PHYSICS OF PLASMAS 15, 072516 (2008)

An analytical demonstration of coupling schemes between magnetohydrodynamic codes and eddy current codes

Yueqiang Liu,[1,a)] R. Albanese,[2] A. Portone,[3] G. Rubinacci,[2] and F. Villone[4]
[1]*Euratom/UKAEA Fusion Association, Culham Science Centre, Abingdon OX14 3DB, United Kingdom*
[2]*ENEA/CREATE, Universita Federico II di Napoli, Via Claudio 21, Napoli, Italy*
[3]*EFDA-CSU Barcelona, C/Josep Pla 2, B3, 08019 Barcelona, Spain*
[4]*ENEA/CREATE, DAEIMI, Universita di Cassino, Via di Biasio 43, Cassino (FR), Italy*

(Received 21 April 2008; accepted 23 June 2008; published online 25 July 2008)

In order to model a magnetohydrodynamic (MHD) instability that strongly couples to external conducting structures (walls and/or coils) in a fusion device, it is often necessary to combine a MHD code solving for the plasma response, with an eddy current code computing the fields and currents of conductors. We present a rigorous proof of the coupling schemes between these two types of codes. One of the coupling schemes has been introduced and implemented in the CARMA code [R. Albanese, Y. Q. Liu, A. Portone, G. Rubinacci, and F. Villone, IEEE Trans. Magn. **44**, 1654 (2008); A. Portone, F. Villone, Y. Q. Liu, R. Albanese, and G. Rubinacci, Plasma Phys. Controlled Fusion **50**, 085004 (2008)] that couples the MHD code MARS-F [Y. Q. Liu, A. Bondeson, C. M. Fransson, B. Lennartson, and C. Breitholtz, Phys. Plasmas **7**, 3681 (2000)] and the eddy current code CARIDDI [R. Albanese and G. Rubinacci, Adv. Imaging Electron Phys. **102**, 1 (1998)]. While the coupling schemes are described for a general toroidal geometry, we give the analytical proof for a cylindrical plasma. [DOI: 10.1063/1.2959129]

I. INTRODUCTION

In a fusion device such as a tokamak, it is well known that an unstable external kink mode, driven either by the plasma current or pressure, produces an external magnetic field perturbation, which induces stabilizing image currents in the surrounding conducting structures, such as the vacuum vessels made of low resistivity materials. The image currents, flowing in the conducting walls, can effectively reduce the growth rate of the external kink mode, from the Alfvénic time scale (microseconds) to the characteristic current decay time of the wall, which is typically a few milliseconds in present tokamaks, and a fraction of a second in ITER.[1]

As a result of slowing down of the mode growth rate, additional stabilizing mechanisms or techniques can be applied to this mode, which is called a resistive wall mode (RWM), to make it completely stable. For instance, the RWM can be stabilized by toroidal rotation of the plasma,[2–4] or by active control using magnetic coils surrounding the plasma.[5–8] Complete suppression of the pressure driven RWM helps maintain the plasma pressure above the no-wall ideal beta limit, giving the possibility of operating a future fusion power plant at higher power production and in steady state.

Due to the strong coupling of the RWM to the external conducting structures, modeling the stability and control of this mode inevitably involves both magnetohydrodynamic (MHD) calculations for the plasma and the eddy current calculations for the conductors. While the plasma response can often be modeled as a linear perturbation with a single toroidal mode number n ($n=1$, 2, or 3 for typical RWM), the external conductors, both resistive walls and control coils for feedback stabilization, normally have three-dimensional (3D) features, which may significantly modify the 3D results. Accurate modeling of the conductors requires solving 3D eddy current problems, which are typically not considered in many free boundary MHD codes. For instance, in the linear MHD codes MARS-F[9] or KINX,[10] the resistive wall is modeled as an axisymmetric thin shell, and the feedback coils as an infinite number of current circuits along the toroidal angle, to produce a single n field perturbation. For the RWM simulations, it is often necessary to combine a MHD code with a 3D eddy current code.

The eddy current problem (i.e., quasistatic Maxwell equations), with shell-like conductors, can be efficiently solved in the integral formulation, where the unknown variables are the eddy current density in the conductors. Examples of this approach include the CARIDDI[11] and VALEN[12] codes. An alternative approach is to solve the Maxwell equations in differential form, and an efficient solver is possible using the ungauged vector potential as the unknown.[13]

In this work, we describe and prove the coupling schemes for integral eddy current solvers. More specifically, we study the coupling scheme behind the CARMA code,[14,15] which combines the MHD code MARS-F and the eddy current code CARIDDI. We also propose and prove a new scheme that can be beneficial in certain specific cases. These schemes can be used to couple a generic MHD code with a generic eddy current code. The schemes are described in a general toroidal geometry in Sec. II, and their validity is rigorously shown for a cylindrical plasma in Sec. III. Section IV summarizes the results.

a)Electronic mail: yueqiang.liu@ukaea.org.uk.

II. COUPLING SCHEMES IN GENERAL TOROIDAL GEOMETRY

We consider two ways to couple a MHD code and an eddy current code. In the first approach, the MHD code computes response matrices of the plasma (described below) that are used as the input to the eddy current code, which subsequently solves the whole RWM (stability and control) problem without involving the MHD code anymore. We refer to this approach as the forward coupling scheme.

In the second approach, referred to as the backward coupling, the eddy current code computes response matrices of the conducting structures that are used as input to the MHD code, which subsequently solves the whole RWM problem without involving the eddy current calculations anymore.

The basic assumptions, for these two coupling schemes to work, are that (i) the MHD code uses the normal component of the magnetic field (or flux) as the inhomogeneous boundary condition to compute the plasma response field; (ii) the computational boundary, which serves as the coupling surface between the MHD code and the eddy current code, can be the plasma boundary or an arbitrary surface outside the plasma but inside all conductors surrounding the plasma; (iii) the eddy current code solves the quasimagnetostatic Maxwell equations in the integral form, i.e., with the solution variables being the eddy current density in the conductors.

There might be approaches where the two codes are coupled in an iterative manner. Normally these approaches result in a less efficient computation, and sometimes have trouble with numerical convergence for the eigenvalue.[16] We do not consider them in this work.

A. Forward coupling scheme from MHD to eddy current formulation

This scheme has been described in detail in Refs. 14 and 15. Below we give a short description of the procedure, in view of the analytical proof later on. We consider only the RWM with a single toroidal mode number n.

(1) Choose M independent normal magnetic field perturbations $\mathbf{b}_N$, e.g., M poloidal harmonics with a unit amplitude. Using these normal fields as the inhomogeneous boundary condition at the coupling surface, solve the MHD equations M times for the plasma response. Find an $M\times M$ matrix $\overleftrightarrow{\mathbf{K}}$ relating the tangential field $\mathbf{b}_T$, at the inner side of the coupling surface, to the normal field $\mathbf{b}_N$,

$$\mathbf{b}_T = \overleftrightarrow{\mathbf{K}}\mathbf{b}_N. \tag{1}$$

(2) Solve the same problem without the plasma (i.e., vacuum solution), and find a similar matrix $\overleftrightarrow{\mathbf{K}}^{\mathrm{vac}}$,

$$\mathbf{b}_T^{\mathrm{vac}} = \overleftrightarrow{\mathbf{K}}^{\mathrm{vac}}\mathbf{b}_N. \tag{2}$$

(3) Compute an equivalent surface current density $\mathbf{I}_{\mathrm{eqv}}$ at the coupling surface, which reproduces the same magnetic field outside the coupling surface as that from the perturbed plasma current,

$$\mathbf{I}_{\mathrm{eqv}} = \mu_0^{-1}(\mathbf{b}_T - \mathbf{b}_T^{\mathrm{vac}}) = \mu_0^{-1}(\overleftrightarrow{\mathbf{K}} - \overleftrightarrow{\mathbf{K}}^{\mathrm{vac}})\mathbf{b}_N = \overleftrightarrow{\mathbf{F}}\mathbf{b}_N. \tag{3}$$

Note that since the perturbed field, as well as the equivalent surface current, generally has two tangential components, we define and compute two sets of matrices $\overleftrightarrow{\mathbf{K}}$ and $\overleftrightarrow{\mathbf{K}}^{\mathrm{vac}}$. These are full matrices with elements independent of the poloidal angle.

(4) Using the Biot–Savart law either over the equivalent surface current $\mathbf{I}_{\mathrm{eqv}}$ or over the perturbed plasma current, compute the normal field perturbation produced by the plasma only,

$$\mathbf{b}_N^{\mathrm{pl}} = \overleftrightarrow{\mathbf{H}}\mathbf{I}_{\mathrm{eqv}} = \overleftrightarrow{\mathbf{H}}\overleftrightarrow{\mathbf{F}}\mathbf{b}_N = \overleftrightarrow{\mathbf{G}}\mathbf{b}_N. \tag{4}$$

(5) Since the total normal field $\mathbf{b}_N$ consists of the plasma contribution $\mathbf{b}_N^{\mathrm{pl}}$ and that from the external conductors $\mathbf{b}_N^{\mathrm{ex}}$, we may write

$$\mathbf{b}_N^{\mathrm{ex}} = (\overleftrightarrow{\mathbf{E}} - \overleftrightarrow{\mathbf{G}})\mathbf{b}_N, \tag{5}$$

where $\overleftrightarrow{\mathbf{E}}$ is the identity matrix.

(6) In the presence of the surface current on the coupling surface, and without any additional voltage sources, the eddy current code eventually solves a lumped circuit-like system of equations,

$$\overleftrightarrow{\mathbf{R}}\mathbf{I} + \gamma\overleftrightarrow{\mathbf{L}}\mathbf{I} + \gamma\mathbf{U} = 0, \tag{6}$$

where $\overleftrightarrow{\mathbf{R}}$ is the resistance matrix, $\mathbf{I}$ the eddy currents in the conductors, γ the growth rate of the mode, $\overleftrightarrow{\mathbf{L}}$ the self-inductance matrix of the conductors, $\mathbf{U}$ the magnetic flux at the position of conductors, produced by the equivalent surface current source $\mathbf{I}_{\mathrm{eqv}}$,

$$\mathbf{U} = \overleftrightarrow{\mathbf{M}}\mathbf{I}_{\mathrm{eqv}}. \tag{7}$$

(7) Compute the inductance matrix between the conductor eddy currents $\mathbf{I}$ and the produced normal field at the coupling surface,

$$\mathbf{b}_N^{\mathrm{ex}} = \overleftrightarrow{\mathbf{Q}}\mathbf{I}. \tag{8}$$

(8) Finally, combining Eqs. (7), (3), (5), and (8) yields

$$\mathbf{U} = \mathbf{MF}(\overleftrightarrow{\mathbf{E}} - \overleftrightarrow{\mathbf{G}})^{-1}\mathbf{QI} = \overleftrightarrow{\mathbf{L}}^{\mathrm{pl}}\mathbf{I}, \tag{9}$$

which, when substituted into the eddy current equation (6), effectively modifies the inductance matrix $\overleftrightarrow{\mathbf{L}}$ to $\overleftrightarrow{\mathbf{L}} + \overleftrightarrow{\mathbf{L}}^{\mathrm{pl}}$, and the modification is caused by the plasma response. Solving Eq. (6) as an eigenvalue problem gives the growth rate of the RWM. Alternatively, adding a voltage source for the active coils to the right-hand side (RHS) of Eq. (6), in combination with an appropriate feedback law, allows investigation of the RWM feedback stabilization.

The above procedure is designed for a linear plasma response. If the MHD response occurs on a time scale much slower than the Alfvén time, which is normally the case for the RWM, we can compute a static response matrix $\overleftrightarrow{\mathbf{L}}^{\mathrm{pl}}$ neglecting the plasma inertia effect. In general cases, the matrix $\overleftrightarrow{\mathbf{L}}^{\mathrm{pl}}$ should depend on the mode growth rate, hence we have to solve a nonlinear eigenvalue problem.

B. Backward coupling scheme from eddy current to MHD formulation

The description of the backward coupling procedure is simpler than that of the forward one.

(1) Assuming M independent surface current distributions $\mathbf{I}_{\mathrm{eqv}}$ along the coupling surface, for a *given* growth rate γ (initial guess), solve the eddy current problem for the conductors as a driven problem

$$\overleftrightarrow{\mathbf{R}}\mathbf{I} + \gamma\overleftrightarrow{\mathbf{L}}\mathbf{I} = -\gamma\overleftrightarrow{\mathbf{M}}\mathbf{I}_{\mathrm{eqv}}, \tag{10}$$

where matrices $\overleftrightarrow{\mathbf{R}}$, $\overleftrightarrow{\mathbf{L}}$, and $\overleftrightarrow{\mathbf{M}}$ have the same meaning as that in Eqs. (6) and (7). The solution of Eq. (10) can be written as

$$\mathbf{I} = -\gamma(\overleftrightarrow{\mathbf{R}} + \gamma\overleftrightarrow{\mathbf{L}})^{-1}\overleftrightarrow{\mathbf{M}}\mathbf{I}_{\mathrm{eqv}} = \overleftrightarrow{\mathbf{W}}\mathbf{I}_{\mathrm{eqv}}. \tag{11}$$

(2) Using the Biot–Savart law, compute both the total normal and tangential magnetic fields just *outside* the coupling surface, produced by the eddy currents $\mathbf{I}$ and the equivalent surface currents $\mathbf{I}_{\mathrm{eqv}}$,

$$\mathbf{b}_N = \overleftrightarrow{\mathbf{S}}_1\mathbf{I} + \overleftrightarrow{\mathbf{S}}_2\mathbf{I}_{\mathrm{eqv}} = (\overleftrightarrow{\mathbf{S}}_1\mathbf{W} + \overleftrightarrow{\mathbf{S}}_2)\mathbf{I}_{\mathrm{eqv}} = \overleftrightarrow{\mathbf{S}}\mathbf{I}_{\mathrm{eqv}}, \tag{12}$$

$$\mathbf{b}_T = \overleftrightarrow{\mathbf{P}}_1\mathbf{I} + \overleftrightarrow{\mathbf{P}}_2\mathbf{I}_{\mathrm{eqv}} = (\overleftrightarrow{\mathbf{P}}_1\mathbf{W} + \overleftrightarrow{\mathbf{P}}_2)\mathbf{I}_{\mathrm{eqv}} = \overleftrightarrow{\mathbf{P}}\mathbf{I}_{\mathrm{eqv}}. \tag{13}$$

Note that the two tangential components of the surface current $\mathbf{I}_{\mathrm{eqv}}$ have to satisfy the divergence-free condition, which helps us to express the magnetic fields in Eqs. (12) and (13) via the toroidal component of the surface current only. Hence, for a general toroidal case, we will replace $\mathbf{I}_{\mathrm{eqv}}$ in Eqs. (12) and (13) by its toroidal component, and the matrices $\overleftrightarrow{\mathbf{S}}$ and $\overleftrightarrow{\mathbf{P}}$ are redefined accordingly. Eventually, we obtain a single matrix $\overleftrightarrow{\mathbf{S}}$, and generally two sets of matrices $\overleftrightarrow{\mathbf{P}}$, for the two tangential components of the magnetic field.

(3) Use the relation

$$\mathbf{b}_T = \overleftrightarrow{\mathbf{P}}\overleftrightarrow{\mathbf{S}}^{-1}\mathbf{b}_N = \overleftrightarrow{\mathbf{T}}\mathbf{b}_N \tag{14}$$

as the boundary condition just *inside* the coupling surface for the MHD code, and solve for a nonlinear eigenvalue problem for γ. Note that the nonlinearity comes from the boundary condition (14), where the matrix $\overleftrightarrow{\mathbf{T}}$ is a function of γ at the resistive wall time scale.

To make this procedure practical, we need to be able to approximate the dependence of $\overleftrightarrow{\mathbf{T}}(\gamma)$ by analytical functions. This is possible as explained in the next section. A more elegant approach is to try to derive a relation, which is linear in γ, between $\mathbf{b}_N$ and $\mathbf{b}_T$. Indeed, a pure algebraic manipulation of Eqs. (10), (12), and (13) gives

$$\begin{aligned}
&[\overleftrightarrow{\mathbf{S}}_2^{-1} + \gamma(\overleftrightarrow{\mathbf{S}}_2^{-1}\overleftrightarrow{\mathbf{S}}_1 - \overleftrightarrow{\mathbf{P}}_2^{-1}\overleftrightarrow{\mathbf{P}}_1)\overleftrightarrow{\mathbf{L}}_1^{-1}(\overleftrightarrow{\mathbf{L}} - \overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{S}}_2^{-1}\overleftrightarrow{\mathbf{S}}_1)^{-1}\overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{S}}_2^{-1}]\mathbf{b}_N \\
&\quad = [\overleftrightarrow{\mathbf{P}}_2^{-1} + \gamma(\overleftrightarrow{\mathbf{S}}_2^{-1}\overleftrightarrow{\mathbf{S}}_1 - \overleftrightarrow{\mathbf{P}}_2^{-1}\overleftrightarrow{\mathbf{P}}_1)\overleftrightarrow{\mathbf{L}}_1^{-1} \\
&\qquad \times(\overleftrightarrow{\mathbf{L}} - \overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{P}}_2^{-1}\overleftrightarrow{\mathbf{P}}_1)^{-1}\overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{P}}_2^{-1}]\mathbf{b}_T,
\end{aligned} \tag{15}$$

where

$$\overleftrightarrow{\mathbf{L}}_1 = (\overleftrightarrow{\mathbf{L}} - \overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{S}}_2^{-1}\overleftrightarrow{\mathbf{S}}_1)^{-1}\overleftrightarrow{\mathbf{R}} - (\overleftrightarrow{\mathbf{L}} - \overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{P}}_2^{-1}\overleftrightarrow{\mathbf{P}}_1)^{-1}\overleftrightarrow{\mathbf{R}}. \tag{16}$$

This type of boundary condition leads to a linear eigenvalue problem after the backward coupling to the MHD code.

The backward coupling can be slightly adapted to include the feedback coils. We demonstrate this by assuming a simple current control logic,

$$\mathbf{I}_f = -\overleftrightarrow{\mathbf{K}}_f b_s, \tag{17}$$

where $\mathbf{I}_f$ is the current in the active coils, $\mathbf{b}_s$ the array of sensor signals, which are (normal or tangential) magnetic fields measured by coils outside the coupling surface, and $\overleftrightarrow{\mathbf{K}}_f$ is the feedback gain, generally being a matrix.

With the inclusion of feedback coils, Eq. (10) becomes

$$\overleftrightarrow{\mathbf{R}}\mathbf{I} + \gamma\overleftrightarrow{\mathbf{L}}\mathbf{I} = -\gamma\overleftrightarrow{\mathbf{M}}\mathbf{I}_{\mathrm{eqv}} - \gamma\overleftrightarrow{\mathbf{M}}_f\mathbf{I}_f, \tag{18}$$

where $\overleftrightarrow{\mathbf{M}}_f$ is the mutual inductance matrix between the active coils and the wall. We can also compute the mutual inductances between various currents and the sensor coils, which give

$$\mathbf{b}_s = (\overleftrightarrow{\mathbf{M}}_1\mathbf{I} + \overleftrightarrow{\mathbf{M}}_2\mathbf{I}_{\mathrm{eqv}} + \overleftrightarrow{\mathbf{M}}_3\mathbf{I}_f). \tag{19}$$

Combining Eqs. (17) and (19), and substituting into Eq. (18), we arrive at

$$\begin{aligned}
&\overleftrightarrow{\mathbf{R}}\mathbf{I} + \gamma[\overleftrightarrow{\mathbf{L}} - \overleftrightarrow{\mathbf{M}}_f(\overleftrightarrow{\mathbf{E}} + \overleftrightarrow{\mathbf{K}}_f\overleftrightarrow{\mathbf{M}}_3)^{-1}\overleftrightarrow{\mathbf{K}}_f\overleftrightarrow{\mathbf{M}}_1]\mathbf{I} \\
&\quad = -\gamma[\overleftrightarrow{\mathbf{M}} - \overleftrightarrow{\mathbf{M}}_f(\overleftrightarrow{\mathbf{E}} + \overleftrightarrow{\mathbf{K}}_f\overleftrightarrow{\mathbf{M}}_3)^{-1}\overleftrightarrow{\mathbf{K}}_f\overleftrightarrow{\mathbf{M}}_2]\mathbf{I}_{\mathrm{eqv}}.
\end{aligned} \tag{20}$$

Equation (20) has the same form as Eq. (10), hence all the coupling steps following Eq. (10) remain the same with the inclusion of the feedback coils.

III. ANALYTICAL PROOF IN CYLINDRICAL GEOMETRY

The coupling procedures described in Sec. II are designed for a general toroidal geometry. In the cylindrical limit, we show rigorously that these procedures yield the correct growth rate for the RWM. Moreover, this proof serves as an explicit demonstration of how the coupling matrices are computed—all the matrices described in Sec. II are calculated here analytically. Though only a single wall is considered, the proof can be extended with multiple walls and feedback coils.

A. Equilibrium and stability

We choose a cylindrical plasma equilibrium with circular cross section and step-like equilibrium current density $\mathbf{J}$ (Shafranov equilibrium).[17] In the cylindrical coordinate (r,θ,z), we assume $\mathbf{J}=J_z\hat{z}$, where

$$J_z(r)=\begin{cases}J_0, & r<r_0\\ 0, & r>r_0\end{cases}, \tag{21}$$

where $r_0\leqslant a$ is the width of the current channel, and a is the plasma minor radius. The poloidal equilibrium field is computed as

$$B_\theta(r)=\begin{cases}\frac{1}{2}\mu_0 J_0 r, & r<r_0\\ \frac{1}{2}\mu_0 J_0\frac{r_0^2}{r}, & r>r_0\end{cases}. \tag{22}$$

Assuming a constant toroidal field $B_z(r)=B_0$, we obtain the safety factor q

$$q=\begin{cases}q_0, & r<r_0\\ q_0\frac{r^2}{r_0^2}, & r>r_0\end{cases}, \tag{23}$$

where $q_0=2B_0/(R\mu_0 J_0)$; R is an equivalent major radius for a torus.

This equilibrium has been extensively used for analytical study of the RWM stability and control.[18–21]

The equation governing the ideal kink stability of a general, zero pressure, cylindrical equilibrium is well known.[22] We list some key steps in the derivation here, which are relevant to our further discussions on the coupling schemes.

Assuming a single mode perturbed magnetic flux function

$$\psi(r,\theta,z)=\psi(r)e^{im\theta-inz/R}, \tag{24}$$

in large aspect ratio approximation, we can write the perturbed field as

$$\mathbf{b}=\nabla\times(\psi\hat{z})=\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{r}-\frac{\partial\psi}{\partial r}\hat{\theta}. \tag{25}$$

The perturbed current density is

$$\mathbf{j}=\frac{1}{\mu_0}\nabla\times\mathbf{b}=-\frac{1}{\mu_0}\left(\hat{z}\nabla_\perp^2\psi+\frac{in}{R}\nabla_\perp\psi\right). \tag{26}$$

Consider the perturbed momentum equation

$$\rho\gamma\mathbf{v}=-\nabla p+\mathbf{J}\times\mathbf{b}+\mathbf{j}\times\mathbf{B}, \tag{27}$$

where ρ is the plasma density. Applying the $\nabla\times$ operator to both sides of Eq. (27), and taking the z component, we arrive at the torque balance equation

$$\nabla_\perp^2\psi-\frac{\mu_0 m}{B_\theta(m-nq)}\frac{dJ_z}{dr}\psi=i\gamma\frac{\mu_0 r}{B_\theta(m-nq)}\nabla\times(\rho\mathbf{v})\cdot\hat{z}. \tag{28}$$

For an incompressible plasma, the perturbed velocity $\mathbf{v}$ can be expressed as

$$\mathbf{v}=\nabla\times(\varphi\hat{z}), \tag{29}$$

where φ can be related to ψ via Faraday's law for an ideal plasma

$$\gamma\mathbf{b}=\nabla\times(\mathbf{v}\times\mathbf{B}). \tag{30}$$

We obtain

$$\varphi=-i\gamma\frac{r}{B_\theta(m-nq)}\psi. \tag{31}$$

Substituting Eqs. (31) and (29) into Eq. (28), we arrive at

$$\nabla_\perp^2\psi-\frac{\mu_0 m}{B_\theta(m-nq)}\frac{dJ_z}{dr}\psi=-\gamma^2\frac{\mu_0 r}{B_\theta(m-nq)}\left[\rho\nabla_\perp^2\frac{r\psi}{B_\theta(m-nq)}+\frac{d\rho}{dr}\frac{d}{dr}\left(\frac{r\psi}{B_\theta(m-nq)}\right)\right]. \tag{32}$$

For the Shafranov equilibrium, assuming also a step function for the plasma density profile, Eq. (32) becomes the same as the vacuum equation for ψ,

$$\nabla_\perp^2\psi=0 \tag{33}$$

in the plasma regions $0\leqslant r<r_0$ and $r_0<r\leqslant a$. Integrating Eq. (32) across $r=r_0$ gives the jump condition for ψ

$$\left.\frac{r[\psi']}{\psi}\right|_{r_0}+\frac{2m}{m-nq_0}=(\gamma\tau_A)^2\frac{q_0^2}{(m-nq_0)^2}\left.\frac{r\psi'}{\psi}\right|_{r_0-}, \tag{34}$$

where $\tau_A=\sqrt{\mu_0\rho_0}R/B_0$; ρ_0 is the density at the plasma center.

The jump condition (34), combined with the vacuum-like solution of Eq. (33) inside the plasma, gives the growth/damping rate of the ideal kink mode without the wall,[22]

$$(\gamma\tau_A)^2=\frac{2}{q_0^2}(m-nq_0)[\nu-(m-nq_0)], \tag{35}$$

where $\nu\equiv\mathrm{sgn}(m)$.

In the presence of a resistive wall, at $r=r_w>a$, we have another jump condition for ψ

$$\left.\frac{r[\psi']}{\psi}\right|_{r_w}=2\gamma\tau_w, \tag{36}$$

where τ_w is defined as the field penetration time for the $m=1$ mode, $\tau_w=\mu_0\sigma r_w d/2$, σ is the wall conductivity, and d the wall thickness. Equations (34) and (36), together with the solution of the Laplace equation (33) in both the plasma and the vacuum regions, determine the growth rate of the RWM,

$$\frac{\nu}{m-nq_0}-\frac{1}{1-\frac{\gamma\tau_w}{\gamma\tau_w+\mu}\left(\frac{r_0}{r_w}\right)^{2\mu}}=\frac{(\gamma\tau_A)^2}{2}\frac{q_0^2}{(m-nq_0)^2}, \tag{37}$$

where $\mu=|m|$.

Equation (37) is the dispersion relation of the RWM stability with the plasma inertia. At $r_w=\infty$ or $\tau_w=0$, we recover the no-wall ideal kink growth rate, Eq. (35). At $\tau_w=\infty$, we obtain the growth rate of the ideal kink with an ideal wall. At $\tau_A=0(\rho_0=0)$, we get the RWM growth rate without the plasma inertia

$$\gamma\tau_w=-\mu\frac{1-\nu(m-nq_0)}{1-\nu(m-nq_0)-(r_0/r_w)^{2\mu}}. \tag{38}$$

Note that since normally at the resistive wall time scale, $\gamma\tau_w$ is the order of unity, and $\tau_A\ll\tau_w$, $\gamma\tau_A\ll 1$, and the RHS of Eq. (37) (the correction due to the plasma inertia) can be neglected. This is exactly the approximation made in our forward coupling scheme, when we compute the static plasma response matrices.

B. Forward coupling

In the following, we try to follow the steps listed in Sec. II A, which lead to the recovering of the RWM growth rate satisfying Eq. (37). The difference is that instead of matrices, we will have scalars because of a single poloidal mode number. However, we still use matrix-like notations as in the previous section.

For a given total normal field at the coupling surface $r=r_v, a\leqslant r_v<r_w$, as the boundary condition, the field solution within the coupling surface (i.e., the plasma response) is

$$\psi=\begin{cases}\psi_{\rm pl}\left(\frac{r}{r_0}\right)^{\mu}+\psi_{\rm ex}\left(\frac{r}{r_v}\right)^{\mu}, & r\leqslant r_0\\ \psi_{\rm pl}\left(\frac{r}{r_0}\right)^{-\mu}+\psi_{\rm ex}\left(\frac{r}{r_v}\right)^{\mu}, & r_0<r\leqslant r_v\\ \psi|_{r_v}=\psi_0,\end{cases}, \tag{39}$$

where ψ_0 corresponds to the inhomogeneous boundary condition.

At the coupling surface,

$$\psi_0=\psi_{\rm pl}\left(\frac{r_0}{r_v}\right)^{\mu}+\psi_{\rm ex}, \tag{40}$$

where the first term on the RHS represents the plasma contribution and the second term represents the contribution from the external conductors (the wall in this case).

Substitution of the solution (39) into the jump condition (34) gives

$$\frac{\nu}{m-nq_0}-\frac{\psi_{\rm pl}}{\psi_{\rm pl}+\psi_{\rm ex}(r_0/r_v)^{\mu}}=\frac{(\gamma_0\tau_A)^2}{2}\frac{q_0^2}{(m-nq_0)^2}, \tag{41}$$

where γ_0 is an assumed mode growth rate. For a static response, we let $\gamma_0=0$.

Let $\psi_{\rm pl}(r_0/r_v)^{\mu}=\overleftrightarrow{\mathbf{G}}\psi_0$ and $\psi_{\rm ex}=(\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}})\psi_0$. Equation (41) leads to a condition for $\overleftrightarrow{\mathbf{G}}$,

$$\frac{\overleftrightarrow{\mathbf{G}}}{\overleftrightarrow{\mathbf{G}}+(\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}})(r_0/r_v)^{2\mu}}=\frac{\nu}{m-nq_0}-\frac{(\gamma_0\tau_A)^2}{2}\frac{q_0^2}{(m-nq_0)^2}\equiv C, \tag{42}$$

from which $\overleftrightarrow{\mathbf{G}}$ is calculated.

Let us consider the problem of invertibility of $(\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}})$, which is essential in computing the final coupling matrix $\overleftrightarrow{\mathbf{L}}^{\rm pl}$ [Eq (9)]. From Eq. (42), we calculate

$$\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}}=\frac{1-C}{1-C[1-(r_0/r_v)^{2\mu}]}. \tag{43}$$

Equations (42) and (37) show that, at the exact solution, i.e., when $\gamma_0=\gamma$ (including the plasma inertia effect) or $\tau_A=0$ (neglecting the plasma mass),

$$C=\frac{1}{1-\frac{\gamma\tau_w}{\gamma\tau_w+\mu}\left(\frac{r_0}{r_w}\right)^{2\mu}}. \tag{44}$$

This means that $C=1$ is equivalent to the condition of $\gamma\tau_w=0$. Consequently, as soon as we are considering massless plasma, or our initial guess for the growth rate is close enough to the true value with inclusion of the inertial effect, $(\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}})$ is not invertible *if and only if* the RWM is at the exact stability margin. In this case, the plasma perturbation does not cause any eddy currents in the external conductors. Since this occurs only at a peculiar equilibrium condition [see Eq. (38)], we conclude that $(\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}})$ is generally invertible. The invertibility does not depend on the radial position of the coupling surface. We also notice, from Eqs. (43) and (44), that $(\overleftrightarrow{\mathbf{E}}-\overleftrightarrow{\mathbf{G}})$ does not approach infinity (i.e., $\overleftrightarrow{\mathbf{L}}^{\rm pl}$ does not vanish) if $\gamma\tau_w>0$ and $r_v<r_w$.

With the same boundary condition, the vacuum solution in step 2 of the forward coupling procedure is simple,

$$\psi^{\rm vac}=\psi_0\left(\frac{r}{r_v}\right)^{\mu}, \quad 0\leqslant r\leqslant r_v. \tag{45}$$

Thus, the equivalent surface current at the coupling surface $r=r_v$ is computed as

$$\mathbf{I}_{\rm eqv}=\mu_0^{-1}(b_\theta-b_\theta^{\rm vac})|_{r_v}=\frac{2\mu}{\mu_0 r_v}\overleftrightarrow{\mathbf{G}}\psi_0. \tag{46}$$

This gives us an explicit expression for the $\overleftrightarrow{\mathbf{F}}$ matrix as defined in Eq. (3),

$$\overleftrightarrow{\mathbf{F}}=-\frac{2i}{\nu\mu_0}\overleftrightarrow{\mathbf{G}}. \tag{47}$$

It is easy to check that the field produced by the current (46), outside the coupling surface $r>r_v$, is exactly the same as that produced by the perturbed plasma current,

$$\psi^{\rm plasma}=\psi_{\rm pl}\left(\frac{r}{r_0}\right)^{-\mu}, \quad r>r_v. \tag{48}$$

Now we consider the circuit equation for the eddy current in the wall, Eq. (6). The resistance and the self-inductance (per unit length) of the wall are

$$\overleftrightarrow{\mathbf{R}}=\frac{1}{\sigma d}, \quad \overleftrightarrow{\mathbf{L}}=\frac{\mu_0 r_w}{2\mu}. \tag{49}$$

Without the plasma correction, the wall eddy current decay rate is

$$\gamma=-\overleftrightarrow{\mathbf{L}}^{-1}\overleftrightarrow{\mathbf{R}}=-\frac{\mu}{\tau_w}, \tag{50}$$

where τ_w is again defined as $\tau_w=\mu_0\sigma r_w d/2$.

The matrix $\overleftrightarrow{\mathbf{Q}}$ from Eq. (8) is calculated as

$$\overleftrightarrow{\mathbf{Q}}=\frac{\mathbf{b}_N^{\rm ex}}{\mathbf{I}}=\frac{im\psi_{\rm ex}/r_v}{(r_w/r_v)^{\mu}\psi_{ex}\overleftrightarrow{\mathbf{L}}^{-1}}=\frac{i\mu_0}{2\nu}\left(\frac{r_v}{r_w}\right)^{\mu-1}. \tag{51}$$

Finally, the matrix $\overleftrightarrow{\mathbf{M}}$ is calculated by using Eq. (46) and noting that

$$\mathbf{U} = \psi_{\text{pl}}\left(\frac{r_0}{r_w}\right)^{\mu} = \mathbf{G}\psi_0\left(\frac{r_v}{r_w}\right)^{\mu}, \tag{52}$$

which gives us

$$\overleftrightarrow{\mathbf{M}} = \frac{\mu_0 r_v}{2\mu}\left(\frac{r_v}{r_w}\right)^{\mu}. \tag{53}$$

Combining Eqs. (53), (47), (43), and (51), we obtain an explicit form of the plasma correction to the inductance matrix $\overleftrightarrow{\mathbf{L}}^{\text{pl}}$,

$$\overleftrightarrow{\mathbf{L}}^{\text{pl}} = \overleftrightarrow{\mathbf{M}}\overleftrightarrow{\mathbf{F}}(\overleftrightarrow{\mathbf{E}} - \overleftrightarrow{\mathbf{G}})^{-1}\overleftrightarrow{\mathbf{Q}} = \frac{\mu_0 r_w}{2\mu}\left(\frac{r_0}{r_w}\right)^{2\mu}\frac{C}{1-C}. \tag{54}$$

We give two comments on the property of $\overleftrightarrow{\mathbf{L}}^{\text{pl}}$. (i) $\overleftrightarrow{\mathbf{L}}^{\text{pl}}$ is independent of the choice of radial position r_v for the coupling surface, as it should be. (ii) When the plasma inertia is included, Eq. (54) shows that $\overleftrightarrow{\mathbf{L}}^{\text{pl}}$ is a rational function of $(\gamma_0\tau_A)^2$. The specific structure of this rational function can be used for constructing the frequency-dependent response matrices in the toroidal computations.

Substituting Eqs. (54) and (49) into Eq. (6), we derive

$$C = \frac{1}{1 - \frac{\gamma\tau_w}{\gamma\tau_w + \mu}\left(\frac{r_0}{r_w}\right)^{2\mu}}. \tag{55}$$

This, combined with the definition of C in Eq. (42), gives the dispersion relation for the RWM growth rate. This dispersion relation is identical to that for the true growth rate of the RWM (37) if $\gamma_0 = \gamma$. Without the plasma inertia, Eq. (55) already determines the true growth rate of the RWM. With the plasma inertia, an iterative process over γ_0 is required in order to find the true growth rate.

C. Backward coupling

In the cylindrical limit, the solution to Eq. (10), with coefficients determined by Eqs. (49) and (53), is

$$\mathbf{I} = -\left(\frac{r_v}{r_w}\right)^{\mu+1}\frac{\gamma\tau_w}{\gamma\tau_w + \mu}\mathbf{I}_{\text{eqv}} = \overleftrightarrow{\mathbf{W}}\mathbf{I}_{\text{eqv}}. \tag{56}$$

The total field outside the coupling surface is

$$\psi = \psi_w\left(\frac{r}{r_w}\right)^{\mu} + \psi_{\text{eqv}}\left(\frac{r}{r_v}\right)^{-\mu}, \quad r_v \leq r \leq r_w, \tag{57}$$

where

$$\psi_w = \frac{\mu_0 r_w}{2\mu}\mathbf{I}, \quad \psi_{\text{eqv}} = \frac{\mu_0 r_v}{2\mu}\mathbf{I}_{\text{eqv}}. \tag{58}$$

This gives

$$\mathbf{b}_N|_{r_v+} = \frac{im}{r_v}\,\psi|_{r_v+} = \frac{i\mu_0}{2\nu}\left[1 - \left(\frac{r_v}{r_w}\right)^{2\mu}\frac{\gamma\tau_w}{\gamma\tau_w + \mu}\right]\mathbf{I}_{\text{eqv}} = \overleftrightarrow{\mathbf{S}}\mathbf{I}_{\text{eqv}}, \tag{59}$$

$$\mathbf{b}_T|_{r_v+} = -\left.\frac{\partial\psi}{\partial r}\right|_{r_v+} = \frac{\mu_0}{2}\left[1 + \left(\frac{r_v}{r_w}\right)^{2\mu}\frac{\gamma\tau_w}{\gamma\tau_w + \mu}\right]\mathbf{I}_{\text{eqv}} = \overleftrightarrow{\mathbf{P}}\mathbf{I}_{\text{eqv}}. \tag{60}$$

Hence

$$\left.\frac{\mathbf{b}_T}{\mathbf{b}_N}\right|_{r_v+} = \overleftrightarrow{\mathbf{P}}\overleftrightarrow{\mathbf{S}}^{-1} = \frac{\nu}{i}\frac{\gamma\tau_w + \mu + \gamma\tau_w(r_v/r_w)^{2\mu}}{\gamma\tau_w + \mu - \gamma\tau_w(r_v/r_w)^{2\mu}}, \tag{61}$$

$$\left.\frac{r_v\psi'}{\psi}\right|_{r_v+} = -\mu\frac{\gamma\tau_w + \mu + \gamma\tau_w(r_v/r_w)^{2\mu}}{\gamma\tau_w + \mu - \gamma\tau_w(r_v/r_w)^{2\mu}}. \tag{62}$$

Note that $\overleftrightarrow{\mathbf{S}}$ is invertible as long as $r_v < r_w$ and $\gamma\tau_w \geq 0$. It is trivial to check that Eq. (15) yields the same expression (61).

Now we solve the MHD eigenvalue problem in the plasma region, with the boundary condition

$$\left.\frac{r_v\psi'}{\psi}\right|_{r_v-} = \left.\frac{r_v\psi'}{\psi}\right|_{r_v+}, \tag{63}$$

as defined in Eq. (62).

The solution has the form

$$\psi = \begin{cases} \psi_{\text{pl}}\left(\frac{r}{r_0}\right)^{\mu} + \psi_{\text{ex}}\left(\frac{r}{r_v}\right)^{\mu}, & r \leq r_0 \\ \psi_{\text{pl}}\left(\frac{r}{r_0}\right)^{-\mu} + \psi_{\text{ex}}\left(\frac{r}{r_v}\right)^{\mu}, & r_0 < r \leq r_v \end{cases}, \tag{64}$$

which should satisfy the jump condition (34) at $r = r_0$.

The above boundary and the jump conditions are sufficient to derive the equation for the RWM growth rate,

$$\frac{\nu}{m - nq_0} - \frac{1}{1 - \frac{\gamma\tau_w}{\gamma\tau_w + \mu}\left(\frac{r_0}{r_w}\right)^{2\mu}} = \frac{(\gamma\tau_A)^2}{2}\frac{q_0^2}{(m - nq_0)^2}, \tag{65}$$

which is identical to the RWM dispersion relation (37).

In contrast to the forward coupling scheme, where we solve eventually a linear eigenvalue problem if the plasma inertia is neglected, in the backward coupling we generally have to solve a nonlinear eigenvalue problem (in the plasma region) even without the plasma inertia. This is because the response of the conducting structures is essentially frequency-dependent at the frequency range for the RWM. On the other hand, Eqs. (59) and (60) show that the frequency dependence of the coupling matrices on the growth rate (or frequency) can be approximated again by rational functions. It is also possible to achieve a linear eigenvalue problem by computing a linear boundary condition of type (15), which requires a somewhat heavy manipulation of the system matrices in the eddy current solver.

IV. SUMMARY AND DISCUSSIONS

We presented two possible schemes that can be used to couple a MHD code and an eddy current code. We demonstrated, in cylindrical geometry, rigorously how and why these schemes work. The forward coupling scheme has been implemented in the CarMa code, and numerical simulations show that this scheme works well for studying the RWM stability and control, in the presence of 3D conducting structures.[14]

There are both advantages and drawbacks for these two schemes. The forward coupling scheme allows solution of a linear eigenvalue problem if we assume a static plasma response. However, a dynamic response may be necessary if the plasma is close to the ideal pressure limit, so that the inertial effect cannot be neglected, or if the toroidal plasma rotation is included. Nevertheless, in the framework of ideal MHD, this scheme allows us to compute the plasma response matrices for different toroidal mode numbers separately, and then investigate easily the geometrical coupling of these modes due to 3D conductors. The feedback implementation is also straightforward for this scheme.

The backward coupling scheme always requires computing dynamic response matrices from the conductors, and hence makes the eigenvalue problem generally nonlinear. Rational function approximation should be a useful tool to simplify the problem. The nonlinearity can be eliminated by computing a linear boundary condition with respect to the growth rate, at the expense of heavy manipulation of the system matrices in the eddy current code. The obvious advantage of the backward coupling scheme is that the plasma inertial effects, the rotation effects, as well as possible advanced kinetic effects can all easily be included in the study, since the final system of equations are solved in the plasma region. A drawback of this scheme is that we cannot study the geometrical coupling effects of different n's, thus the response of 3D structures always needs to be decomposed in the Fourier harmonics along the toroidal angle of the torus. However, this scheme is more suitable to couple an eddy current code to a nonlinear MHD code, since the conductor response is still linear, and therefore does not need to be recomputed during the nonlinear time stepping of the MHD equations.

It is also possible to combine the forward and the backward coupling schemes to obtain a new scheme that further modularizes the whole computation. For instance, one can compute the coupling matrix $\overleftrightarrow{\mathbf{F}}(\gamma)$ (for the toroidal component of the surface current) from Eq. (3), using the MHD code, and compute the coupling matrix $\overleftrightarrow{\mathbf{S}}(\gamma)$ (defined for the toroidal component of the surface current) from Eq. (12). Then solve the following nonlinear eigenvalue problem:

$$\overleftrightarrow{\mathbf{F}}(\gamma)\overleftrightarrow{\mathbf{S}}(\gamma)=\overleftrightarrow{\mathbf{E}} \tag{66}$$

to obtain the growth rate of the RWM. It is straightforward to check analytically, using Eqs. (47), (43), and (59), that this procedure yields the correct eigenvalue.

This combined scheme avoids solving a nonlinear eigenvalue problem in either the MHD or the eddy current codes. Instead, the two codes only compute coupling matrices. However, for an efficient solution of Eq. (66), it is desirable to produce the analytical dependence of $\overleftrightarrow{\mathbf{F}}$ and $\overleftrightarrow{\mathbf{S}}$ on γ in a matrix-wise (i.e., not element-wise) manner. Moreover, numerical tests are needed to check whether the computed eigenvalue from Eq. (66) is sensitive to the rational function approximation of $\overleftrightarrow{\mathbf{F}}(\gamma)$ and $\overleftrightarrow{\mathbf{S}}(\gamma)$.

ACKNOWLEDGMENTS

This work was partly funded by the United Kingdom Engineering and Physical Sciences Research Council and by the European Communities under the Contract of Association between EURATOM and UKAEA. The views and opinions expressed herein do not necessarily reflect those of the European Commission. This work was carried out within the framework of the European Fusion Development Agreement. The support from Italian MIUR under PRIN grant is also acknowledged.

Y.Q.L. is grateful to Dr. V. D. Pustovitov for very useful suggestions on improving the manuscript.

[1] R. Aymar, P. Barabaschi, and Y. Shimomura, Plasma Phys. Controlled Fusion **44**, 519 (2002).
[2] A. Bondeson and D. J. Ward, Phys. Rev. Lett. **72**, 2709 (1994).
[3] Y. Q. Liu, A. Bondeson, M. S. Chu, J.-Y. Favez, Y. Gribov, M. Gryaznevich, T. C. Hender, D. F. Howell, R. J. La Haye, and J. B. Lister, Nucl. Fusion **45**, 1131 (2005).
[4] A. M. Garofalo, M. S. Chu, E. D. Fredrickson, M. Gryaznevich, T. H. Jensen, L. C. Johnson, R. J. La Haye, G. A. Navratil, M. Okabayashi, J. T. Scoville, E. J. Strait, A. D. Turnbull, and DIII-D Team, Nucl. Fusion **41**, 1171 (2001).
[5] Y. Q. Liu, A. Bondeson, Y. Gribov, and A. Polevoi, Nucl. Fusion **44**, 232 (2004).
[6] Y. Q. Liu, M. S. Chu, A. M. Garofalo, Y. Gribov, M. Gryaznevich, T. C. Hender, D. F. Howell, R. J. La Haye, M. Okabayashi, S. D. Pinches, H. Reimerdes, P. de Vries, and EFDA-JET contributors, Phys. Plasmas **13**, 056120 (2006).
[7] M. Okabayashi, N. Pomphrey, and R. E. Hatcher, Nucl. Fusion **38**, 1607 (1998).
[8] E. J. Strait, J. M. Bialek, I. N. Bogatu, M. S. Chance, M. S. Chu, D. H. Edgell, A. M. Garofalo, G. L. Jackson, R. J. Jayakumar, T. H. Jensen, O. Katsuro-Hopkins, J. S. Kim, R. J. La Haye, L. L. Lao, M. A. Makowski, G. A. Navratil, M. Okabayashi, H. Reimerdes, J. T. Scoville, A. D. Turnbull, and DIII-D Team, Phys. Plasmas **11**, 2505 (2004).
[9] Y. Q. Liu, A. Bondeson, C. M. Fransson, B. Lennartson, and C. Breitholtz, Phys. Plasmas **7**, 3681 (2000).
[10] L. Degtyarev, A. Martynov, S. Medvedev, F. Troyon, L. Villard, and R. Gruber, Comput. Phys. Commun. **103**, 10 (1997).
[11] R. Albanese and G. Rubinacci, Adv. Imaging Electron Phys. **102**, 1 (1998).
[12] J. Bialek, A. H. Boozer, M. E. Mauel, and G. A. Navratil, Phys. Plasmas **8**, 2170 (2001).
[13] Y. Q. Liu, A. Bondeson, R. Bergstrom, C. Johnson, M. Larson, and K. Samuelsson, IEEE Trans. Magn. **38**, 449 (2002).
[14] R. Albanese, Y. Q. Liu, A. Portone, G. Rubinacci, and F. Villone, IEEE Trans. Magn. **44**, 1654 (2008).
[15] A. Portone, F. Villone, Y. Q. Liu, R. Albanese, and G. Rubinacci, Plasma Phys. Controlled Fusion **50**, 085004 (2008).
[16] F. Villone, G. Rubinacci, Y. Q. Liu, and Y. Gribov, in *Proceedings of the 32nd EPS Conference on Plasma Physics and Controlled Fusion, Tarragona, 2005*, edited by C. Hidalgo and B. Ph. van Milligen (European Physical Society, Madrid, 2005), ECA Vol. 29C, P. P-5.049.
[17] V. D. Shafranov, Sov. Phys. Tech. Phys. **15**, 175 (1970).
[18] J. M. Finn, Phys. Plasmas **2**, 198 (1995).
[19] A. B. Mikhailovskii, *Instabilities in a Confined Plasma* (Institute of Physics, Bristol, 1998), p. 391.
[20] A. B. Mikhailovskii and V. D. Pustovitov, Plasma Phys. Rep. **26**, 477 (2000).
[21] A. Bondeson, Y. Q. Liu, D. Gregoratto, Y. Gribov, and V. D. Pustovitov, Nucl. Fusion **42**, 768 (2002).
[22] J. A. Wesson, Nucl. Fusion **18**, 87 (1978).