MAXIMUM VARIANCE OF KTH RECORDS

Monika Klimczak *

Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

email: moka@mat.uni.torun.pl

Tomasz Rychlik †

Nicolaus Copernicus University and Polish Academy of Sciences,

Chopina 12/18, 87-100 Toruń, Poland

email: trychlik@impan.gov.pl

Abstract

Papadatos (1995) provided sharp bounds for the variances of order statistics in population variance units. This paper presents similar results for the variances of *k*th record values.

Keywords: variance, Hoeffding's identity, *k*th record, order statistic, sharp bound.

1 Introduction.

Let $\{X_n\}$, $n \geq 1$, be a sequence of independent and identically distributed random variables with continuous distribution function F and finite variance σ^2. For arbitrarily fixed positive integer k, we form the sequence of *k*th greatest order statistics $\{X_{n+1-k:n}\}$, $n \geq k$, which is nondecreasing. We define *k*th record statistics $R_n^{(k)}$, according to the definition introduced by Dziubdziela and Kopociński [6], in the following way:

$$R_n^{(k)} = X_{L_n^{(k)}:L_n^{(k)}+k-1}, \qquad n \geq 0, \tag{1}$$

*Corresponding author

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where $L_n^{(k)}$ are the *n*th occurrence times of *k*th records defined as:

$$L_0^{(k)} = 1,$$

$$L_{n+1}^{(k)} = \min\{j > L_n^{(k)} : \; X_{L_n^{(k)}:L_n^{(k)}+k-1} < X_{j:j+k-1}\}.$$

It is well known (cf. [1], [10]) that the distribution function on the *n*th value of *k*th record is a composition

$$G(F(x)) = G_n^{(k)}(F(x))$$

of the parent function F with the distribution function

$$G_n^{(k)}(x) = 1 - (1-x)^k \sum_{j=0}^{n} \frac{(-\ln(1-x))^j \, k^j}{j!} \tag{2}$$

of the respective record values from the standard uniform sequence. Therefore

$$ER_n^{(k)} = \int_0^1 F^{-1}(x) \, g_n^{(k)}(x) \, dx, \tag{3}$$

where

$$g_n^{(k)}(x) = \frac{k^{n+1}}{n!} \left[-\ln(1-x)\right]^n \, (1-x)^{k-1}, \qquad k \geq 1, \;\; n \geq 0.$$

is density function of the *n*th value of *k*th record of the i.i.d. standard uniform sequence and F^{-1} is the quantile function of the original distribution function F.

The aim of paper is to find the maximum of the variance of *k*th record statistics in terms of population variance units, i.e. to determine the best constant $\sigma_n^2(k)$ in

$$VarR_n^{(k)} \leq \sigma_n^2(k) \cdot \, \sigma^2, \qquad n \geq 1, \;\; k \geq 1, \tag{4}$$

where $\sigma_n^2(k)$ depends on k and n only. If $k = 1$, a finite constant cannot be founded which is concluded from the example presented below. Theorem 1 provides optimal finite constants for the other cases and describes distributions which attain the bounds.

The theory of record and *k*th record values is still developing. We can mention a few results concerning the bounds on the moments. Using the Schwarz inequality, Nagaraja [11] presented the mean - variance bounds on the expectations of standard records for the case of i.i.d. sequences with general and symmetric distributions. Grudzień and Szynal [7] derived analogous non sharp bounds for *k*th records. Sharp bounds for the moments of *k*th record values based on greatest convex minorants (Moriguti's method) were obtained by Raqab [14]. Raqab [15] also derived *p*th absolute moment bounds on the expectations of first records in general and symmetric populations based on the Hölder inequality. Similar results for the *k*th records are given in Raqab and Rychlik [16]. Expectations of the second records from symmetric populations are evaluated in Raqab and Rychlik [17]. Rychlik [18] and Danielak [3] presented bounds for the differences of adjacent and nonadjacent classic records, respectively, coming from various families of parent distributions. Similar results for *k*th records are given in Danielak and Raqab [4, 5]. By now, no bounds have been presented on the variances of records. Our methods of proof are similar to those of Papadatos [12] who obtained analogous bounds on the variances of arbitrary order statistics, improving the results of Young [19] and Lin and Huang [8]. Papadatos [13] refined his results for the case of symmetric populations.

2 Main results

We first show that $\sigma_n^2(1) = +\infty$. To this end it suffices to consider i.i.d. random variables with the distribution function

$$F(x) = 1 - \frac{e^2}{x^2 (\ln x)^{\frac{3}{2}}}, \quad x \geq e,$$

and density function

$$f(x) = \frac{e^2 (4 \ln x + 3)}{2x^3 (\ln x)^{\frac{5}{2}}}, \quad x > e.$$

Elementary calculations show that $VarX_1 < \infty$ and $VarR_n^{(1)} = \infty$, $n = 1, 2, \ldots$, which gives the desired statement. The distribution is a modification of an example in Nagaraja $[11, \mathrm{p.}\, 177]$.

Considering the cases $k \geq 2$, we first introduce the following notation. Set

$$f_1(x) = \frac{G(x)}{x}, \quad 0 < x < 1, \tag{5}$$

$$f_2(y) = \frac{1 - G(y)}{1 - y}, \quad 0 < y < 1, \tag{6}$$

$$f(x, y) = f_1(x) \cdot f_2(y) = \frac{G(x)(1 - G(y))}{x(1 - y)}, \quad 0 < x \leq y < 1, \tag{7}$$

and

$$f(u) = f(u, u) = \frac{G(u)(1 - G(u))}{u(1 - u)}, \quad 0 < u < 1, \tag{8}$$

where $G(x) = G_n^{(k)}(x)$, $k \geq 2$, $n \geq 1$, is given in (2).

Lemma 1 *(cf. Papadatos 1995, Lemma 2.1) Let* $k > 1$ *and* $n \geq 1$. *There exist unique numbers* $\rho_1 = \rho_1(k, n)$, $\rho_2 = \rho_2(k, n)$ *such that*

$$0 < \rho_1 < 1 - e^{-\frac{n}{k-1}} < \rho_2 < 1$$

and

(i) $f_1(x) = \dfrac{G(x)}{x}$ *strictly increases in* $(0, \rho_2)$ *and strictly decreases in* $(\rho_2, 1)$*,*

(ii) $f_2(y) = \dfrac{1 - G(y)}{1 - y}$ *strictly increases in* $(0, \rho_1)$ *and strictly decreases in* $(\rho_1, 1)$*.*
Moreover,

(iii) $\sup\limits_{0 \le x \le y \le 1} \dfrac{G(x)[1 - G(y)]}{x(1 - y)} = \max\limits_{\rho_1 \le u \le \rho_2} \dfrac{G(u)[1 - G(u)]}{u(1 - u)}.$

Proof. (i) Consider function $f_1(x)$ given in the form (5). Then

$$f_1'(x) = \frac{x\,g(x) - G(x)}{x^2}.$$

The derivative of the numerator

$$[xg(x) - G(x)]' = xg'(x) = \frac{k^{n+1}}{n!}(-\ln(1 - x))^{n-1}x\,(1 - x)^{k-2}[n + (k - 1)\ln(1 - x)]$$

is positive if $x < 1 - e^{-\frac{n}{k-1}}$ and negative if $x > 1 - e^{-\frac{n}{k-1}}$. Furthermore

$$\lim_{x \to 0^+}(xg(x) - G(x)) = 0 \quad \text{and} \quad \lim_{x \to 1^-}(xg(x) - G(x)) = -1.$$

Hence function $f_1'(x)$ has a unique zero at $\rho_2 = \rho_2(k, n) \in (1 - e^{-\frac{n}{k-1}}, 1)$ and $f_1'(x) > 0$ for $x \in (0, \rho_2)$ and $f_1'(x) < 0$ for $x \in (\rho_2, 1)$. Therefore function $f_1(x)$ strictly increases in $(0, \rho_2)$ and decreases in $(\rho_2, 1)$.
(ii) Similarly we can show that function $f_2(y)$ defined in (6) strictly increases in $(0, \rho_1)$ and strictly decreases in $(\rho_1, 1)$ for a unique $\rho_1 = \rho_1(k, n)$ such that

$$0 < \rho_1 < 1 - e^{-\frac{n}{k-1}}.$$

Indeed,

$$f_2'(y) = \frac{-g(y)(1 - y) + 1 - G(y)}{(1 - y)^2}.$$

The derivative of the numerator

$$[-g(y)(1 - y) + 1 - G(y)]' = (y - 1)g'(y) = \frac{k^{n+1}}{n!}(-\ln(1 - y))^{n-1}(1 - y)^{k-1}[-n - (k - 1)\ln(1 - y)]$$

is positive if $y > 1 - e^{-\frac{n}{k-1}}$ and negative if $y < 1 - e^{-\frac{n}{k-1}}$. Furthermore

$$\lim_{x \to 0^+}(-g(y)(1 - y) + 1 - G(y)) = 1 \quad \text{and} \quad \lim_{x \to 1^-}(-g(y)(1 - y) + 1 - G(y)) = 0.$$

Hence function $f_2'(y)$ has a unique zero at $\rho_1 = \rho_1(k,n) \in (1-e^{-\frac{n}{k-1}}, 1)$ and $f_2'(y) > 0$ for $y \in (0,\rho_1)$ and $f_2'(y) < 0$ for $y \in (\rho_1, 1)$. Therefore function $f_1(y)$ strictly increases in $(0,\rho_1)$ and decreases in $(\rho_1, 1)$.

(iii) Consider the set

$$\{(x,y) : 0 \le x \le y \le 1\} \subset A \cup B \cup C,$$

where

$$\begin{aligned} A &= \{(x,y) : 0 \le x \le \rho_1, \, 0 \le y \le 1\}, \\ B &= \{(x,y) : 0 \le x \le 1, \, \rho_2 \le y \le 1\}, \\ C &= \{(x,y) : \rho_1 \le x, y \le \rho_2\}. \end{aligned}$$

If $0 \le x \le \rho_1$ and $0 \le y \le 1$, then

$$\frac{G(x)[1-G(y)]}{x(1-y)} \le \frac{G(x)[1-G(\rho_1)]}{x(1-\rho_1)} \le \frac{G(\rho_1)[1-G(\rho_1)]}{\rho_1(1-\rho_1)}.$$

If $0 \le x \le 1$ and $\rho_2 \le y \le 1$, then

$$\frac{G(x)[1-G(y)]}{x(1-y)} \le \frac{G(\rho_2)[1-G(y)]}{\rho_2(1-y)} \le \frac{G(\rho_2)[1-G(\rho_2)]}{\rho_2(1-\rho_2)}.$$

In the case $\rho_1 \le x \le \rho_2$ and $\rho_1 \le y \le \rho_2$, we have

$$\frac{G(x)[1-G(y)]}{x(1-y)} \le \frac{G(x)[1-G(x)]}{x(1-x)}$$

and

$$\frac{G(x)[1-G(y)]}{x(1-y)} \le \frac{G(y)[1-G(y)]}{y(1-y)}.$$

Therefore

$$\sup_{0 \le x \le y \le 1} \frac{G(x)[1-G(y)]}{x(1-y)} = \max_{\rho_1 \le u \le \rho_2} \frac{G(u)[1-G(u)]}{u(1-u)}.$$

The supremum of the right side of this equality is attained, because function $f(u)$ given in (8) is continuous in the closed interval $[\rho_1, \rho_2]$. □

Definition 1 *For $k \ge 2$, $n \ge 1$, we define*

$$\sigma_n^2(k) = \max_{\rho_1 \le x \le \rho_2} \left[\frac{G(x)\,(1-G(x))}{x\,(1-x)} \right] = \frac{G(x_0)[1-G(x_0)]}{x_0(1-x_0)}$$

for some $\rho_1 \le x_0 \le \rho_2$.

Theorem 1 *Put $k \geq 2$ and $n \geq 1$. Let $R_n^{(k)}$ be the nth value of the kth record from an i.i.d. sequence of random variables with an arbitrary distribution function F with a finite variance σ^2. Then we have*

$$VarR_n^{(k)} \leq \sigma_n^2(k)\cdot\, \sigma^2. \tag{9}$$

Equality is attained in (9) if $F(x)$ is a two point distribution function

$$F(x) = \begin{cases} 0 & x < a \\ x_0 & a \leq x < b \\ 1 & b \leq x \end{cases} \tag{10}$$

for some $a < b$, where $x_0 = x_0(k,n)$ is a point in $[\rho_1, \rho_2]$ such that the function $f(x) = \dfrac{G(x)\,(1-G(x))}{x\,(1-x)}$ attains its maximum $\sigma_n^2(k)$.

Precisely, bounds (9) are attained in limit by the sequences of continuous distributions which tend weakly to (10).

Proof. From the Hoeffding identity for the covariance of a pair of random variables X, Y that has the form

$$Cov(X,Y) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} [H(x,y) - H(x,\infty)H(\infty,y)]\, dydx, \tag{11}$$

where $H(x,y)$ is the bivariate distribution function of the pair, we obtain

$$Var(X) = 2\iint\limits_{x\leq y} F(x)\,(1-F(y))\, dydx. \tag{12}$$

Alternative versions and generalizations of (11) and (12) were presented in Bassan et al ([2], Section 2). Hence

$$\begin{aligned} VarR_n^{(k)} &= 2\iint\limits_{x\leq y} G(F(x))\ [1-G(F(y))]\ dydx \\ &= 2\iint_{0<F(x)\leq F(y)<1} G(F(x))\ [1-G(F(y))]\ dydx = \\ &= 2\iint_{0<F(x)\leq F(y)<1} \frac{G(F(x))\,(1-G(F(y)))}{F(x)\,(1-F(y))}\, F(x)\,(1-F(y))dydx \\ &\leq \frac{G(x_0)\,(1-G(x_0)}{x_0\,(1-x_0)}\, 2\iint_{0<F(x)\leq F(y)<1} F(x)\,(1-F(y))dydx = \\ &= \frac{G(x_0)\,(1-G(x_0)}{x_0\,(1-x_0)}\, 2\iint_{x\leq y} F(x)\,(1-F(y))dydx = \sigma_n^2(k)\cdot VarX_1. \end{aligned}$$

We have an equality in (9) iff

$$x_0(1-x_0)G(F(x))(1-G(F(y))) = G(x_0)(1-G(x_0))F(x)(1-F(y))$$

on the set $\{x \leq y : 0 < F(x) \leq F(y) < 1\}$ almost everywhere. This happens if both $F(x)$ and $F(y)$ take on only one value x_0 in the interval $(0,1)$, which characterizes two point distribution (10). $\square$

Tables 1 nad 2 provide values $\sigma_n^2(k) = \sup\limits_F \frac{Var_F R_n^{(k)}}{Var_F X_1}$ and $x_0(k,n)$ describing the two point distribution attaining the bounds for $1 \leq n \leq 10$ and $2 \leq k \leq 7$. For comparison, we also present the values of ratios $Var_F R_n^{(k)}/Var_F X_1$ for the standard exponential, uniform distribution and Pareto distribution $F_\alpha(x) = 1 - x^{-\alpha},\ x \geq 1$, with shape parameter $\alpha = 3$. They represent distributions with various tail behavior. For the exponential distribution, we have

$$Var_E R_n^{(k)}/Var_E X_1 = \frac{n}{k^2}, \tag{13}$$

which can be found, e.g., in Nevzorov $[10, \text{p.}\, 96]$.

By simple calculations, we also obtain

$$Var_U R_n^{(k)}/Var_U X_1 = 12\left[\left(\frac{k}{k+2}\right)^{n+1} - \left(\frac{k}{k+1}\right)^{2(n+1)}\right] \tag{14}$$

and

$$Var_P R_n^{(k)}/Var_P X_1 = \frac{4}{3}\left[\left(\frac{k}{k-\frac{2}{\alpha}}\right)^{n+1} - \left(\frac{k}{k-\frac{1}{\alpha}}\right)^{2(n+1)}\right] \tag{15}$$

for the uniform and Pareto distributions, respectively. For each entry (n,k), we present a column of five values. The numbers assigned by E, U, P represent (13), (14), (15), respectively. The next value assigned by B and printed in bold, provides the bound, and the last one is $x_0(k,n)$. For example, if $n = 4$ and $k = 5$, the values for exponential, uniform and Pareto distributions are equal to:

$$\begin{aligned} Var_E R_4^{(5)}/Var_E X_1 &= 0,160000, \\ Var_U R_4^{(5)}/Var_U X_1 &= 0,293148, \\ Var_P R_4^{(5)}/Var_P X_1 &= 0,068860, \end{aligned}$$

respectively. Also, we have

$$\begin{aligned} \sigma_4^2(5) &= 1.059840, \\ x_0(4,5) &= 0.631712. \end{aligned}$$

This means that the optimal variance bound for $R_4^{(5)}$ is

$$VarR_4^{(5)} \leq 1.059840 \cdot \sigma^2$$

and is attained by any two point distribution that assigns probability 0.631712 to the smaller value. Analysis of the tables lead us to the following conclusions. The bounds become larger, if k is fixed and n increases, and the same holds for the exponential and Pareto distributions which have infinite right tails. For the uniform distributions with a finite right en, the *k*th records approach the right end for increasing n and become less dispersed then. If k increases, then the variance ratios for the exponential and Pareto distributions decrease, and the bounds behave similarly. Relations for the uniform distributions are more complicated. If k is small, then uniform record values are least dispersed, and Pareto ones have the greatest variances. For large k, the relative variances are ordered conversely. This is closely related with the expected location of the *k*th records in the supports of the distributions and the probability concentration in the respective regions. For instance, the Pareto distribution is more concentrated on the left on the domain, where *k*th records with large k occur most likely. Therefore these are less dispersed than the *k*th records with small indices. Note that for each particular distribution, and parameters n and k, the variance ratios are far from the bounds. It can be also observed that $x_0(k,n)$ increases in n and decreases in k. The values correspond to the location of respective records values in the domain of distributions. Splitting the domain into two parts remote from the possible values of the record provides the maximal dispersion.

Remark 1 *We have not proven that there exists a point $x_0 = x_0(k,n)$ such that the function*

$$f(x) = \frac{G(x)\,(1-G(x))}{x\,(1-x)}$$

strictly increases in $(0, x_0)$ and strictly decreases in $(x_0, 1)$. This would imply that the maximum point is unique and is easily determined by solving equation $f'(x) = 0$. It is still the open problem, although all the numerical examples confirm the hypothesis.

References

[1] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998). *Records*, Wiley, New York.

[2] Bassan, B., Denuit, M. and Scarsini, M. (1999). Variability orders and mean differences, *Statist. Probab. Lett.* **45**, 121-130.

[3] Danielak, K. (2004). Distribution - free bounds for expectations of increments of records, *J. Statist. Plann. Inference*, to appear.

[4] Danielak, K. and Raqab, M. Z. (2004). Sharp bounds for expectations of *k*th record increments, *Austr. New Zeal. J. Statist.*, to appear.

[5] Danielak, K. and Raqab, M. Z. (2004). Sharp bounds on expectations of *k*th record spacings from restricted families, submitted for publication.

[6] Dziubdziela, W. and Kopociński, B. (1976). Limitimg properties of the *k*th record values, *Zastos. Mat.* **15**, 187-190.

[7] Grudzień, Z. and Szynal, D. (1985). On the expected values of *k*th record values and associated characterizations of distributions, in: *Probability and Statistical Decision Theory*, Vol. A (F. Konecny, J. Mogyoródy, and W. Wertz, eds.) Reidel, Dordrecht, 119-127.

[8] Lin, G. and Huang, J. (1989). Variances of sample medians, *Statist. Probab. Lett.* **8**, 143-146.

[9] Moriguti, S. (1953). A modification of Schwarz's inequality with applications to distributions, *Ann. Math. Statist.* **24**, 107-113.

[10] Nevzorov, V. B. (2000). *Records: Mathematical Theory,* American Mathematical Society, Providence, RI.

[11] Nagaraja, H. N. (1978). On the expected values of records, *Austr. J. Statist.* **20**, 176-182.

[12] Papadatos, N. (1995). Maximum variance of order statistics, *Ann. Inst. Statist. Math.* **47**, 185-193.

[13] Papadatos, N. (1997). A note on maximum variance of order statistics from symmetric populations, *Ann. Inst. Statist. Math.* **49**, 117-121.

[14] Raqab, M. Z. (1997). Bounds based on the greatest convex minorants for moments of record values, *Statist. Probab. Lett.* **36**, 35-41.

[15] Raqab, M. Z. (2000). On the moments of record values, *Commun. Statist. Theory Meth.* **29**, 1631-1647.

[16] Raqab, M. Z. and Rychlik, T. (2002). Sharp bounds for the mean of *k*th record value, *Commun. Statist. Theory Meth.* **31**, 1927-1937.

[17] Raqab, M. Z. and Rychlik, T. (2004). Sharp bound on the expectations of second record values from symmetric populations, *Statistics* **38**, 49-57.

[18] Rychlik, T. (1997). Evaluating improvements of records, *Appl. Math. (Warsaw)* **24**, 315-324.

[19] Yang, H. (1982). On the variances of median and some other order statistics, *Bull. Inst. Math. Acad. Sinica* **10**, 197-204.

Table 1: Extreme values of $Var_F R_n^{(k)}/Var_F X_1$, $2 \leq k \leq 7$, $1 \leq n \leq 5$ and respective variance ratios for the exponential, uniform and Pareto distributions.

n	k	2	3	4	5	6	7
1	E	0.250000	0,111111	0,062500	0,040000	0,027777	0,020408
	U	0,629628	0,523128	0,418128	0,335400	0,272700	0,225084
	P	0,235200	0,068340	0,031604	0,018068	0,011658	0,008134
	B	**1,037078**	**1,034055**	**1,166995**	**1,332005**	**1,509444**	**1,692976**
	x_0	0,630973	0,384983	0,273331	0,211249	0,171969	0,144940
2	E	0,500000	0,222222	0,125000	0,080000	0,055555	0,040816
	U	0,446508	0,456252	0,409824	0,354408	0,303672	0,260544
	P	0,518688	0,130771	0,056653	0,031191	0,019639	0,013468
	B	**1,550262**	**1,049699**	**1,000852**	**1,039912**	**1,109983**	**1,194410**
	x_0	0,864099	0,631401	0,482914	0,388173	0,323684	0,277257
3	E	0,750000	0,333333	0,187500	0,120000	0,083333	0,061224
	U	0,281784	0,353844	0,357108	0,332880	0,300588	0,268092
	P	1,016911	0,222436	0,090272	0,047864	0,029407	0,019823
	B	**2,714505**	**1,298755**	**1,056034**	**1,002071**	**1,008864**	**1,042249**
	x_0	0,947486	0,777350	0,631599	0,525566	0,448018	0,389625
4	E	1,000000	0,444444	0,250000	0,160000	0,111111	0,081633
	U	0,166896	0,257352	0,291756	0,293148	0,278952	0,258624
	P	1,869351	0,354715	0,134850	0,068860	0,041283	0,027353
	B	**5,002766**	**1,741590**	**1,218672**	**1,059840**	**1,008134**	**1,000814**
	x_0	0,979102	0,864287	0,736766	0,631712	0,549319	0,484472
5	E	1,250000	0,555555	0,312500	0,200000	0,138888	0,102040
	U	0,095018	0,179760	0,228864	0,247836	0,248532	0,239520
	P	3,299366	0,543044	0,193386	0,095101	0,055635	0,036234
	B	**9,408865**	**2,430987**	**1,475595**	**1,180121**	**1,062378**	**1,014421**
	x_0	0,991573	0,916621	0,811330	0,713721	0,631785	0,564433

Table 2: Extreme values of $Var_F R_n^{(k)} / Var_F X_1$, $2 \leq k \leq 7$, $6 \leq n \leq 10$ and respective variance ratios for the exponential, uniform and Pareto distributions.

n	k	2	3	4	5	6	7
6	E	1,500000	0,666666	0,375000	0,240000	0,166666	0,122449
	U	0,052644	0,122112	0,174564	0,203736	0,215280	0,215664
	P	5,662337	0,808286	0,269630	0,127695	0,072896	0,046664
	B	**17,860167**	**3,468883**	**1,838765**	**1,356583**	**1,157645**	**1,064191**
	x_0	0,996593	0,948447	0,864381	0,777157	0,698939	0,631836
7	E	1,750000	0,777777	0,437500	0,280000	0,194444	0,142857
	U	0,028608	0,081288	0,130452	0,164064	0,182688	0,190224
	P	9,520640	1,178553	0,368262	0,167962	0,093561	0,058870
	B	**34,071324**	**5,013705**	**2,333652**	**1,592720**	**1,290344**	**1,142990**
	x_0	0,998625	0,967971	0,902261	0,826302	0,753661	0,688671
8	E	2,000000	0,888888	0,500000	0,320000	0,222222	0,163265
	U	0,015312	0,053280	0,095976	0,130068	0,152604	0,165180
	P	15,760035	1,691620	0,495120	0,217475	0,118209	0,073109
	B	**65,205997**	**7,302645**	**2,997940**	**1,897557**	**1,461548**	**1,248567**
	x_0	0,999451	0,980031	0,929401	0,864437	0,798288	0,736613
9	E	2,250000	1,000000	0,562500	0,360000	0,250000	0,186735
	U	0,008112	0,034500	0,069744	0,101856	0,125904	0,141648
	P	25,769919	2,398116	0,657462	0,278107	0,147507	0,089672
	B	**125,101063**	**10,687522**	**3,883387**	**2,284568**	**1,675101**	**1,381184**
	x_0	0,999782	0,987521	0,948907	0,894078	0,834715	0,777074
10	E	2,500000	1,111111	0,625000	0,400000	0,277777	0,204082
	U	0,004260	0,022128	0,050184	0,078960	0,102852	0,120264
	P	41,721886	3,365750	0,864308	0,352088	0,182226	0,108886
	B	**240,528237**	**15,688761**	**5,059462**	**2,771702**	**1,936955**	**1,542041**
	x_0	0,999914	0,992189	0,962965	0,917151	0,864475	0,811302