Using intensifying hedges to reduce size of multi-adjoint concept lattices with heterogeneous conjunctors

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Abstract. In this work we focus on the use of intensifying hedges as a tool to reduce the size of the recently introduced multi-adjoint concept lattices with heterogeneous conjunctors.

1 Introduction and preliminaries

Formal concept analysis (FCA) is a very active topic for several research groups throughout the world [6, 1, 3, 5, 7, 8, 10, 11]. In this work, the authors aim to merge recent advances obtained in this area: on the one hand, the use of hedges as structures which allow to modulate the size of fuzzy concept lattices [4] and, on the other hand, the consideration of heterogeneous conjunctors in the general approach to fuzzy FCA so-called multi-adjoint framework [9].

One of the key features of the latter approach is that some quasi-closure operators arise which, although do not directly allow to prove the complete lattice structure of the resulting set of concepts as usual, i.e. in terms of a Galois connection, actually do provide means to manually build the operators for suprema and infima of a set of concepts. The core notion in [9] is that of P-connected pair of posets which, in some sense, turns out to be a more abstract notion than a truth-stressing hedge. As a consequence of this observation, due to Radim Belohlavek, we now focus on the use of the specific properties of hedges in order to import some results related to the size of fuzzy concept lattices to the more general framework of [9].

The structure of the paper is the following: in Section 2 the preliminary definitions are introduced, interested readers will obtain further comment on the intuitions underlying the definitions in the original papers [4, 9]; the main results are presented in Section 3.

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2 Preliminaries

In this section, we introduce the basic definitions and preliminary results which will be used later in the core of this work.

Definition 1. *Let* $(L, \preceq, \top, \bot)$ *be a complete lattice, a truth-stressing hedge in* L *is a mapping* $*\colon L \to L$ *satisfying, for each* $x, y \in L$,

$$*(\top) = \top, \tag{1}$$

$$*(x) \preceq x, \tag{2}$$

$$x \preceq y \ \textit{implies} \ *(x) \preceq *(y), \tag{3}$$

$$*(*(x)) = *(x) \tag{4}$$

$\mathrm{fix}(*)$ *denotes set of fixed points of* $*$ *in* L*, i.e.* $\mathrm{fix}(*) = \{a \in L \mid *(a) = a\}$.

In [3, 4] truth-stressing hedges were used to decrease size of a concept lattice (in fact, the truth-stressing hedges were defined on a residuated lattice).

Later in this work, we will need the following lemmas.

Lemma 1. *Let* $(L, \preceq)$ *be a complete lattice, for any mapping* $*\colon L \to L$ *satisfying* (2), (3), *and* (4) *we have, for each* $x_i \in L$,

$$\bigvee_{i\in I} *(x_i) = *(\bigvee_{i\in I} *(x_i)) \quad \textit{and} \quad *(\bigwedge_{i\in I} *(x_i)) = *(\bigwedge_{i\in I} x_i). \tag{5}$$

In addition, if we have $x_j = \bigvee_{i\in I} x_i$ *for some* $j \in I$ *then*

$$*(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} *(x_i). \tag{6}$$

Similarly, if we have $x_j = \bigwedge_{i\in I} x_i$ *for some* $j \in I$ *then*

$$*(\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} *(x_i). \tag{7}$$

Lemma 2. (a) *Let* $*\colon L \to L$ *be a mapping satisfying* (2), (3), *and* (4). *Then* $\mathrm{fix}(*)$ *is a* $\vee$*-subsemilattice of* L.

(b) *Let* K *be a* $\vee$*-subsemilattice of* L *then the mapping* $*_K\colon L \to L$ *defined by*

$$*_K(x) = \bigvee\{y \in K \mid y \leq x\}$$

satisfies (2), (3), *and* (4).

(c) $*_{\mathrm{fix}(*)} = *$ *and* $\mathrm{fix}(*_K) = K$.

By Lemma 2 the set $\mathrm{fix}(*)$ of truth-stressing hedge $*$ is a $\vee$-subsemilattice. Now we will introduce the basic notions of multi-adjoint concept lattices with heterogeneous conjunctors, in order to show how both frameworks, hedges and heterogeneous conjunctors, can be merged.

Firstly, let us introduced a bit of terminology: in the rest of this work we will call a mapping $*\colon L \to L$ satisfying (2), (3), and (4) an **intensifying hedge**, following the terminology introduced in [2]. In terms of interior structures $(L, \preceq)$, a mapping satisfying (2)–(4) is an interior operator on the lattice of truth degrees.

The two main notions on which multi-adjoint concept lattices with heterogeneous conjunctors is defined are given below: the P-connection between posets, and the adjoint triples.

Definition 2. *Given the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P, \leq)$, we say that P_1 and P_2 are* P-connected *if there exist non-decreasing mappings $\psi_1\colon P_1 \to P$, $\phi_1\colon P \to P_1$, $\psi_2\colon P_2 \to P$ and $\phi_2\colon P \to P_2$ verifying that $\phi_1(\psi_1(x)) = x$, and $\phi_2(\psi_2(y)) = y$, for all $x \in P_1$, $y \in P_2$.*

Definition 3. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be posets, and consider mappings $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$, then $(\&, \swarrow, \nwarrow)$ is an* adjoint triple *with respect to P_1, P_2, P_3 if: $x \leq_1 z \swarrow y$ iff $x \,\&\, y \leq_3 z$ iff $y \leq_2 z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$.*

From Lemma 2 we immediately obtain the following proposition:

Corollary 1. *Consider the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and $(P, \leq)$, and assume that L_1 and L_2 are P-connected, then:*

(a) If $\psi_1 \circ \phi_1$ is contractive (i.e. satisfies (2)) then P_1 is isomorphic to a $\vee$-subsemilattice of P.
(b) If $*\colon P_1 \to P_1$ is an intensifying hedge (i.e. satisfies properties (2), (3), and (4)) then the composition $\psi_1 \circ * \circ \phi_1\colon P \to P$ is an intensifying hedge in $\mathrm{fix}(\psi_1 \circ \phi_1)$.

Lemma 3. *Let $(L, \preceq), (L_1, \preceq_1), (L_2, \preceq_2)$ be lattices and let $(\&, \swarrow, \nwarrow)$ be an adjoint triple. For $a, a_i \in L_1$, $b, b_i \in L_2$, we have*

$$\bigvee_{i\in I}(a_i \,\&\, b) = (\bigvee_{1\, i\in I} a) \,\&\, b \quad \textit{and} \quad \bigvee_{i\in I}(a \,\&\, b_i) = a \,\&(\bigvee_{2\, i\in I} b_i) \tag{8}$$

Definition 4. *A multi-adjoint frame is a tuple*

$$(L_1, L_2, P, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n)$$

where L_i are complete lattices and P i a poset, such that $(\&_i, \swarrow^i, \nwarrow_i)$ is an adjoint triple with respect to L_1, L_2, P for all $i = 1, \ldots, n$.

Definition 5. *Let $(L_1, L_2, P, \&_1, \ldots, \&_n)$ be a multi-adjoint frame, a* multi-adjoint context *is a tuple (A, B, R, σ) such that A and B are non-empty sets (usually interpreted as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1, \ldots, n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.*

Given a complete lattice $(L, \preceq)$ such that L_1 and L_2 are L-connected, a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$, and a context (A, B, R, σ), we can define the mappings ${}^{\uparrow_{c\sigma}}\colon L^B \to L^A$ and ${}^{\downarrow^{c\sigma}}\colon L^A \to L^B$ defined for all $g \in L^B$ and $f \in L^A$ as follows:

$$g^{\uparrow_{c\sigma}}(a) = \psi_1(\inf\{R(a,b) \swarrow^{\sigma(b)} \phi_2(g(b)) \mid b \in B\}) \tag{9}$$

$$f^{\downarrow^{c\sigma}}(b) = \psi_2(\inf\{R(a,b) \nwarrow_{\sigma(b)} \phi_1(f(a)) \mid a \in A\}) \tag{10}$$

The notion of concept is defined as usual. A *concept* is a pair $\langle g, f\rangle$ satisfying $g \in L^B$, $f \in L^A$ and that $g^{\uparrow_{c\sigma}} = f$ and $f^{\downarrow^{c\sigma}} = g$.

Definition 6. *Given the complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L, \preceq)$, where L_1 and L_2 are L-connected, the set of* multi-adjoint L-connected concepts *associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \ldots, \&_n)$ and context (A, B, R, σ) is given by $\mathfrak{M}_L = \{\langle g, f\rangle \mid \langle g, f\rangle$ is a concept$\}$.*

The main theorem of concept lattices in [9], proves that $\mathfrak{M}_L$ has the structure of a complete lattice:

Theorem 1 ([9]). *Given complete lattices $(L_1, \preceq_1)$, $(L_2, \preceq_2)$ and $(L, \preceq)$, where L_1 and L_2 are L-connected, a context (A, B, R, σ), and a multi-adjoint frame $(L_1, L_2, L, \&_1, \ldots, \&_n)$, the multi-adjoint L-connected concept lattice $\mathfrak{M}_L$ is actually a complete lattice with the meet and join operators $\curlywedge, \curlyvee\colon \mathfrak{M}_L \times \mathfrak{M}_L \to \mathfrak{M}_L$ defined below, for all $\langle g_1, f_1\rangle, \langle g_2, f_2\rangle \in \mathfrak{M}_L$,*

$$\langle g_1, f_1\rangle \curlywedge \langle g_2, f_2\rangle = \langle \psi_2 \circ \phi_2(g_1 \wedge g_2), (f_1 \vee f_2)^{\downarrow^c\uparrow_c}\rangle$$
$$\langle g_1, f_1\rangle \curlyvee \langle g_2, f_2\rangle = \langle (g_1 \vee g_2)^{\uparrow_c\downarrow^c}, \psi_1 \circ \phi_1(f_1 \wedge f_2)\rangle$$

The order $\preceq$ which corresponds to $\curlywedge$ and $\curlyvee$ is defined as

$$\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle \quad \text{iff} \quad \phi_2(g_1) \leq \phi_2(g_2) \quad (\text{iff } \phi_1(f_2) \leq \phi_1(f_1))$$

In what follows $\mathfrak{M}$ denotes multi-adjoint L-connected concept lattice of given context (A, B, R, σ). We will also omit subscript $\sigma(b)$ and write just $\swarrow$ instead of $\swarrow^{\sigma(b)}$.

3 Reducing the size of multi-adjoint concept lattices

The size of the concept lattice $\mathfrak{M}$ can be reduced either by a suitable selection of a $\vee$-subsemilattice of L_1 (and/or L_2) and the use of a restriction of $\&$. The following proposition says that the selection of $\vee$-subsemilattices of L_1 (resp. L_2) yields a reduction of size of concept lattice and, moreover, preserves intents (or extents) of the original concept lattice, meaning that each intent of the reduced concept lattice is an intent of the original concept lattice.

Proposition 1. *Let* $\mathbf{A} = (L_1, L_2, P, \&_1, \dots, \&_n), \mathbf{A}' = (K_1, L_2, P, \&'_1, \dots, \&'_n)$ *be multi-adjoint frames, s.t.* K_1 *is a* $\vee$*-subsemilattice of* L_1*, and* $\&'_1, \dots, \&'_n$ *are restrictions of* $\&_1, \dots, \&_n$ *to* $K_1 \times L_2$ *and* $\psi'_1 = \psi_1$, $\psi'_2 = \psi_2$, $\phi'_2 = \phi_2$, $\phi'_1 = *_{K_1} \circ \phi_1$*, where* $*_{K_1}$ *is the hedge associated to* K_1 *as introduced in Lemma 2. Then,* $\mathrm{Int}(\mathfrak{M}_{\mathbf{A}'}) \subseteq \mathrm{Int}(\mathfrak{M}_{\mathbf{A}})$ *where* $\mathrm{Int}(\mathfrak{M})$ *denotes the set of intents in* $\mathfrak{M}$.

Proof (sketch). We have $z \nwarrow' x = z \nwarrow x$, for each $x \in K_1, z \in P$, whence $f^{\downarrow'} = f^{\downarrow}$, for each $f \colon A \to \psi_1(K_1)$ where $\psi_1(K_1) \in L$ is image of ψ_1 (note that $\swarrow'$ is well-defined since K_1 is $\vee$-subsemilattice) and thus by Proposition 16 in [9] $\mathrm{Ext}(\mathfrak{M}_{\mathbf{A}'}) \subseteq \mathrm{Ext}(\mathfrak{M}_{\mathbf{A}})$. $\square$

Remark 1. One can state a dual proposition to Proposition 1 for intents. Let $\mathbf{A} = (L_1, L_2, P, \&_1, \dots, \&_n), \mathbf{A}' = (L_1, K_2, P, \&'_1, \dots, \&'_n)$ be multi-adjoint frames, s.t. K_2 is a $\vee$-subsemilattice of L_2, and $\&'_1, \dots, \&'_n$ are restrictions of $\&_1, \dots, \&_n$ to $L_1 \times K_2$ and $\phi'_2 = *_{K_2} \circ \phi_2$.

The following proposition says that by selection of $\vee$-subsemilattices of both L_1 and L_2 we obtain a reduction of the size as well. However, the preservation of intents (or extents) is lost.

Proposition 2. *Let* $\mathbf{A} = (L_1, L_2, L, \&_1, \dots, \&_n), \mathbf{A}' = (K_1, K_2, L, \&'_1, \dots, \&'_n)$ *be multi-adjoint frames, s.t.* K_1 *is a* $\vee$*-subsemilattice of* L_1*,* K_2 *is a* $\vee$*-subsemilattice of* L_2*, and* $\&'_1, \dots, \&'_n$ *are restrictions of* $\&_1, \dots, \&_n$ *to* $K_1 \times K_2$*, and* $\phi'_1 = *_{K_1} \circ \phi_1, \phi'_2 = *_{K_2} \circ \phi_2$*. Then we have* $|\mathfrak{M}_{\mathbf{A}'}| \leq |\mathfrak{M}_{\mathbf{A}}|$.

Proof (sketch). By applying Proposition 1 and Remark 1 we obtain the result. $\square$

In the next result we show how to generate new adjoint triples using hedges.

Lemma 4. *Assume* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple with respect to* L_1*,* L_2*,* P*, and* $*_1 \colon L_1 \to L_1$*,* $*_2 \colon L_2 \to L_2$ *are hedges, then* $x \,\&^*\, y = *_1(x) \,\&\, *_2(y)$ *has two residuated implications* $\swarrow^*, \nwarrow_*$ *which form a new adjoint triple with respect to* L_1*,* L_2*,* P*, if and only if the following equalities hold:*

$$*_1(z \swarrow *_2(y)) = *_1(\bigvee\{x \mid x \,\&^*\, y \leq z\}) \tag{11}$$

$$*_2(z \nwarrow *_1(x)) = *_2(\bigvee\{y \mid x \,\&^*\, y \leq z\}) \tag{12}$$

Proof. "$\Rightarrow$": Let $(\&^*, \swarrow^*, \nwarrow_*)$ be an adjoint triple. We have

$$x \,\&^*\, y \leq z \quad \text{iff} \quad y \preceq_2 z \nwarrow_* x$$

by definition. In particular, we obtain

$$*_1(x) \,\&^*\, *_2(y) \leq z \quad \text{iff} \quad *_2(y) \preceq_2 z \nwarrow_* *_1(x)$$

and $*_1(x) \,\&^*\, *_2(y) = *_1(*_1(x)) \,\&\, *_2(*_2(y)) = *_1(x) \,\&\, *_2(y) = x \,\&^*\, y$. Hence, we have

$$x \,\&^*\, y \leq z \quad \text{iff} \quad *_2(y) \preceq_2 z \nwarrow_* *_1(x)$$

From (3) and (4) we obtain that

$$*_2(y) \preceq_2 z \nwarrow_* *_1(x) \quad \text{implies} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow_* *_1(x))$$

and due to (2) we have

$$*_2(y) \preceq_2 *_2(z \nwarrow_* *_1(x)) \quad \text{implies} \quad *_2\,(y) \preceq_2 z \nwarrow_* *_1(x)$$

Therefore, we have

$$x \,\&^*\, y \leq z \quad \text{iff} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow_* *_1(x)). \tag{13}$$

Analogously, we obtain

$$*_1(x) \,\&\, *_2(y) \leq z \quad \text{iff} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow *_1(x)) \tag{14}$$

By setting $y = (z \nwarrow *_1(x))$, in Equation (13), and $y = (z \nwarrow_* *_1(x))$, in Equation (15), we obtain equivalent inequalities $*_2(z \nwarrow *_1(x)) \preceq *_2(z \nwarrow_* *_1(x))$, $*_2(z \nwarrow *_1(x)) \succeq *_2(z \nwarrow_* *_1(x))$ respectively. Thus we have

$$*_2(z \nwarrow *_1(x)) = *_2(z \nwarrow_* *_1(x)).$$

Which is equal to (12). The first equation (11) can be obtained dually.

"$\Leftarrow$": Assume (12) holds true. By properties of adjointness, to show that $\&^*$ generates an adjoint triple we need to show that

$$R = \{y \mid *_1(x) \,\&\, *_2(y) \leq z\}$$

has a greatest element.

In the previous part, we proven that

$$*_1(x) \,\&\, *_2(y) \leq z \quad \text{iff} \quad *_2\,(y) \preceq_2 *_2(z \nwarrow *_1(x)) \tag{15}$$

hence $R = \{y \mid *_2(y) \preceq *_2(z \nwarrow *_1(x))\}$. Now, if R has no greatest element, i.e. $\bigvee R \notin R$, then we have $*_2(\bigvee R) \not\preceq *_2(z \nwarrow *_1(x))$ which is a contradiction with the assumption. By the contradiction we proved that R has a greatest element. $\square$

Proposition 3. *Let $\mathbf{A} = (L_1, L_2, P, \&_1, \dots, \&_n)$ be a multi-adjoint frame $*_1, *_2$ be hedges on L_1 and L_2, respectively. Let $\mathbf{A}' = (\mathrm{fix}(*_1), \mathrm{fix}(*_2), P, \&'_1, \dots, \&'_n)$ s.t. $\&'_1, \dots, \&'_n$ are restrictions of $\&_1, \dots, \&_n$ to $\mathrm{fix}(*_1) \times \mathrm{fix}(*_2)$, and $\phi'_1 = *_1 \circ \phi_1, \phi'_2 = *_2 \circ \phi_2$. Let $\mathbf{A}^* = (L_1, L_2, P, \&^*_1, \dots, \&^*_n)$ be a multi-adjoint frame where $\&^*_i$ is defined by $a \,\&^*_i\, b = *_1(a) \,\&_i\, *_2(b)$, for all $i \in \{1, \dots, n\}$, and the conditions in Lemma 4 are satisfied. Then $(\mathfrak{M}_{A'}, \preceq')$ and $(\mathfrak{M}_{A^*}, \preceq^*)$ are isomorphic.*

Proof. Let $\mathbb{K} = (A, B, R, \sigma)$ be a formal context, denote by ${}^\uparrow, {}^\downarrow$ concept-forming operators induced by $\mathbb{K}$ and $\mathbf{A}'$ and denote by ${}^\Uparrow, {}^\Downarrow$ concept-forming operators induced by $\mathbb{K}$ and $\mathbf{A}^*$. Furthermore, denote compositions $\psi_1 \circ *_1 \circ \phi_1$ and $\psi_2 \circ *_2 \circ \phi_2$ by $\bullet_1$ and $\bullet_2$ respectively.

For each mapping $g : B \to L$ we have

$$\begin{aligned}
\bullet_1(g^{\Uparrow}(a)) &= \bullet_1(\psi_1 \bigwedge\nolimits_1 (R(a,b) \swarrow^{*} \phi_2(g(b)))) \\
&= \psi_1 *_1 (\phi_1 \psi_1 \bigwedge\nolimits_1 (R(a,b) \swarrow^{*} (\phi_2(g(b))))) \\
&= \psi_1 \bigwedge\nolimits_1 *_1 (\bigvee\nolimits_1 \{x \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b))\}) \\
&\overset{(\Delta)}{=} \psi_1 \bigwedge\nolimits_1 (\bigvee\nolimits_1 \{*_1(x) \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b))\}) \\
&= \psi_1 \bigwedge\nolimits_1 (\bigvee\nolimits_1 \{x \in \mathrm{fix}(*_1) \mid x \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b))\}) \\
&= \psi_1 \bigwedge\nolimits_1 (R(a,b) \swarrow' *_2(\phi_2(g(b)))) \\
&= \psi_1 \bigwedge\nolimits_1 (R(a,b) \swarrow' \phi_2 \psi_2 *_2 (\phi_2 g(b))) \\
&= \psi_1 \bigwedge\nolimits_1 (R(a,b) \swarrow' \phi_2 \bullet_2 (g(b))) \\
&= (\bullet_2 \circ g)^{\uparrow}(a)
\end{aligned}$$

where (Δ) is due to Lemma 1 (6) and the fact that & generates adjoint triple and thus $\bigvee_1 \{x \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b))\})$ has a greatest elements. Dually, we have $\bullet_2 \circ (f^{\Downarrow}) = (\bullet_1 \circ f)^{\downarrow}$ for each mapping $f : A \to L$. From that we have

$$g^{\uparrow} = \bullet_1 \circ (g^{\Uparrow}) \quad \text{and} \quad f^{\downarrow} = \bullet_2 \circ (f^{\Downarrow})$$

for each $g : B \to \mathrm{fix}(\bullet_2)$, $f : A \to \mathrm{fix}(\bullet_1)$. As a result of the previous equalities, we have that $\bullet_2$ is a surjective mapping $\mathrm{Ext}(\mathfrak{M}_{A^*}) \to \mathrm{Ext}(\mathfrak{M}_{A'})$ and $\bullet_1$ is a surjective mapping $\mathrm{Int}(\mathfrak{M}_{A^*}) \to \mathrm{Int}(\mathfrak{M}_{A'})$. In addition, for $g \in \mathrm{Ext}(\mathfrak{M}_{A^*})$ we have

$$\begin{aligned}
\bullet_2(g)^{\Uparrow}(a) &= \psi_1 \bigwedge\nolimits_1 R(a,b) \swarrow^{*} \phi_2 \psi_2 *_2 \phi_2(g(b)) \\
&= \psi_1 \bigwedge\nolimits_1 \bigvee\nolimits_2 \{x \mid *_1(x) \mathbin{\&} *_2 *_2 (\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \bigwedge\nolimits_1 \bigvee\nolimits_2 \{x \mid *_1(x) \mathbin{\&} *_2(\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \bigwedge\nolimits_1 R(a,b) \swarrow^{*} \phi_2(g(b))) \\
&= g^{\Uparrow}(a)
\end{aligned}$$

and dually $\bullet_1(f)^{\Downarrow} = f^{\Downarrow}$. Putting it together, we have $g = g^{\Uparrow\Downarrow} = \bullet_1(g^{\Uparrow})^{\Downarrow} = \bullet_2(g)^{\uparrow\Downarrow}$ showing that $\uparrow\Downarrow$ is injective; whence $\bullet_1, \bullet_2$ are bijections.

To show that $\bullet_1, \bullet_2$ are order-preserving let $\langle g_1, f_1 \rangle, \langle g_2, f_2 \rangle \in \mathfrak{M}_{A^*}$. An extent of $\langle g_1, f_1 \rangle \wedge \langle g_2, f_2 \rangle$ is equal to $\psi_2 \phi_2(g_1 \wedge g_2)$ by the main Theorem in [9].

For $g_1, g_2 \in \mathrm{Ext}(\mathfrak{M}_{A^*})$ we have

$$\begin{aligned}
\bullet_2 \psi_2 \phi_2(g_1 \wedge g_2) &= \psi_2 *_2 \phi_2 \psi_2 \phi_2(g_1 \wedge g_2) \\
&= \psi_2 *_2 \phi_2(g_1 \wedge g_2) \\
&= \psi_2 *_2 \phi_2 \bullet_2 (g_1 \wedge g_2) \\
&\overset{(\Delta)}{=} \psi_2 \phi_2'(\bullet_2(g_1) \wedge \bullet_2(g_2))
\end{aligned}$$

Equality (Δ) is due to Corollary 1(b) since note that g_1, g_2 are fixpoints of $\psi_2 \circ \phi_2$.

Now, let $g_1, g_2 \in \mathrm{Ext}(\mathfrak{M}_{A'})$. We have

$$\begin{aligned}
(\psi_2 \phi_2'(g_1 \wedge g_2))^{\uparrow\Downarrow} &= (\psi_2 *_2 \phi_2(g_1 \wedge g_2))^{\uparrow\Downarrow} \\
&= (\bullet_1(g_1 \wedge g_2))^{\uparrow\Downarrow} \\
&= (\bullet_1(g_1^{\uparrow\downarrow} \wedge g_2^{\uparrow\downarrow}))^{\uparrow\Downarrow} \\
&= (\bullet_1(\bullet_1(g_1^{\uparrow\Downarrow}) \wedge \bullet_1(g_2^{\uparrow\Downarrow})))^{\uparrow\Downarrow} \\
&\overset{(\Delta)}{=} (\bullet_1 \bullet_1 (g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow}))^{\uparrow\Downarrow} \\
&= (\bullet_1(g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow}))^{\uparrow\Downarrow} \\
&= (g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow})^{\Uparrow\Downarrow} \\
&\overset{(\nabla)}{=} \psi_2 \phi_2(g_1^{\uparrow\Downarrow} \wedge g_2^{\uparrow\Downarrow})
\end{aligned}$$

Equality (Δ) is due to Corollary 1(b) since g_1, g_2 are fixpoints of $\psi_2 \circ \phi_2$; equality (∇) is due to [9, Lemma 21].

This proves that $\bullet_1, \bullet_2, \uparrow\Downarrow$, and $\downarrow\Uparrow$ are order-preserving. □

Example 1. Consider the multi-adjoint frame depicted in Fig. 1 (structures are the same as in [9, Example 3 (Fig. 2)] (where all $\&_i$'s coincide). Figure 2 depicts a formal context with two objects and two attributes, together with their associated multi-adjoint concept lattice.

Concept lattices with truth-stressing hedges

In this part, we follow the way in which the hedges are used in [4], i.e. we generalize concept-forming operators using intensifying hedges. Then we show how this is related to the theory described above.

We define the concept-forming operators as follows

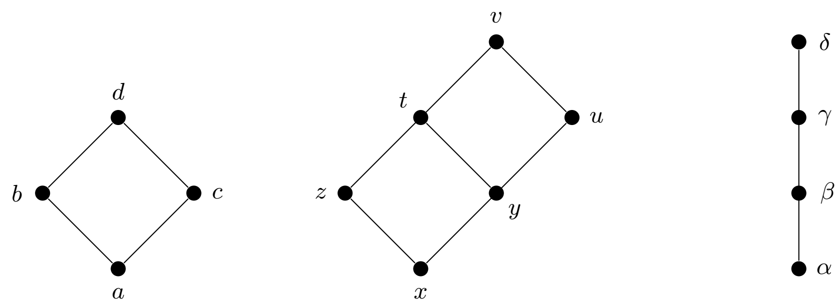
$$\begin{aligned}
g^{\triangle}(a) &= \psi_1 {\textstyle\bigwedge_1}_{b \in B} R(a,b) \swarrow *_2(\phi_2(g(b))), \\
f^{\nabla}(b) &= \psi_2 {\textstyle\bigwedge_2}_{a \in A} R(a,b) \nwarrow *_1(\phi_1(f(a))).
\end{aligned}$$

Note that this is not strictly the same approach as used in Proposition 3 since $\swarrow$ and $\nwarrow$ are residua of the original adjoint operators $\&$, not the altered operators $\&^*$. In fact, generally there is no base operation $\&$ such that $(\cdot) \swarrow *_2(\cdot)$ and $(\cdot) \nwarrow *_1(\cdot)$ are its residua, since we do not generally have

$$x \leq z \swarrow *_2(y) \text{ iff } y \leq z \nwarrow *_1(x)$$

for each $x \in L_1, y \in L_2, z \in L$.

Lemma 5. *Assume* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple,* $*_1, *_2$ *are intensifying hedges, and* $\swarrow^{\diamond}, \nwarrow_{\diamond}$ *being defined as* $z \swarrow^{\diamond} y = z \swarrow *_2 y$*, and* $z \nwarrow_{\diamond} x = z \nwarrow *_1 x$*; then*



	a	b	c	d
ψ_1	x	t	u	v

	α	β	γ	δ
ψ_2	x	y	t	v

	x	y	z	t	u	v
ϕ_1	a	b	a	b	c	d

	x	y	z	t	u	v
ϕ_2	α	β	γ	γ	δ	δ

$\&$	α	β	γ	δ
a	x	x	x	x
b	x	y	v	v
c	x	y	y	t
d	x	y	v	u

$\swarrow$	α	β	γ	δ
t	d	d	c	c
u	d	d	d	d
v	d	d	d	b
x	d	a	a	a
y	d	d	c	a
z	d	a	a	a

$\nwarrow$	a	b	c	d
t	δ	β	δ	β
u	δ	δ	δ	δ
v	δ	δ	γ	γ
x	δ	α	α	α
y	δ	β	γ	β
z	δ	α	α	α

Fig. 1. L_1 (top left), L (top middle), L_2 (top right), connection operators $\phi_1, \phi_2, \psi_1, \psi_2$ (middle), adjoint triple ($\langle \&, \nwarrow, \swarrow \rangle$) (bottom).

	1	2
1	u	v
2	v	y

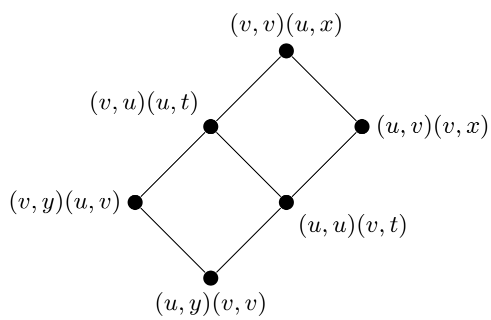


Fig. 2. Multi-adjoint formal context with two objects and two attributes (left) and the multi-adjoint concept lattice associated to the context (right).

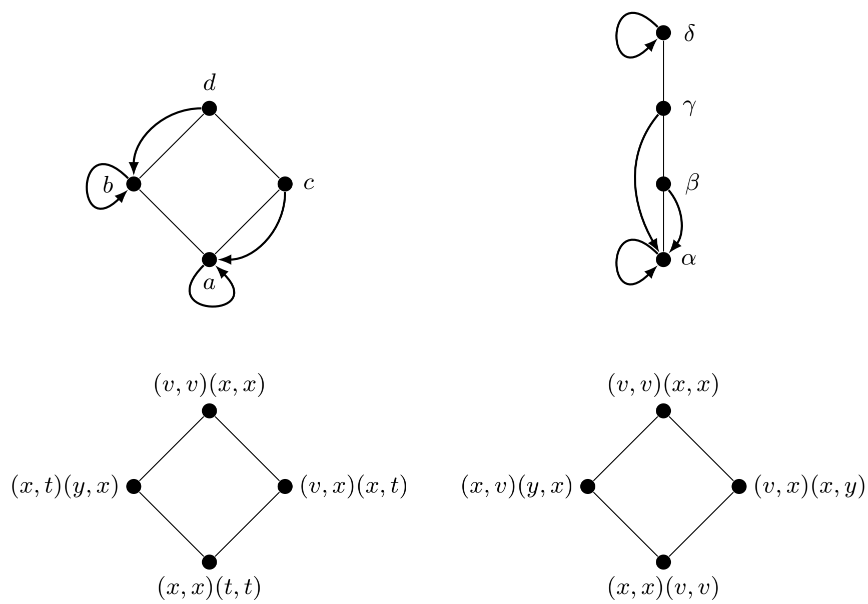


Fig. 3. Intensifying hedge on L_1 (top left) and L_2 (top right); concept lattices $\mathfrak{M}_{A'}$ (bottom left), $\mathfrak{M}_{A^*}$ (bottom right) of the formal context in Fig. 2; labels of nodes of $\mathfrak{M}_{A'}$ and $\mathfrak{M}_{A^*}$ represent characteristic vectors of corresponding extents and intents.

$\swarrow^{\diamond}, \nwarrow_{\diamond}$ *are part of an adjoint triple with conjunctor* $\&^{\diamond}$ *if and only if for all* x, y *the following equality holds*

$$x \,\&\, *_2(y) = *_1(x) \,\&\, y$$

and, in this case the previous value is the definition of $\&^{\diamond}$.

Proof. For all x, y, z, on the one hand, we have

$$*_1(x) \,\&\, y \leq z \quad \text{iff} \quad y \preceq_2 z \nwarrow *_1(x) \quad \text{iff} \quad y \preceq_2 z \nwarrow_{\diamond} x.$$

On the other hand, we have

$$x \,\&\, *_2(y) \leq z \quad \text{iff} \quad x \preceq_1 z \swarrow *_2(y) \quad \text{iff} \quad x \preceq_1 z \swarrow^{\diamond} y.$$

Thus we have

$$y \preceq_2 z \nwarrow_{\diamond} x \quad \text{iff} \quad x \preceq_1 z \swarrow^{\diamond} y$$

is equivalent to

$$x \,\&\, *_2(y) \leq z \quad \text{iff} \quad *_1(x) \,\&\, y \leq z,$$

which is equivalent to $x \,\&\, *_2(y) = *_1(x) \,\&\, y$. □

However, the concept-forming operators $\vartriangle, \triangledown$ are in one-to-one correspondence with concept-forming operators $\uparrow, \downarrow$ with restrictions of L_1 and L_2 to subsemilattices $\mathrm{fix}(*_1)$ and $\mathrm{fix}(*_2)$:

$$\begin{aligned}
\bullet_1(g^{\vartriangle}(a)) &= \bullet_1(\psi_1 \textstyle\bigwedge_{b \in B} R(a,b) \swarrow *_2\phi_2(g(b))) \\
&= \psi_1 *_1 (\textstyle\bigwedge_{1\, b \in B} R(a,b) \swarrow *_2(\phi_2(g(b)))) \\
&= \psi_1 \textstyle\bigwedge_{1\, b \in B} *_1 (R(a,b) \swarrow *_2(\phi_2(g(b)))) \\
&= \psi_1 \textstyle\bigwedge_{1\, b \in B} *_1 \bigvee_1 \{x \mid x \,\&\, *_2(\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \textstyle\bigwedge_{1\, b \in B} \bigvee_1 \{*_1(x) \mid x \,\&\, *_2(\phi_2(g(b))) \leq R(a,b)\} \\
&\overset{(\Delta)}{=} \psi_1 \textstyle\bigwedge_{1\, b \in B} \bigvee_1 \{*_1(x) \mid *_1(x) \,\&\, *_2(\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \textstyle\bigwedge_{1\, b \in B} \bigvee_1 \{x \in \mathrm{fix}(*_1) \mid x \,\&\, *_2(\phi_2(g(b))) \leq R(a,b)\} \\
&= \psi_1 \textstyle\bigwedge_{1\, b \in B} \bigvee_1 \{x \in \mathrm{fix}(*_1) \mid x \,\&\, \phi_2'(g(b)) \leq R(a,b)\} \\
&= \psi_1 \textstyle\bigwedge_{1\, b \in B} R(a,b) \swarrow \phi_2'(g(b)) \\
&= g^{\uparrow}(a)
\end{aligned}$$

where equality (Δ) holds because each x satisfying $x \,\&\, y \leq z$ satisfies $*_2(x) \,\&\, y \leq z$ as well; and because each $*_2(x)$ such that $*_2(x) \,\&\, y \leq z$ there is x' (explicitly, $*_2(x)$) with $*_2(x') = *_2(x)$ such that $x' \,\&\, y \leq z$. Dually, one can show $\bullet_2(f^{\triangledown}) = f^{\downarrow}$.

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