.10}$$

Example. Univariate interpolation by periodic spline functions of order ν (degree $\nu-1$) with knots at the meshpoints, if ν is even, or half-way in between, if ν is odd, defines an interpolation operator with attenuation factors

$$\tau_n = \frac{\psi(n/N)}{\sum_{k\in\mathbb{Z}} \psi(k+n/N)} \quad \text{where} \quad \psi(x) = \left(\frac{\sin \pi x}{\pi x}\right)^{\nu} \tag{5.11}$$

(see, e.g., Gautschi [11], p. 388, for ν even). In view of $\sin(k+n/N)\pi = (-1)^k \sin n\pi/N$ this simplifies to

$$\tau_n = \left[\sum_{k\in\mathbb{Z}} \left((-1)^k \frac{n/N}{k+n/N}\right)^{\nu}\right]^{-1}. \tag{5.12}$$

ψ is the Fourier transform (in our normalization) of Schoenberg's central cardinal B-spline M_ν, a result Schoenberg attributes to Laplace [20, p. 12]. Therefore, (5.11) is a special case of (5.10). (Since $H(x) = M_\nu(Nx)$, the argument n in (5.10) has to be replaced by n/N). However, (5.11) is also a special case of (5.4): In fact, $\psi(x)/\Sigma\psi(k+x)$ is the Fourier transform of the cardinal spline interpolant to the unit pulse $\{\delta_{0k}\}_{k\in\mathbb{Z}}$ [19]. This derivation of (5.11) is due to Reinsch [11, p. 338]. (The factor $N_0 = N$ in (5.4) is again a consequence of replacing the meshsize 1 by $1/N$.) Assuming ν even, Gautschi derived an elegant recursive formula for computing these attenuation factors, thus avoiding the infinite sum in (5.12) [11, pp. 377, 388]. A similar recursive formula holds for odd ν [12]. A nearly as efficient way to compute these attenuation factors consists in determining the DFT coefficients $\hat{h}_n$ of the periodically extended cardinal B-spline $h(x) = \sum_{k\in\mathbb{Z}} M_\nu(Nx - Nk)$ (due to the compact support the sum always contains just a finite number of nonzero terms, usually 0 or 1); then according to (5.10), $\tau_n = \psi(n/N)/\hat{h}_n$, i.e., $\hat{h}_n$ is equal to the infinite sum in (5.11).

6. Box Splines

Interesting examples for the theory in Sect. 5 are provided by *box splines*, which include tensor product splines and certain finite elements as special cases. So the following treatment covers a variety of very useful examples. Box splines were introduced by de Boor and DeVore [2] and further studied mainly by de Boor and Höllig [3–5], Chui and Wang [6, 7], and Dahmen and Micchelli [8–10]. We consider here *interpolation by translates of periodically extended centered box splines*. (The use of centered box splines is suggested by the univariate case, where for interpolation by even degree splines the interpolation points have to be chosen between the knots, preferably in the middle.)

Given $m \geq D$ vectors $\mathbf{x}^1, \ldots, \mathbf{x}^m \in \Xi$ which span R, the *centered box spline* $B(\mathbf{x}) = B(\mathbf{x}|\mathbf{x}^1, \ldots, \mathbf{x}^m)$ is implicitly defined by requiring that for every $G \in C(R)$ with compact support

$$\int_{[-\frac{1}{2},\frac{1}{2}]^m} G\left(\sum_{\ell=1}^{m} \lambda_\ell \mathbf{x}^\ell\right) d\lambda_1 \ldots d\lambda_m = \int_R G(\mathbf{x})\, B(\mathbf{x}|\mathbf{x}^1, \ldots, \mathbf{x}^m)\, d\mathbf{x}. \tag{6.1}$$

B is a bounded function of compact support. (Our definition differs slightly from the standard one in that the grid Ξ is used instead of Z.)

We denote the span of a set $A=\{a_1, a_2, \ldots\}$ of vectors (in some linear space) by $\langle A\rangle$ or $\langle a_1, a_2, \ldots\rangle$, and $|A|$ is the cardinality of A.
Let $X:=\{\mathbf{x}^1, \ldots, \mathbf{x}^m\}$ and

$$\delta:=\delta(X):=\max\{l; \langle X\setminus Y\rangle = R \quad \text{for all} \quad Y\subset X \quad \text{with} \quad |Y|=l\}.$$

$\delta(X)$, which clearly satisfies $\delta(X)\leq m-D$, determines the smoothness of the spline space $\mathscr{S}(X):=\langle\{B(\cdot-\mathbf{j}/\mathbf{N})\}_{\mathbf{j}\in Z}\rangle$; one has [3]

$$\mathscr{S}(X)\subset C^{\delta-1}(R)\setminus C^{\delta}(R),$$

and, if $\mathscr{P}_l$ denotes the set of polynomials of total degree at most l on R,

$$\mathscr{P}_\delta\subset\mathscr{S}(X), \qquad \mathscr{P}_{\delta+1}\not\subset\mathscr{S}(X).$$

By choosing $G(\mathbf{x})=e^{-2\pi i(\mathbf{u},\mathbf{x})}$ in (6.1) one sees that the Fourier transform $\hat{B}=\mathscr{F}_R B$ of B in our normalization is

$$\hat{B}(\mathbf{u})=\prod_{l=1}^{m}\frac{\sin(\mathbf{u},\mathbf{x}^l)\pi}{(\mathbf{u},\mathbf{x}^l)\pi}, \tag{6.2}$$

with

$$\frac{\sin(\mathbf{u},\mathbf{x}^l)\pi}{(\mathbf{u},\mathbf{x}^l)\pi}:=1 \quad \text{if} \quad (\mathbf{u},\mathbf{x}^l)=0. \tag{6.3}$$

As an example for Definition 5.1, we consider now periodically extended box splines:

$$b(\mathbf{x}):=\sum_{\mathbf{j}\in Z}B(\mathbf{x}-\mathbf{j}). \tag{6.4}$$

As we have seen in Sect. 5, interpolation of arbitrary periodic data on Ξ by a linear combination of translates $\{b(\cdot-\mathbf{k}/\mathbf{N})\}_{\mathbf{k}\in K}$ of these periodically extended box splines is possible if and only if

$$\hat{b}_{\mathbf{n}}:=(\mathscr{F}_{\mathbf{N}}\{b(\mathbf{k}/\mathbf{N})\}_{\mathbf{k}\in K})_{\mathbf{n}}\neq 0, \qquad \forall\mathbf{n}\in K, \tag{6.5a}$$

or, equivalently,

$$\sum_{\mathbf{j}\in Z}\hat{B}(\mathbf{n}+\mathbf{j}\cdot\mathbf{N})\neq 0, \qquad \forall\mathbf{n}\in K, \tag{6.5b}$$

and in this case the attenuation factors for this interpolation process are given by (5.10),

$$\tau_{\mathbf{n}}=\frac{\hat{B}(\mathbf{n})}{\hat{b}_{\mathbf{n}}}=\frac{\hat{B}(\mathbf{n})}{\sum_{\mathbf{j}\in Z}\hat{B}(\mathbf{n}+\mathbf{j}\cdot\mathbf{N})} \qquad (\mathbf{n}\in Z). \tag{6.6}$$

As in the univariate case, the last expression can be simplified. First, note that

$$\hat{B}(\mathbf{u})=0 \Leftrightarrow \exists l: (\mathbf{u},\mathbf{x}^l)\in\mathbb{Z}\setminus\{0\}. \tag{6.7}$$

Second, assume $\hat{B}(\mathbf{n}) \neq 0$. Then $(\mathbf{n}, \mathbf{x}^l) \in \mathbb{Z}$ is equivalent to $(\mathbf{n}, \mathbf{x}^l) = 0$; hence, $(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l) = 0$ implies $(\mathbf{n}, \mathbf{x}^l) = 0$ since $(\mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l) \in \mathbb{Z}$. Consequently, if we set

$$\frac{(\mathbf{n}, \mathbf{x}^l)}{(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l)} := (-1)^{(\mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l)} \quad \text{if} \quad (\mathbf{n}, \mathbf{x}^l) = (\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l) = 0, \tag{6.8}$$

we get, if $\hat{B}(\mathbf{n}) \neq 0$,

$$\hat{B}(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}) = \hat{B}(\mathbf{n})(-1)^{(\mathbf{j} \cdot \mathbf{N}, \Sigma \mathbf{x}^l)} \prod_{l=1}^{m} \frac{(\mathbf{n}, \mathbf{x}^l)}{(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l)}. \tag{6.9}$$

Hence,

$$\tau_{\mathbf{n}} = \begin{cases} 0 \quad \text{if} \quad \hat{B}(\mathbf{n}) = 0, \\ \left[\sum_{\mathbf{j} \in Z} (-1)^{(\mathbf{j} \cdot \mathbf{N}, \Sigma \mathbf{x}^l)} \prod_{l=1}^{m} \frac{(\mathbf{n}, \mathbf{x}^l)}{(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l)} \right]^{-1} \quad \text{otherwise.} \end{cases} \tag{6.10}$$

Summarizing, we obtain

Lemma 5. *Interpolation on the grid Ξ by a linear combination of the translates $\{b(\cdot - \mathbf{k}/\mathbf{N})\}_{\mathbf{k} \in K}$ of the periodically extended box spline* (6.4) *is possible for arbitrary data if and only if* (6.5a) *or, equivalently,* (6.5b) *holds. The attenuation factors are then given by* (6.6) *or, equivalently, by* (6.10), *where the convention* (6.8) *must be observed.*

Although the formula (6.10) may turn out to be useful in theoretical considerations, for the computation of the attenuation factors one will in practice use the first expression in (6.6), which essentially requires the evaluation of the periodic box spline b on Ξ and the application of a D-dimensional FFT to the obtained values. (In the case of tensor product splines one should, of course, apply Theorem 4 instead.) The values of the box spline at the grid points can be computed either by applying the recurrence relation [3] or – in the bivariate case – by using the Bernstein-Bézier representation [1, 17, 18].

It would be interesting and useful to know a necessary and sufficient condition in terms of $\mathbf{x}^1, \ldots, \mathbf{x}^m$ for (6.5) to hold. Here is a necessary one, which follows from a result of de Boor and Höllig [3] adapted to our situation (different grid, periodicity):

Lemma 6. *If $\{\mathbf{x}^1, \ldots, \mathbf{x}^m\} \subset \Xi$ contains a basis $\{\mathbf{x}^{\kappa(1)}, \ldots, \mathbf{x}^{\kappa(d)}\}$ of R for which*

$$\Delta := |\det(\mathbf{N} \cdot \mathbf{x}^{\kappa(1)}, \ldots, \mathbf{N} \cdot \mathbf{x}^{\kappa(D)})| \neq 1 \tag{6.11}$$

and

$$\mathbf{e}^d := (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{Z}\mathbf{x}^{\kappa(1)} + \ldots + \mathbb{Z}\mathbf{x}^{\kappa(D)}, \qquad d = 1, \ldots, D, \tag{6.12}$$

then the functions $\{b(\cdot - \mathbf{k}/\mathbf{N})\}_{\mathbf{k} \in K}$ are linearly dependent and (6.5) *does not hold.*

Proof. As shown in [3], both the sets $\{B(\cdot - \mathbf{j}/\mathbf{N})\}_{\mathbf{j} \in Z}$ and $\{\Delta B(\cdot - \Sigma j_d \mathbf{x}^{\kappa(d)})\}_{\mathbf{j} \in Z}$ provide a partition of unity. Hence this is also true for

$$\{\sum_{\mathbf{j} \in Z} B(\cdot - \mathbf{j} - \mathbf{k}/\mathbf{N})\}_{\mathbf{k} \in K} = \{b(\cdot - \mathbf{k}/\mathbf{N})\}_{\mathbf{k} \in K} \tag{6.13}$$

and, in view of (6.12), the sum over the functions in the second set considered can be similarly reformulated as a sum of translates of Δb. Hence the constant 1 can be represented (and interpolated) by two different linear combinations of translates of b. Since interpolation on Ξ by translates of b corresponds to a linear map between two spaces of the same dimension $N_0 = \Pi N_d$, uniqueness of the interpolant is equivalent to existence of an interpolant for arbitrary data. □

Note that the breakdown of interpolation described in Lemma 6 is not or not only due to cancellation in one of the infinite sums in (6.5b), but due to all terms being zero in such a sum. In fact, the assumptions of Lemma 6 persist if $\{\mathbf{x}^1, \ldots, \mathbf{x}^m\}$ is replaced by $\{\mathbf{x}^1, \ldots, \mathbf{x}^m, \mathbf{x}^1, \ldots, \mathbf{x}^m\}$, so that each direction appears an even number of times. But then all terms in the sum in (6.10) are nonnegative and, likewise, all nonvanishing terms in each sum in (6.5b) have the same sign.

From this it follows, on the other hand, that if directions are chosen such that $\sum_{\mathbf{j} \in Z} |\hat{B}(\mathbf{n} + \mathbf{j} \cdot \mathbf{N})| > 0$ for all $\mathbf{n} \in K$ and each direction appears an even number of times among $\mathbf{x}^1, \ldots, \mathbf{x}^m$, then interpolation is possible and is numerically stable. (An example is the tensor product of odd degree splines.)

Here is, as an illustration for Lemma 6, a simple, but fairly general example, where interpolation is in general impossible:

Example. Let $\mathbf{x}^l := \mathbf{e}^l / N_l := (0, \ldots, 0, \ 1/N_l, \ 0, \ldots, 0)^T$, $l = 1, \ldots, D-1$, and $\mathbf{x}^D := (*, *, \ldots, *, \nu/N_D)^T$ with $\nu \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $N_D/\nu \in \mathbb{Z}$. Further directions $\mathbf{x}^{D+1}, \ldots, \mathbf{x}^m$ can be chosen arbitrarily. Then, for $\mathbf{n} := (N_D/\nu)\, \mathbf{e}^D := (0, \ldots, 0, \ N_D/\nu)$, we have $\hat{B}(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}) = 0$, $\forall \mathbf{j} \in Z$, and hence $\hat{b}_{\mathbf{n}} = 0$. [*Proof.* If $\mathbf{j} \notin \langle \mathbf{e}^D \rangle = \langle \mathbf{n} \rangle$, $\mathbf{j} \neq \mathbf{0}$, then $(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^l) \in \mathbb{Z} \setminus \{0\}$ for some l, $1 \leqq l < D$. If $\mathbf{j} \in \langle \mathbf{n} \rangle$, then $(\mathbf{n} + \mathbf{j} \cdot \mathbf{N}, \mathbf{x}^D) = 1 + \mathbf{j}_D \nu \neq 0$.]

Contrary to appearance condition (6.11) does not depend on the meshsize determined by $\mathbf{N}$ (since $\mathbf{N} \cdot \mathbf{x}^l$ does not), while condition (6.12) does depend on $\mathbf{N}$ (because $\mathbf{x}^l$ does). Since one wants to choose box spline types (determined by $\mathbf{N} \cdot \mathbf{x}^1, \ldots, \mathbf{N} \cdot \mathbf{x}^m$) which work on every 1-periodic grid, (6.11) should not hold, i.e., every basis $\{\mathbf{x}^{\kappa(1)}, \ldots, \mathbf{x}^{\kappa(D)}\} \subseteq \{\mathbf{x}^1, \ldots, \mathbf{x}^m\}$ should satisfy

$$\Delta := |\det(\mathbf{N} \cdot \mathbf{x}^{\kappa(1)}, \ldots, \mathbf{N} \cdot \mathbf{x}^{\kappa(D)})| = 1. \tag{6.14}$$

De Boor and Höllig [3] showed that this condition is necessary for the set $\{B(\cdot - \mathbf{j}/\mathbf{N})\}_{\mathbf{j} \in Z}$ to be globally linearly independent (i.e., linear combinations of infinitely many members are identically zero only when all coefficients vanish). Dahmen and Micchelli [8] and Rong-qing Jia [14] proved that the condition is also sufficient (even if the coefficients are allowed to grow arbitrarily). Consequently, (6.14) implies that the functions (6.13) are linearly independent. But from this we cannot conclude that then these functions are linearly independent on Ξ (which would mean that there is always a unique interpolant), though this seems to be very likely.

In the bivariate case $R = \mathbb{R}^2$ (i.e., $D = 2$) condition (6.14) leaves, up to obvious symmetries, only the vectors $\mathbf{N} \cdot \mathbf{x} = (1, 0)$, $(0, 1)$ and $(1, 1)$ as candidates for elements of X. Let μ_1, μ_2, μ_3 denote the multiplicities with which they appear.

De Boor et al. [5] showed that for arbitrary $\mu_1, \mu_2, \mu_3 \geqq 0$ with $\langle X \rangle = R$ cardinal interpolation or, in our normalization, interpolation on Ξ from $\mathscr{S}(X)$ is always possible and unique. Consequently, interpolation of periodic data is also always possible and unique.

Through reference to standard results on Toeplitz operators the question of cardinal interpolation is in [5] first reduced to that of proving

$$\sum_{\mathbf{j} \in Z} B(\mathbf{j}/\mathbf{N})\, e^{-2\pi i \langle \mathbf{j}, \mathbf{x} \rangle} \neq 0, \qquad \forall \mathbf{x} \in Q, \tag{6.15}$$

which, in view of the Poisson summation formula, is equivalent to

$$\sum_{\mathbf{j} \in Z} \hat{B}(\mathbf{N} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{j}) \neq 0, \qquad \forall \mathbf{x} \in Q. \tag{6.16}$$

In [5] the nonvanishing of this sum is then established in the bivariate case mentioned above. Note that in our condition (6.5b) for the periodic case, Q is just replaced by the finite subset $K/\mathbf{N}$. Hence, (6.16) clearly implies (6.5).

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