of $1/1.5 \times 10^4$ and the total performance is found to be satisfactory for practical use. To give better accuracy, the frequency timed by gating a crystal oscillator must be higher, so that many pulses to be counted are generated and much higher accuracy is theoritically possible. Alternatively, the measuring time needed to be taken more longer, so that the number of data collected in a computer is increased. Also, with the improvement of the lever system including swivels, very high accuracy up to $\frac{1}{5} \times 10^5$ can be guaranteed.

5 Conclusions

To enable us to appreciate the qualities of the GWMD, the principle and the dynamical characteristics of the gyroscopic method were reviewed briefly, and some remarkable features as a weight measuring device were described. Through experimental results, the performance of the GWMD was demonstrated by comparison to comertially available one in Stevens-Wöhwa.

However, some technical problems in the GWMD must be taken into account for future development. The first problem is to forecast the area of industries where the GWMD will be needed as a highly precise weight measurement. For measurement for small amount of weights less than 20 to 30 N, electromagnetic force-balance types can be sufficiently worked with accuracy as same as the GWMD. On the other hand, for measurement for weight up to 3×10^5 N, the existing weighing scales can sufficiently achieved a repeatable accuracy of ± 100 N. The exceptionally high resolving power such as the GWMD is not required in this area. For measurement of middle class of weights, cheap load cells on the market can be worked with adequate accuracy up to 1/10,000.

Secondly, the GWMD is prohibitively costly. The gyroscopic instruments have been manufactured by specialist firms and are now used for special purposes where limited accuracy is required such as flight and navigation control and other aeronautical instruments. The demand for gyroscopic instruments is slight and the manufacturing cost becomes highly expensive.

The application of the GWMD is now limited to the specific areas of industries. When the importance of the GWMD is fully appreciated by industries in the future, this device will be utilized in many areas not at present visualized. The GWMD appears to have a considerable potential for providing standards of weight measurement.

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References

Arnold, R. N., and Maunder, L., 1961, *Gyrodynamics and its Engineering Applications*, Academic Press, pp. 85–93.

Hashimoto, Y., 1982, "Application of Gyroscopic Method for Weighing Instruments," *Technical Journal of Material Test*, Vol. 27, No. 4, pp. 257–267.

Sato, C., and Kanamori, H., 1986, "Stability of Gyroscopic Force Measuring Apparatus," *Proc. of SICE' 86* (Tokyo), pp. 569–570.

Shilling, G. D., 1963, *Process Dynamics and Control*, Holt, Rinehart and Winston, New York, pp. 107–122.

Stevens, C. & Son Ltd., 1979, "Stevens-Wöhwa Gyroscopic Force Measuring System" German Federal Republic Patent, No. P21034048.

Uchino, H., and Maejima, T., 1982, "Gyroscopic Force Measuring Apparatus and its Applications," *Instrumentation*, Vol. 25, No. 11, pp. 78–84.

Uchino, H., and Maejima, T., 1983, "Gyroscopic Force Measuring Apparatus and Its Applications," *Oil Pressure and Air Pressure*, Vol. 14, No. 2, pp. 20–25.

Wöhwa-Waagenbau, Josef Wöhl GmbH & Co., Postfach 7, D-7114 Pfedelbach.

A Robust Controller for Second-Order Systems Using Acceleration Measurements

C.-H. Chuang,[1] Oliver Courouge,[1] and Jer-Nan Juang[2]

This paper presents a robust control design using strictly positive realness for second-order dynamic systems. A robust strictly positive real controller stabilizes second-order systems with only acceleration measurements. An important property of this design is that the stabilization is independent of the system plant parameters. The control design connects a virtual system to a given plant such that any strictly positive real controller can be used to achieve robust stability. A spring-mass system is used as an example to demonstrate the robust stability and robust performance of this design.

1 Introduction

Positive Real (PR) systems have many applications for shape and vibration control of large flexible structures. Benhabib et al. (1981) used strictly PR controllers with position and velocity feedback to control large space structures. PR feedback with velocity measurement has been examined by Takahashi and Slater (1986) for control of a flutter mode. McLaren and Slater (1987) used a passive controller with collocated velocity sensors and actuators. Several passive control designs using acceleration, velocity, and position measurements have also been presented (Juang and Phan, 1990 and Juang et al., 1991). A method presented later (Morris and Juang, 1991) used displacement sensors. Bar-Kana et al. (1991) examined direct position plus velocity feedback. A feedforward positive real design has also been studied (Chuang et al., 1992).

Nevertheless, for some spacecraft and civil structures, only acceleration is directly measurable. Even though velocity and position can be obtained by integrating the measured acceleration, exact initial values of velocity and position are needed to achieve asymptotic stability. The bias in acceleration measurement can also reduce the integration accuracy. In this study, we develop a virtual system, which is connected to a strictly positive real (SPR) controller, when only acceleration is directly measurable. Although integration is carried out in the virtual system, initial values of the states of the virtual system can be arbitrary and the closed-loop system is asymptotically stable. Furthermore, the bias in acceleration measurement can be scaled down by the system matrix of the virtual system.

The inputs to the virtual system are only acceleration and the control force applied to the plant. More important, the virtual system is plant model (excluding input matrix) independent, and thus the global system is robustly positive real. An input/output controller can be constructed by using any strictly positive real controller. When the stiffness matrix of the second-order system is positive definite, we show that it is possible to stabilize the displacement if the actuators are properly located. With this design, the displacement is globally asymptotically stable. A spring-mass example with three masses and no damp-

[1] Associate Professor and Graduate Research Assistant, respectively, Georgia Institute of Technology, Atlanta, GA 30332.

[2] Principal Scientist, Spacecraft Dynamics Branch, NASA Langley Research Center, Hampton, VA 23665.

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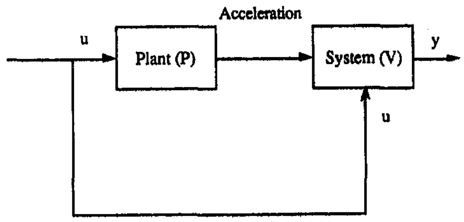

Fig. 1 A virtual system

ing is used to illustrate the design. Robust performance is demonstrated for a spring-mass system with only one mass and one spring.

2 Virtual System Design

The multivariable system (Plant (P)) is described by

$$M\ddot{x} + D\dot{x} + Kx = Bu \quad (1)$$

where u is an $m \times 1$ control vector, x is an $n \times 1$ state vector, M is an $n \times n$ symmetric positive definite matrix, D and K are $n \times n$ symmetric positive semi-definite matrices, and B is an $n \times m$ matrix. Let a virtual system (V) be defined by the following equation

$$\dot{x}_a = \Lambda\ddot{x} + \bar{B}u \quad (2)$$

where Λ is a $p \times n$ matrix, $\bar{B}$ is a $p \times m$ matrix, and x_a is a $p \times 1$ vector. The following theorem allows us to compute an output y that will make the global system (a combined system of the given plant and the virtual system) positive real. Note that since Λ has $p \times n$ dimensions, the number of state variables for the virtual system x_a can be made smaller than the number of the plant state variables x.

Theorem 1: Let

$$2H_v\Lambda = B^T$$
$$\bar{B}^T M_a^T = 2H_v \quad (3)$$

where M_a is a $p \times p$ positive semi-definite matrix. If

$$y = H_v\dot{x}_a \quad (4)$$

then the system with input u and output y is positive real. This scheme is illustrated in Fig. 1.

Proof: Let

$$X^T = [x_1^T \quad x_2^T \quad x_3^T \quad x_4^T] = [x^T \quad \dot{x}^T \quad x_a^T \quad \dot{x}_a^T] \quad (5)$$

The equations describing the global system may be rewritten as

$$\begin{cases} \dot{X} = f(X) + g(X)u \\ Y = h(X) \end{cases} \quad (6)$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -M^{-1}Dx_2 - M^{-1}Kx_1 \\ x_4 \\ -\Lambda M^{-1}Dx_2 - \Lambda M^{-1}Dx_1 \end{bmatrix},$$

$$g(X) = \begin{bmatrix} 0 \\ M^{-1}B \\ 0 \\ \Lambda M^{-1}B + \bar{B} \end{bmatrix}, \quad h(X) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ H_v x_4 \end{bmatrix} \quad (7)$$

Let a function $\phi(X)$ be

$$\phi(X) = \tfrac{1}{2}\dot{x}^T M\dot{x} + \tfrac{1}{2}x^T Kx + \tfrac{1}{2}(\dot{x}_a - \Lambda\dot{x})^T M_a(\dot{x}_a - \Lambda\dot{x}) \quad (8)$$

where M_a is positive semi-definite. The sum of the first two terms corresponds to the stored energy of the plant. The third term is added to achieve a positive real design. The function $\phi(X)$ can be written using (5) as

$$\phi(X) = \tfrac{1}{2}x_2^T Mx_2 + \tfrac{1}{2}x_1^T Kx_1 + \tfrac{1}{2}(x_4 - \Lambda x_2)^T M_a(x_4 - \Lambda x_2) \quad (9)$$

$\phi(X)$ is a positive function and $\phi(0) = 0$. For a positive real system, it must be shown (Moylan, 1974) that

$$h(x) = \tfrac{1}{2}g^T(x)\nabla\phi(x) \quad (10)$$

and there exists a function $1(X)$ such that

$$\nabla^T\phi(X)f(X) = -1^T(X)1(X) \quad (11)$$

The calculation for (11) can be simplified by considering

$$\nabla^T\phi(X)f(X) = \left.\frac{d\phi(X)}{dt}\right|_{u=0}$$
$$= \dot{x}^T(M\ddot{x} + Kx) + \frac{1}{2}(\ddot{x}_a - \Lambda\ddot{x})^T M_a(\dot{x}_a - \Lambda\dot{x})$$
$$\left. + \frac{1}{2}(\dot{x}_a - \Lambda\dot{x})^T M_a(\ddot{x}_a - \Lambda\ddot{x})\right|_{u=0} \quad (12)$$

when $u = 0$, the last two terms of (12) are zero and therefore

$$\nabla^T\phi(X)f(X) = \dot{x}^T(M\ddot{x} + Kx)|_{u=0} = -\dot{x}^T D\dot{x} = -x_2^T Dx_2 \quad (13)$$

Since D is positive semi-definite, it is possible to find a matrix R such that $D = R^T R$. The above equality becomes

$$\nabla^T\phi(X)f(X) = -(Rx_2)^T(Rx_2) = -1^T(X)1(X) \quad (14)$$

where $1(X) = Rx_2$.

Since the first and third rows of $g(x)$ in Eq. (7) are zero, only the partial derivatives with respect to velocity are needed to evaluate Eq. (10). Therefore,

$$\frac{\partial\phi}{\partial x_2} = x_2^T M + (\Lambda x_2 - x_4)^T M_a\Lambda$$
$$\frac{\partial\phi}{\partial x_4} = (x_4 - \Lambda x_2)^T M_a \quad (15)$$

These lead to

$$2h(X) = (M^{-1}B)^T\left(\frac{\partial\phi}{\partial x_2}\right)^T + (\Lambda M^{-1}B + \bar{B})^T\left(\frac{\partial\phi}{\partial x_4}\right)^T$$
$$= (B^T - \bar{B}^T M_a^T\Lambda)x_2 + \bar{B}^T M_a^T x_4 \quad (16)$$

$h(X)$ will be equal to $H_v x_4$ in Eq. (7) if the following equations are satisfied

$$B^T - \bar{B}^T M_a^T\Lambda = 0$$
$$\bar{B}^T M_a^T = 2H_v \quad (17)$$

Those equations can be rewritten as

$$2H_v\Lambda = B^T$$
$$\bar{B}^T M_a^T = 2H_v \quad (18)$$

Therefore, the system described by Eqs. (1–4) is positive real (Moylan, 1974). ■

Note that although Kalman-Yakubovich-Popov theorem (Anderson, 1967) can be applied to a linear system, the nonlinear

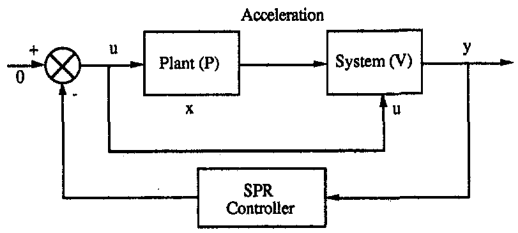

Fig. 2 A SPR Controller for the plant and the Virtual System

positive realness theory (Moylan, 1974) allows an easy construction of $\phi(x)$ for this problem particularly. There are two ways to solve Eq. (18). Given H_v and B, possible Λ, M_a, and $\bar{B}$ can be solved directly. Then it must be checked that M_a is positive semi-definite. Another way consists of choosing B, Λ, and a positive semi-definite M_a and then solving for possible $\bar{B}$ and H_v. Note that the choice of the virtual system in Eq. (2) is independent of the plant parameters M, D, and K. This means that the virtual system will make the global system positive real regardless of the uncertainty in M, D, and K.

3 Controller Design

If the output of the global system is chosen according to Theorem 1, then the global system is positive real. Thus, the closed-loop system is uniformly, asymptotically stable with zero input if the controller is strictly positive real (McLaren and Slater, 1986). That is, for this case

$$\lim_{t\to\infty} (H_v \dot{x}_a) = 0 \tag{19}$$

The following Theorem 2 may be used to obtain zero x when time goes to infinity.

Theorem 2: Assume that Theorem 1 is used to make the global system PR. Furthermore, assume that

(i) $B^T\dot{x} = 0$ and $u = 0$ imply $\dot{x} = 0$.
(ii) K is positive definite.
(iii) The system is connected to an SPR closed-loop controller.

Then $\lim_{t\to\infty} x(t) = 0$.

Figure 2 shows the control scheme for the plant (P) and virtual system (V).

Theorem 2 allows us to design a robust controller for plant (P). No knowledge of the constant matrices M, D, or K is required. Furthermore, the only measurements needed are acceleration and input. Acceleration can be easily measured for many practical systems by using accelerometers. The input u may be obtained by measuring the output of the SPR controller.

The proof of Theorem 2 uses the following Lemma 1.

Lemma 1: Let $\epsilon(t) \in R$ and $\lim_{t\to\infty} \epsilon(t) = 0$. Then, if x satisfies a differential equation

$$D\dot{x} + Kx = \epsilon \tag{20}$$

where D is positive semi-definite and K is positive definite, then $\lim_{t\to\infty} x(t) = 0$.

Proof: Let m denote the rank of D. There exists an invertible $n \times n$ matrix P such that

$$D^* = PDP^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix} \tag{21}$$

Note that since D is positive semi-definite, the zero eigenvalues appear on D^*. P is an orthogonal matrix consisting of the eigenvalues of the D matrix. Therefore, D_{22} is an $m \times m$ positive definite matrix. Let K^* be

$$K^* = PKP^{-1} = \begin{bmatrix} K_{11}^* & K_{12}^* \\ K_{21}^* & K_{22}^* \end{bmatrix} \tag{22}$$

The dynamic equation can be rewritten as

$$PDP^{-1}(P\dot{x}) + PKP^{-1}(Px) = P\epsilon(t) \tag{23}$$

Let $y = Px$ and $\eta(t) = P\epsilon(t)$. The system is now described by

$$D^*\dot{y} + K^*y = \eta(t) \tag{24}$$

Let $y = [y_1^T\ y_2^T]^T$ and $\eta = [\eta_1^T\ \eta_2^T]^T$. Equation (24) leads to

$$\begin{cases} K_{11}^* y_1 + K_{12}^* y_2 = \eta_1(t) \\ D_{22}\dot{y}_2 + K_{21}^* y_1 + K_{22}^* y_2 = \eta_2(t) \end{cases} \tag{25}$$

The first equation of (25) can be solved in terms of y_1. Therefore,

$$\begin{cases} y_1 = -K_{11}^{*-1}K_{12}^* y_2 + K_{11}^{*-1}\eta_1(t) \\ D_{22}\dot{y}_2 + (K_{22}^* - K_{21}^*K_{11}^{*-1}K_{12}^*)y_2 = \eta_2(t) + K_{21}^*K_{11}^{*-1}\eta_1(t) \end{cases} \tag{26}$$

Note that since K is positive definite, K_{11}^* is invertible. D_{22} and $(K_{22}^* - K_{21}^*K_{11}^{*-1}K_{12}^*)$ are positive definite matrices. Thus y_2 may be considered as the output of a strictly stable system. The output of a strictly stable system converges to zero; y_2 will therefore go to zero. The first equality in Eq. (26) shows that y_1 also goes to zero. Consequently, y converges to zero and so does x.

Proof of Theorem 2: Refer to Fig. 2, $(-u)$ is the output of an SPR controller. Since an SPR controller is always strictly stable, when y goes to zero, u also goes to zero. Furthermore, we have

$$2H_v\ddot{x}_a = 2H_v\Lambda\ddot{x} + 2H_v\bar{B}u \tag{27}$$

Since $2H_v\Lambda = B^T$, this equation reduces to

$$B^T\ddot{x} = 2H_v\ddot{x}_a - 2H_v\bar{B}u \tag{28}$$

where $2H_v\bar{B}u$ goes to zero as u goes to zero. Furthermore, $y = H_v\dot{x}_a$ converges to zero as time increases.

Let x_c denote the state of the SPR controller. Since the SPR controller is linear, the system can be described by

$$\dot{x}_c = Rx_c + Sy$$
$$y_c = Tx_c \tag{29}$$

where R, S, and T are constant matrices. Therefore, the global system becomes

$$M\ddot{x} + D\dot{x} + Kx = -BTx_c$$
$$\dot{y} = (H_v\Lambda)\ddot{x} - (H_v\bar{B}T)x_c$$
$$\dot{x}_c = Rx_c + Sy \tag{30}$$

Further define $x = [x^T\ \dot{x}^T\ x_c^T\ y^T]^T$. Equation (30) is rewritten in the form

$$\dot{x} = Ax \tag{31}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -M^{-1}K & -M^{-1}D & -M^{-1}BT & 0 \\ 0 & 0 & R & S \\ 0 & H_v\Lambda & -H_v\bar{B}T & 0 \end{bmatrix} \tag{32}$$

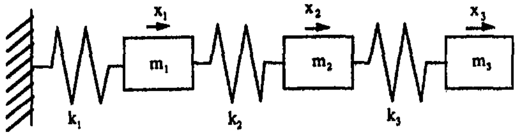

Fig. 3 A spring-mass system

Since A is constant, the solution of y is

$$y = \sum_i \alpha_i e^{(a_i+jb_i)t} p_i(t) \tag{33}$$

where α_i are complex constants and

$$p_i(t) = \sum_{i=1}^{n} \beta_i t^{i-1} \tag{34}$$

Note that n is the dimension of matrix A. Since $\lim_{t\to\infty} y(t) = 0$, $a_i < 0$ for all i. Therefore,

$$\lim_{t\to\infty} \frac{dy}{dt} = \lim_{t\to\infty} \left\{ \sum_i \alpha_i(a_i + jb_i)e^{(a_i+jb_i)t}p_i(t) + \sum_i \alpha_i e^{(a_i+jb_i)t} \frac{dp_i(t)}{dt} \right\}$$

$$= 0 \tag{35}$$

As a consequence, $B^T\ddot{x}$ goes to zero. Furthermore, if $B^T\ddot{x}$ and u go to zero, $\ddot{x}$ goes to zero according to assumption (i) in Theorem 2. The dynamics of the closed-loop system is now

$$D\dot{x} + Kx = Bu - M\ddot{x} = \epsilon(t) \tag{36}$$

where $\epsilon(t)$ vanishes as time increases. Using Lemma 1 we conclude that $x(t)$ goes to zero. ■

4 Examples

Two spring-mass systems are used here to demonstrate controller design.

The first example is a system with three masses, three springs, and no dashpots. The example is shown in Fig. 3. This system needs to be stabilized since it is not asymptotically stable. The dynamic equations describing the system are

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = u_1 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = u_2 \\ m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = u_3 \end{cases} \tag{37}$$

The matrices M, D, and K are

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad D = 0,$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{38}$$

M and K are positive definite as long as none of the masses and the spring constants is equal to zero. Several possible controller designs can be used here. Although $m = p$ is selected in the following example, $m \neq p$ can be selected for the controller design.

m = n = p = 3

For a choice of

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{39}$$

and the control vector $u^T = [u_1\ u_2\ u_3]$, solutions to Eq. (3) are given by

$$\Lambda = I_{3\times3}, \quad H_v = \frac{1}{2}B^T, \quad \bar{B} = \lambda B, \quad M_a = \frac{1}{\lambda}I_{3\times3} \tag{40}$$

where λ is an arbitrary strictly positive real number. As a consequence, the virtual state vector x_a is generated by the differential equation

$$\ddot{x}_a = \ddot{x} + \lambda Bu \tag{41}$$

All the assumptions of Theorem 2 are satisfied for this example. The vector x may therefore be controlled with any SPR feedback controller. A simple choice is a constant controller, that is a controller with a transfer matrix of the form kI, where I is the identity matrix and k is a constant. The following values are used in the simulation: $m_1 = m_2 = m_3 = 1$, $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$. The initial conditions are arbitrarily chosen to be $x_1(0) = 5$, $x_2(0) = -2$, $x_3(0) = 9$, $\dot{x}_1(0) = 3$, $\dot{x}_2(0) = 5$, and $\dot{x}_3(0) = -4$. For vector x_a, we choose initial conditions $x_a = 0$ and $\dot{x}_a = 0$. The constant λ is selected to be 0.5. The gain of the feedback controller is $k = 1$. The three displacements of the three masses are shown in Fig. 4.

The control objective is achieved since the three displacements vanish with time. Nevertheless, this design requires that an actuator be applied to each of the three masses. It is possible to reduce the number of actuators with the following control designs.

m = p = 2

Here only two forces are applied to the system. Thus, there are three possible choices, depending on which masses the forces are applied. Let

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{42}$$

This means that the forces are applied to the mass m_1 and mass m_2 only. The control vector u is $u^T = [u_1\ u_2]$. The vector x_a is now a two-dimensional vector. Equation (3) has the following solution

$$\Lambda = B^T, \quad H_v = \frac{1}{2}I_{2\times2}, \quad \bar{B} = \lambda I_{2\times2}, \quad M_a = \frac{1}{\lambda}I_{2\times2} \tag{43}$$

where λ is an arbitrary strictly positive real number, and where $I_{2\times2}$ denotes the 2×2 identity matrix. Thus x_a can be computed from the following differential equation

$$\begin{bmatrix} \ddot{x}_{a_1} \\ \ddot{x}_{a_2} \end{bmatrix} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{44}$$

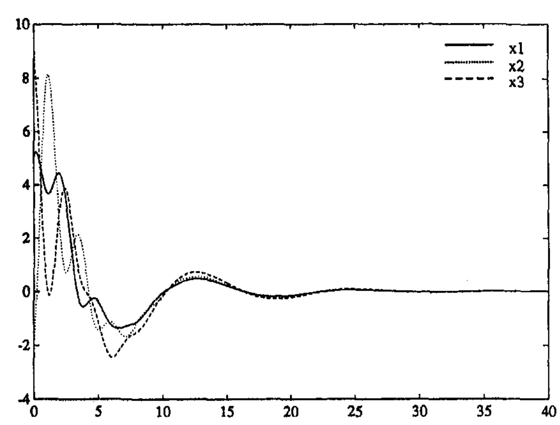

Fig. 4 Displacements of x_1, x_2, and x_3

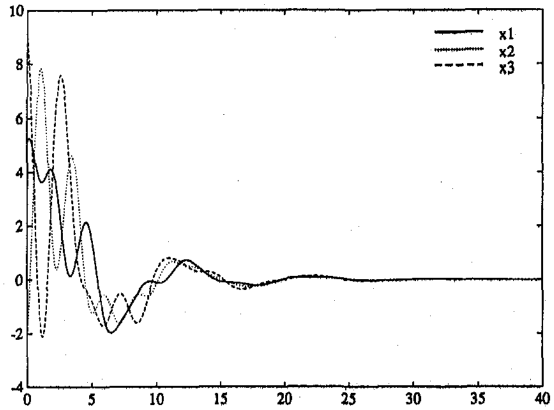

Fig. 5 Displacements of x_1, x_2, and x_3

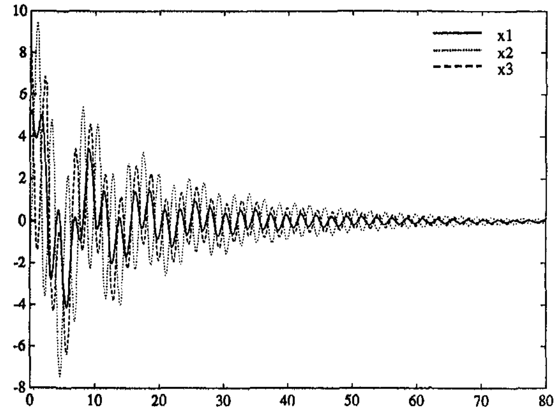

Fig. 6 Displacements of x_1, x_2, and x_3

and the output of the system is $y = (1/\lambda)\dot{x}_a$. In this example, the number of the virtual states is less than the number of the plant states.

An SPR controller must be chosen to control the system. Here again, a constant controller is a possible choice. It's transfer matrix is kI, where k is a positive constant.

It remains to ensure that $B^T\dot{x} = 0$ and $u = 0$ imply $x = 0$. If $B^T\dot{x} = 0$ and $u = 0$, then the dynamical equations of the system become

$$\begin{cases}(k_1 + k_2)x_1 + k_2x_2 = 0 \\ -k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0 \\ m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0\end{cases} \tag{45}$$

By differentiating the second equation and solving for $\dot{x}_3$, we have

$$\dot{x}_3 = -\frac{k_3}{k_2}\dot{x}_1 + \frac{(k_2 + k_3)}{k_3}\dot{x}_2 \tag{46}$$

Since $\dot{x}_1$ and $\dot{x}_2$ are both equal to zero, $\dot{x}_3$ is also equal to zero. Thus, Eq. (45) is reduced to $Kx = 0$. Since K is positive definite, this yields $x = 0$. Therefore, all the assumptions of Theorem 2 are satisfied and x goes to zero.

The closed-loop system is simulated with the same parameter choice as before. The three displacements are shown in Fig. 5. Here again the stabilization is achieved since the three displacements decrease to zero.

It is also possible to stabilize this system with a different distribution of forces. For instance, two controllers are applied to mass 2 and mass 3 or two controllers are applied to mass 1 and mass 3. The results are all similar to Fig. 5.

m = p = 1

Here we design a control system with only one actuator. This actuator may be located on any of the three masses. Let us first apply a force on mass 1, i.e., the matrix $B = [1\ 0\ 0]^T$.

Equation (3) in Theorem 1 has the following solution

$$\Lambda = B^T, \quad H_v = \frac{1}{2}, \quad \bar{B} = \lambda, \quad M_a = \frac{1}{\lambda} \tag{47}$$

where λ is an arbitrary strictly positive real number. The state x_a is calculated by integrating the differential equation

$$\ddot{x}_a = \ddot{x}_1 + \frac{1}{\lambda}u \tag{48}$$

The output of the system is $y = (1/\lambda)\dot{x}_a$.

Here again an SPR controller is chosen to be constant. It's transfer matrix is of the form $G(s) = k$, where k is any strictly positive real number. With this choice x converges to zero.

It should be checked as before that $B^T\dot{x} = 0$ and $u = 0$ imply $\dot{x} = 0$. The procedure is unchanged and once again those assumptions yield $Kx = 0$. Since K is assumed to be positive definite, x must be equal to zero.

The simulation is run with the same choice of initial conditions. The constant λ is still equal to 0.5, and k is equal to 1. The three displacements go to zero as expected (see Fig. 6).

The force could be applied to mass 3. However, if we choose to apply the force on mass 2, the design cannot be completed. In this case, $B = [0\ 1\ 0]^T$. Condition (i) of Theorem 2 is not satisfied for this choice. Thus no controller design can be implemented.

To see the robust stability, let's study the example with $m = n = p = 3$. The system is now perturbed to $m_1 = 1.5$, $m_2 = 2$, $m_3 = 3$, $k_1 = 2$, $k_2 = 1.5$, and $k_3 = 3.5$ while the controller is kept the same as before. The simulation is shown in Fig. 7 which clearly indicates robust stability.

The second example consists of one spring and one mass. The system is described by

$$\begin{aligned} m\ddot{x} + kx &= u \\ y &= \ddot{x} \end{aligned} \tag{49}$$

If a simple integration of the output acceleration is used for the feedback control, it can be shown that

$$\lim_{t\to\infty} x(t) = \frac{d}{k}[\dot{x}(0) - c(0)] \tag{50}$$

where d is the feedback gain, $\dot{x}(0)$ is the true initial velocity, and $c(0)$ is the estimated initial velocity. Since velocity is not measurable, $c(0)$ is not equal to $\dot{x}(0)$. Therefore, asymptotic stability is not achieved.

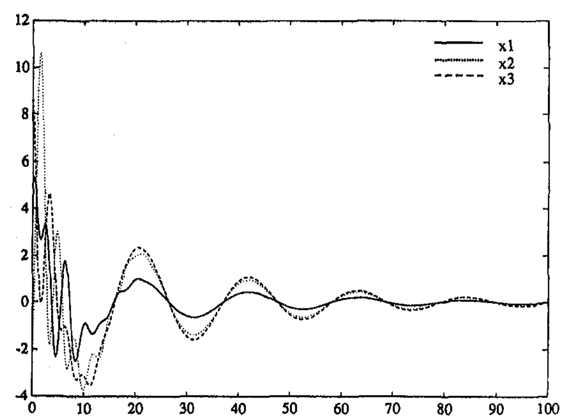

Fig. 7 Displacements of x_1, x_2, and x_3 with uncertainty

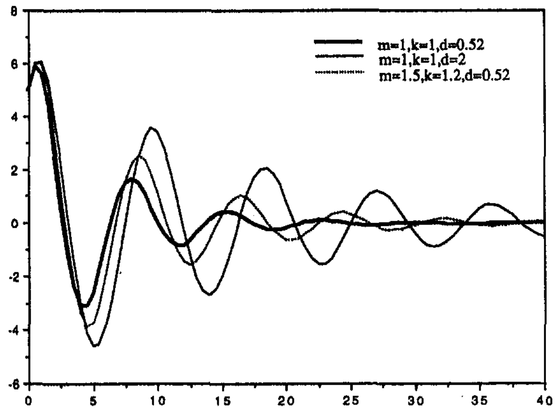

Fig. 8 Displacements of x_1 for three cases

However, if a virtual system is used by selecting

$$\Lambda = 1, \quad H_v = \tfrac{1}{2}, \quad \bar{B} = 1, \quad M_a = 1 \tag{51}$$

The global system is robustly positive real. Using a simple constant feedback with $2d$ as the gain leads to the following closed-loop system

$$m\ddot{x} + kx = u$$
$$\ddot{x}_a = \ddot{x} + u$$
$$u = -d\dot{x}_a \tag{52}$$

It can be shown that

$$\lim_{t\to\infty} \left[\int_0^t x dt - \left(\frac{m+1}{k} \right) \dot{x}(0) \right] = 0$$
$$\lim_{t\to\infty} x(t) = 0 \tag{53}$$

regardless of the initial velocity $\dot{x}(0)$. Therefore, the asymptotic stability of this design is independent of the initial velocity.

The performance of the controller can be obtained by optimization. The real part of the closed-loop eigenvalues can be minimized with respect to the feedback gain. For $m = 1$ and $k = 1$, the optimal feedback gain d is calculated to be $d = 0.52$. Figure 8 shows the responses for the optimal feedback gain $d = 0.52$, and for the feedback gain $d = 2$. It is clear that when $d = 0.52$ the system performs better than the system with $d = 2$. The robust performance is also demonstrated in Fig. 8 in which the mass and spring constants are perturbed to $m = 1.5$ and $k = 1.2$.

5 Conclusions

In this paper, a virtual system has been developed for second-order systems with only acceleration output. The combined system of the virtual system and the second-order system is positive real and allows infinite uncertainty in mass, spring constant, and damping coefficient. The states of the virtual system are not necessarily the same as the states of the plant. The number of the virtual states can be made smaller than the number of the plant states. Furthermore, any strictly positive real controllers can be used to achieve the asymptotic stability of the closed-loop system. This design is of particular interest for practical applications since only acceleration measurement is required. Asymptotic stability can be achieved with infinite uncertainty in the plant parameters and a large set of SPR controllers can be selected to optimize the performance. Two spring-mass systems have been used to demonstrate the virtual systems and controller designs. Extension to robust performance is possible since one of the examples has been shown with some degree of robust performance.

References

Anderson, B. D. O., 1967, "A System Theory Criterion for Positive Real Matrices," *SIAM J. of Control Optim.*, Vol. 5.

Bar-Kana, I., Fischl, R., and Kalata, P., 1991, "Direct Position Plus Velocity Feedback Control of Large Flexible Space Structures," *IEEE Transactions on Automatic Control*, Vol. 36, No. 10.

Benhabib, R. J., Iwens, R. P., and Jackson, R. L., 1981, "Stability of Large Space Structure Control Systems Using Positivity Concepts," *J. of Guidance, Dynamics and Control*, Vol. 4, No. 5.

Chuang, C.-H., Courouge, O., and Juang, J-N., 1992, "Controller Designs for Positive Real Second-Order Systems," *Proceedings of 1st International Motion and Vibration Control.*

Hill, D. and Moylan, P. J., 1976, "The Stability of Nonlinear Dissipative Systems," *IEEE Transactions on Automatic Control.*

Juang, J-N. and Phan, M., 1990, "Robust Controller Designs for Second-Order Dynamic Systems: A Virtual Approach," NASA Technical Memorandum TM-102666.

Juang, J-N., Wu, S-C., Phan, M., and Longman, R. W., 1991, "Passive Dynamic Controllers for Nonlinear Mechanical Systems," NASA Technical Memorandum 104047.

McLaren, M. D. and Slater, G. L., 1987, "Robust Multivariable Control of Large Space Structures Using Positivity," *J. of Guidance, Dynamics and Control*, Vol. 10, No. 4.

Morris, K. A. and Juang, J-N., 1991, "Robust Controller Design for Structures with Displacement Sensors," *Proceedings of the 30th Conference on Decision and Control.*

Moylan, P. J., 1974, "Implications of Passivity in a Class of Nonlinear Systems," *IEEE Transactions on Automatic Control*, Vol. AC-19, No. 4.

Takahashi, M. and Slater, G. L., 1986, "Design of a Flutter Mode Controller Using Positive Real Feedback," *J. of Guidance, Dynamics and Control*, Vol. 9, No. 3.

Wang, Q., Speyer, J. L., and Weiss, H., 1990, "System Characterization of Positive Real Conditions," *Proceedings of the 29th Conference on Decision and Control.*

Numerical Differentiation of Tracking Data of Human Motion: The Virtual Accelerometer[1]

S. B. Bortolami,[2] P. O. Riley,[3] and D. E. Krebs[4]

A method based on the Kalman Filter for deriving whole-body segment accelerations from position tracking data is described. Such a procedure is experimentally calibrated and qualified as a Virtual Accelerometer supplying a resolution of 0.2 m/s² and a relative accuracy of better than 10 percent rms within 0–8 Hz bandwidth. Results are finally compared to accelerometer performances reported by previous investigators.

Introduction

Human motion frequently requires measuring the derivatives of the body's trajectory either for computation of joint forces

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[2] Research Associate, Harvard Medical School, Massachusetts General Hospital Biomotion Laboratory, Boston, MA.

[3] Technical Assistant Director while working at the Massachusetts General Hospital Biomotion Laboratory; Presently, Lecturer at the Massachusetts Institute of Technology, Boston, MA.

[4] Director, Massachusetts General Hospital Biomotion Laboratory; Professor, MGH Institute of Health and Professions; Massachusetts Institute of Technology; Harvard Medical School; Boston, MA.

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