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Traveling Wave Solutions of the Generalized b-family Equation with Dispersion

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Abstract: This paper introduces a family of evolutionary 1+1 PDEs that describe the balance between convection and stretching in the dynamics of 1D nonlinear waves in fluids. It is reversible in time and parity invariant. In the paper, special solutions are discussed for $b = 0$ and for $b \neq 0$, the general solutions are given.When $b = 3$ and $b = -1$, the paper obtains the exact solutions of the generalized $b-$family equation.
Keywords: b-family equation, traveling wave solution

1 Introduction

In [1]-[3], Degasperis and Procesi firstly studied the following family of third order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x \tag{1}$$

where α, c_0, c_1, c_2 and c_3 are real constants and indices denote partial derivatives. When $c_1 = -\frac{\alpha}{2}$, $c_2 = \frac{\varepsilon(\beta-1)}{2}$, $c_3 = \varepsilon$ and replacing c_0 with k, and α^2 with ε in the equation above, we obtain the following equation

$$\begin{cases} (u - \varepsilon u_{xx})_t + k u_x + \alpha u u_x + \gamma u_{xxx} = \varepsilon(\beta u_x u_{xx} + u u_{xxx}), \quad x \in R,\ t > 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

where $u(x,t)$ stands for the fluid velocity in the x direction (or equivalently the height of the free surface of water above a float bottom), k is a constant related to the critical shallow water wave speed, and α, β, ε are dispersion parameters. It is necessary to point out that Eq.(2) is equivalent to Eq.(1) since when $\varepsilon = \alpha^2 = c_3$, $k = c_0$, $\alpha = -2c_1$ and $\beta = 1 + \frac{2c_2}{\varepsilon}$, Eq.(2) turns out to be Eq.(1). To better understand the common properties of the equation (1), we resort to study Eq.(2), which is convenient for us to research. We call it the general shallow water wave equation.

In Eq.(2), if $\alpha - \beta = 1$, $\gamma = -k\varepsilon$,then this equation includes the KdV equation , the Camassa-Holm equation, the Degasperis-Procesi equation and the $b-$family of equations. For convenience, we suppose that $\alpha - \beta = 1$, $\gamma = -k\varepsilon$ from Eq.(2). Thus Eq.(2) changes into the following equation ($b = \beta$)

$$m_t + \underbrace{u m_x}_{convection} + \underbrace{b u_x m}_{streching} = \underbrace{-k m_x}_{dispersion}, \tag{3}$$

in independent variables time and one spatial coordinate x.

Zhou and Tian [4] employ the bifurcation method to dynamical systems to investigate the exact traveling wave solutions for the Fornberg-Whitham equation. The explicit expressions for peakons and periodic

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cusp wave solutions are also obtained. By using bifurcation method, Zhou and Tian [5] successfully find the Fornberg-Whitham equation has a type of traveling wave solutions called kink-like wave solutions and antikink-like wave solutions. Tian, et al [6] converge to the solution to the corresponding BBM equation as the parameter converges to zero. Zhou and Tian [7] obtain a conserbation law which enable us to present a blow-up result by using multiplier technique. A remarkable feature of the physical model is that it has peakon solution which has peak form[8]. Tian and Yin [9] introduce the concept of nonlinear intensity, study a fully nonlinear sine-Gordon equation SG(m, n, p) and obtain a new type of peakon solutions and kink solutions by a direct method. Tian and Yin [10] introduce the fully nonlinear generalized Camassa-Holm equation C(m,n,p) and by using four direct ansatzs, we obtain abundant solutions. Tian and Song[11] consider generalized Camassa-Holm equations and the generalized weakly dissipative Camassa-Holm equations and derive some new exact peaked solitary wave solutions.

The paper is organized as follows. In Section 2, we obtain the conservation law and some general properties. In Section 3, peakons, ramps and cliffs are discussed for $b = 0$ and for $b \neq 0$, general solution is given. With special cases $b = -1$ and $b = 3$, we obtain the exact solutions of eq.(3).

2 Basic notations and general properties

We seek solutions for the fluid velocity u that are defined either on the real line and vanishing at spatial infinity, or on a periodic one dimensional domain. Here $u = g * m$ denotes the convolution (or filtering),

$$u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy, \tag{4}$$

which relates velocity u to momentum density m by integration against kernel $g(x)$ over the real line. We shall choose $g(x)$ to be an even function, so that u and m have the same parity.

The family of Eq.(3) is characterized by the kernel g and the real dimensionless constant b, which is the ratio of stretching to convective transport. As we see, b is also the number of covariant dimensions associated with the momentum density m. The function $g(x)$ will determine the traveling wave shape and length scale for Eq.(3), while the constant b will provide a balance or bifurcation parameter for the nonlinear solution behavior. Special values of b will include the first few positive and negative integers.

Its invariance under space and time translations ensures that Eq.(3) admits traveling wave solutions for any b. We write the traveling wave solutions as $u = u(z)$ and $m = m(z)$, where $z = x - ct$, and let prime $'$ denote d/dz.

Eq.(3) implies a similar reversible, parity invariant equation for the absolute value $|m|$,

$$(1-k/c)|m|_t + u|m|_x + bu_x|m| = 0 \quad \text{and} \quad u = g * m. \tag{5}$$

If $m^{1/b}$ is well-defined, Eq.(3) may be written as the conservation law

$$\partial_t T m^{1/b} + \partial_x (m^{1/b} u), T = 1 - k/c. \tag{6}$$

Eq.(3) for m is reversible, or invariant under $t \to -t$, $u \to -u$. The latter implies $m \to -m$. Hence, the transformation $u(x,t) \to -u(x,-t)$ takes solutions into solutions, and in particular, it reverses the direction and amplitude of the traveling wave $u(x,t) = cg(x-ct)$.

We choose $g(x)$ to be an even function so that m and $u = g * m$ both have odd parity under mirror reflections. Hence, Eq.(3) is invariant under the parity reflections $u(x,t) \to -u(-x,t)$.

Eq.(3) implies a similar reversible, parity invariant equation for the absolute value $|m|$,

$$|m|_t + (u+k)|m|_x + bu_x|m| = 0, u = g * m. \tag{7}$$

Theregfore, the positive and negative components $m_\pm = \frac{1}{2}(m \pm |m|)$ satisfy equation separately. Also, if m is initially zero, it remains so. This is conservation of the signature of m .

3 Traveling wave solutions

For $b = 0$, Eq.(3) is Galilean invariant and its traveling wave solutions satisfy

$$(u(z) - c + k)m'(z) = 0, z = x - ct, \tag{8}$$

where prime $'$ denotes d/dz . Eq.(8) admits generalized functions $m'(z) \simeq \delta(z)$ matched by $u - c + k = 0$ at $z = 0$. The velocity u is given by the integral of the Green's function that relates m and $u = g * m$

$$u - c + k = (c - k)[\int g(y)dy]_0^z. \tag{9}$$

When $g(x) = e^{-|x|/\alpha}$ (the Green's function for the 1D Helmholtz operator), we have $m = u - \alpha^2 u_{xx}$. Consequently, the equation $m' = u' - \alpha^2 u''' = \pm 2\delta(z)$ with $u - c + k = 0$ at $z = 0$ is satisfied by

$$u - c + k = \pm(c - k)[\int e^{-|y|/\alpha}dy]_0^z = \pm(c - k)sgn(z)(1 - e^{-|z|/\alpha}). \tag{10}$$

This represents a rightward moving traveling wave that connects the left state $u - c + k = \pm(c - k)$ to the same two right states.

From what has been discussed above, we have the following theorem:

Theorem 1 *Assume $b = 0$, $u = g * m$, $m = u - \alpha^2 u_{xx}$, Eq.(3) has the solution $u = \pm(c - k)sgn(z)(1 - e^{-|x|/\alpha}) + c - k$.*

We define $p(x) = \frac{1}{2\alpha^2}e^{-|\frac{x}{\alpha^2}|}$,$x \in R$, then $(1 - \alpha^2\partial_x^2)^{-1}f = p * f$ for all $f \in L^2(R)$, where $*$ denotes convolution. Using this identity, we can rewrite Eq.(1) as the following nonlocal form

$$u_t + uu_x - ku_x + \partial_x p * [\frac{3\alpha^2}{2}u_x^2 + 2ku] = 0. \tag{11}$$

The antisymmetric connections $u = \pm(c-k)sgn(z)(1-e^{-|z|/\alpha})$ (with $u-c+k = \mp(c-k)$ connecting to $u - c + k = \pm(c - k)$), with no jump in derivative at $z = 0$, are the regularized shocks (cliffs). While the symmetric connections $u = \pm(c - a)e^{-|z|/\alpha}$, with a jump in derivative at $z = 0$, are the peakons. They propagate rightward but may face either leftward or rightward, because Eq.(3) in the absence of viscosity has no entropy condition that would distinguish between leftward and rightward facing solutions. Besides Eq.(3) also has ramp-like similarity solutions $u \simeq x/t$ when $g(x) = e^{-|x|/\alpha}$ for any b. They may emerge in the initial value problem for the peakon case of Eq.(3) and interact with the peakons and cliffs.

For $b = 0$, the traveling wave Eq.(8) apparently has only the first integral for $m = u - \alpha^2 u_{xx}$

$$(u - c + k)(u - \alpha^2 u'') - \frac{u^2}{2} + \frac{\alpha^2}{2}u'^2 = K. \tag{12}$$

Thus, perhaps surprisingly, we have been unable to find a second integral for the traveling wave equation for peakons when $b = 0$.

Reversibility means that Eq.(3) is invariant under the transformation $u(x, t) \to -u(x, -t)$. Consequently, the rightward traveling waves have leftward moving counterparts under the symmetry $c - k \to -c + k$. The case of constant velocity $u = \pm(c - k)$ is also a solution.

For $b \neq 0$, the conservation law (6) for traveling waves becomes

$$((u - c + k)m^{1/b})' = 0, \tag{13}$$

which yields after one integration

$$(u - c + k)^b m = K, \tag{14}$$

where K is the first integral.

For $g(x) = e^{-|x|/\alpha}$, so $m = u - \alpha^2 u_{xx}$, Eq.(14) becomes

$$(u - c + k)^b(u - \alpha^2 u'') = K. \tag{15}$$

For $u-c+k\neq 0$ we rewrite Eq.(15) as

$$\alpha^2 u'' = u - K(u-c+k)^{-b} \tag{16}$$

and integrate again to give the second integral in two separate cases,

$$\alpha^2 u'^2 = \begin{cases} u^2 - \frac{2K}{1-b}(u-c+k)^{1-b} + 2H, b\neq 1 \\ u^2 - 2Kln(u-c+k) + 2H, b=1. \end{cases} \tag{17}$$

We shall rearrange this into quadratures:

$$\pm\frac{dz}{\alpha} = \frac{du}{[u^2 - 2Kln(u-c+k)+2H]^{\frac{1}{2}}} \quad \text{for} \quad b=1. \tag{18}$$

and

$$\pm\frac{dz}{\alpha} = \frac{du}{[u^2 - \frac{2K}{1-b}(u-c+k)^{1-b}+2H]^{\frac{1}{2}}} \quad \text{for} \quad b\neq 1, \tag{19}$$

For $b=1$ and $K\neq 0$, the integral in Eq.(18) is transcendental.

For $K=0$, the two quadratures Eq.(19) and Eq.(18) are equal, independent of b , and elementary, thereby yielding the traveling wave solutions

$$e^{-|z|/\alpha} = \frac{u+\sqrt{u^2+2H}}{c-k+\sqrt{(c-k)^2+2H}}, \tag{20}$$

with $u-c+k=0$ at $z=0$.

For $H=0$, Eq.(20) recovers the peakon traveling wave.

For $H>0$, Eq.(20) gives a rightward moving traveling wave that is a continuous deformation of the peakon.

For $H>0$ and $c=k$, Eq.(20) gives stationary solutions of the form

$$\frac{u+\sqrt{u^2+2H}}{\sqrt{2H}} = e^{-|z|/\alpha}, \tag{21}$$

then

$$u = \frac{\sqrt{2H}}{2}e^{-|z|/\alpha} - \frac{\sqrt{2H}}{2}e^{|z|/\alpha}. \tag{22}$$

For $b=3$, Eq.(19) is

$$\pm\frac{dz}{\alpha} = \frac{du}{[u^2 + K(u-c+k)^{-2}+2H]^{\frac{1}{2}}}, \tag{23}$$

which for the hyperbolic limit $H=0$ and $c=k$ is

$$\pm\frac{dz}{\alpha} = \frac{du}{[u^2 + Ku^{-2}]^{\frac{1}{2}}}. \tag{24}$$

Eq.(24) gives solutions of the form

$$u = (\frac{1}{2}e^{-2|z|/\alpha} - \frac{K}{2}e^{-2|z|/\alpha})^{\frac{1}{2}}. \tag{25}$$

Theorem 2 *Assume $b=3$, $u=g*m$, $m=u-\alpha^2 u_{xx}$, and $H=0, c=k$, Eq.(3) has the solution $u=(\frac{1}{2}e^{-2|z|/\alpha}-\frac{K}{2}e^{-2|z|/\alpha})^{\frac{1}{2}}$.*

For $b=-1$, Eq.(19) becomes

$$\pm\frac{dz}{\alpha} = \frac{du}{[u^2 - K(u-c+k)^2+2H]^{\frac{1}{2}}}, \tag{26}$$

which integrates to

$$e^{-|z|/\alpha} = \frac{u + \sqrt{u^2 - K(u-c+k)^2 + 2H} + K(c-k)}{c - k + \sqrt{(c-k)^2 + 2H} + K(c-k)} \tag{27}$$

with $u = c - k$ at $z = 0$, then

$$\begin{aligned} u = \frac{1}{2K}(&-2Ke^{-|z|/\alpha} - 2kKe^{-|z|/\alpha} + 2\sqrt{k^2+2H}e^{-|z|/\alpha} \pm 2(2k^2e^{-2|z|/\alpha} - 2ke^{-2|z|/\alpha}\sqrt{k^2+2H} \\ &-k^2K^2e^{-2|z|/\alpha} + 2He^{-2|z|/\alpha} - k^2K^3 - k^2K^2 + 2KH + 2k^2K^3e^{-|z|/\alpha} - 2kK^2e^{-|z|/\alpha} \\ &\sqrt{k^2+2H} + 2kK^2e^{-2|z|/\alpha}\sqrt{k^2+2H} + 2k^2K^2e^{-|z|/\alpha} - k^2K^3e^{-2|z|/\alpha} - 2KHe^{-|z|/\alpha})^{\frac{1}{2}}). \end{aligned} \tag{28}$$

Theorem 3 *Assume $b = -1$, $u = g * m$, $m = u - \alpha^2 u_{xx}$, Eq.(3) has the solution*

$$\begin{aligned} u = \frac{1}{2K}(&-2Ke^{-|z|/\alpha} - 2kKe^{-|z|/\alpha} + 2\sqrt{k^2+2H}e^{-|z|/\alpha} \pm 2(2k^2e^{-2|z|/\alpha} - 2ke^{-2|z|/\alpha}\sqrt{k^2+2H} \\ &-k^2K^2e^{-2|z|/\alpha} + 2He^{-2|z|/\alpha} - k^2K^3 - k^2K^2 + 2KH + 2k^2K^3e^{-|z|/\alpha} - 2kK^2e^{-|z|/\alpha} \\ &\sqrt{k^2+2H} + 2kK^2e^{-2|z|/\alpha}\sqrt{k^2+2H} + 2k^2K^2e^{-|z|/\alpha} - k^2K^3e^{-2|z|/\alpha} - 2KHe^{-|z|/\alpha})^{\frac{1}{2}}). \end{aligned}$$

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