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Research Article

Convergence of a Hybrid Iterative Scheme for Fixed Points of Nonexpansive Maps, Solutions of Equilibrium, and Variational Inequalities Problems

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Let K be a closed, convex, and nonempty subset of a real q-uniformly smooth Banach space E, which is also uniformly convex. For some $\kappa > 0$, let $T_i : K \to E$ $i \in \mathbb{N}$ and $A : K \to E$ be family of nonexpansive maps and κ-inverse strongly accretive map, respectively. Let $G : K \times K \to \mathbb{R}$ be a bifunction satisfying some conditions. Let P_K be a nonexpansive projection of E onto K. For some fixed real numbers $\delta \in (0, 1)$, $\lambda \in (0, (q\kappa/d_q)^{1/(q-1)})$, and arbitrary but fixed vectors $x_1, u \in E$, let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $G(y_n, \eta) + (1/r)\langle \eta - y_n, j_q(y_n - x_n)\rangle \geq 0, \forall \eta \in K$, $x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i\geq 1} \sigma_{in} T_i P_K(y_n - \lambda A y_n)$, $n \geq 1$, where $r \in (0, 1)$ is fixed, and $\{\alpha_n\}, \{\sigma_{i,n}\} \subset (0, 1)$ are sequences satisfying appropriate conditions. If $F := [\cap_{i=1}^{\infty} F(T_i)] \cap \mathrm{VI}(K, A) \cap \mathrm{EP}(G) \neq \emptyset$, under some mild conditions, we prove that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to some element in F.

1. Introduction

Let E be a real normed space and E^* its dual space. For some real number q $(1 < q < \infty)$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \right\}, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between elements of E and elements of E^*.

For $q = 2$, J_2 usually denoted by J is called the normalised duality mapping.

Let E be a real Banach space; a map $A : D(A) \to E$ is said to be *accretive* if for all $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\left\langle Ax - Ay, j_q(x - y) \right\rangle \geq 0. \quad (2)$$

For some real number $\kappa > 0$, A is called *κ-inverse strongly accretive* if for all $x, y \in D(A)$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\left\langle Ax - Ay, j_q(x - y) \right\rangle \geq \kappa \|Ax - Ay\|^q. \quad (3)$$

Observe that a κ-inverse strongly accretive map is $1/\kappa$-Lipschitzian.

Let K be a nonempty, closed, and convex subset of E, and let $A : K \to E$ be an accretive mapping. A variational inequality problem is, searching for $x^* \in K$ such that for some $j_q(v - x^*) \in J_q(v - x^*)$

$$\left\langle Ax^*, j_q(v - x^*) \right\rangle \geq 0, \quad \forall v \in K. \quad (4)$$

Let $G : K \times K \to \mathbb{R}$ be a bifunction on a closed convex nonempty subset K of a real Banach space E; an equilibrium problem is searching for $x^* \in K$ such that

$$G(x^*, v) \geq 0, \quad \forall v \in K. \quad (5)$$

A set of solutions of the problems (4) and (5) are denoted by $\mathrm{VI}(K, A)$ and $\mathrm{EP}(G)$, respectively.

Let P be a mapping of E onto K. Then, P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geq 0$. A mapping P of E into E is said to be a retraction if $P^2 = P$. If a mapping P is a *retraction,* then $Pz = z$ for every $z \in R(P)$, *range* of P. A subset K is said to be *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto

K. A retraction P is said to be *orthogonal* if for each x, $x-P(x)$ is normal to K in the sense of James [1].

It is well known (see [2]) that if E is a Banach space; a projection mapping is a sunny nonexpansive retraction P of E onto K. If E is uniformly smooth and there exists a nonexpansive retraction of E onto K, then there exists a nonexpansive projection of E onto K. If E is a real smooth Banach space, then P is an orthogonal retraction of E onto K if and only if $P(x) \in K$ and $\langle P(x) - x, j_q(P(x) - y)\rangle \leq 0$ for all $y \in K$. It then follows that, for $x, y \in K$, we have $\langle P(x) - x, j_q(P(x) - P(y))\rangle \leq 0$ and $\langle P(y) - y, j_q(P(y) - P(x))\rangle \leq 0$ which implies

$$\|P(x) - P(y)\|^q \leq \langle x - y, j_q(P(x) - P(y))\rangle. \tag{6}$$

An accretive mapping A is said to be maximal if its graph GF(A) is not contained in the graph of any other accretive map. Equivalently, A is maximal accretive if for every $(v_0, w_0) \in E \times E$ such that $\langle w - w_0, j_p(v - v_0)\rangle \geq 0$ holds for all $w \in Av, v \in K$; then, $w_0 \in Av_0$. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x^* \in D(T)$ and $Tx_n \rightarrow p$; then, $Tx^* = p$. The following proposition is known to hold; see, for example, [3].

Proposition 1. *Let $A : K \rightarrow E$ be a κ-inverse strongly accretive map. Let M be defined by*

$$Mv = \begin{cases} Av + N_K v, & v \in K, \\ \emptyset, & v \notin K, \end{cases} \tag{7}$$

where $N_K v = \{w \in E : \langle w, j_q(v - u)\rangle \geq 0$, for all $u \in K\}$; then, M is maximal accretive and $u \in M^{-1}(0)$ if and only if $u \in \mathrm{VI}(K, A)$.

Recently, Maingé [4] studied the Halpern-type scheme for approximation of a common fixed point of *countable infinite* family of nonexpansive mappings in a real Hilbert space.

The present author [3] proved a strong convergence theorem for family of nonexpansive maps and solution of variational inequality problems. Kumam and Jaiboon [5] studied a hybrid iterative method for mixed equilibrium problem and variational inequality problem in the framework of a real Hilbert space.

Various numerous authors studied the problem of approximating solutions of equilibrium and fixed point problems in the framework of a real Hilbert space; see, for example, [5–18] and the references contained therein. In [19], Ceng et al. studied this problem in the framework of a uniformly smooth and uniformly convex Banach space.

Takahashi and Zembayashi [20] (see also [21–23]) studied the problem of approximating solutions of equilibrium problems and fixed points of some nonlinear maps in the framework of real Banach spaces. It is our purpose in this paper to introduce a new hybrid iterative method for approximating a common element in the intersection of the set of fixed points of countable infinite family of nonexpansive mappings, the set of solutions of variational inequality problem, and the set of solutions of equilibrium problem in Banach spaces. Our theorems extend and improve some recent important results, and our method of proof in this paper is of independent interest.

2. Preliminaries

Let $S := \{x \in E : \|x\| = 1\}$ denote a unit sphere of the real Banach space E. E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{8}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$, the limit is attained uniformly for $x \in S$. Let E be a normed space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau\right\}. \tag{9}$$

The space E is called *uniformly smooth* if and only if $\lim_{t \to 0^+}(\rho_E(t)/t) = 0$. For some constant $q > 1$, E is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$, $t > 0$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1, \epsilon = \|x - y\|\right\}. \tag{10}$$

E is called *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A Banach space E is said to be strictly convex if $\|x-y\|/2 < 1$ for $x, y \in E$ with $\|x\| = 1 = \|y\|$ and $x \neq y$.

It is well known that if E is smooth then the duality mapping is singled valued, and if E has uniformly Gâteaux differentiable norm then the duality mapping is norm-to-*weak** uniformly continuous on bounded subset of E. Also, every q-uniformly smooth space is uniformly smooth and has a uniformly *Gâteaux differentiable* norm, and every uniformly convex space is strictly convex.

In the sequel, we will make use of the following results.

Lemma 2 (see Petryshyn [24]). *Let E be a real normed linear space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \quad \forall x, y \in E, j(x + y) \in J(x + y). \tag{11}$$

Theorem 3 (see Goebel and Kirk [25]). *Let E be a real uniformly convex Banach space, K a closed convex subset of E, and $T : K \rightarrow E$ a nonexpansive mapping. Then, $(I - T)$ is demiclosed at zero, where I denotes the identity map.*

Lemma 4 (see Suzuki [26]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence* *in* $[0,1]$ *with* $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. *Suppose that* $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ *for all integers* $n \geq 0$ *and* $\limsup(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. *Then,* $\lim \|y_n - x_n\| = 0$.

Lemma 5 (see Xu [27]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n) a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0, \tag{12}$$

where (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$*; (ii)* $\limsup \sigma_n \leq 0$*; (iii)* $\gamma_n \geq 0$*;* $(n \geq 0)$, $\sum \gamma_n < \infty$. *Then,* $a_n \to 0$ *as* $n \to \infty$.

Lemma 6 (see Xu [28]). *Let* E *be a real* q*-uniformly smooth Banach space for some* $q > 1$*; then, there exists some positive constant* d_q *such that*

$$\|x+y\|^q \leq \|x\|^q + q\left\langle y, j_q(x)\right\rangle + d_q\|y\|^q \tag{13}$$

for all $x, y \in E$ *and* $j_q(x) \in J_q(x)$.

Lemma 7 (see Kamimura and Takahashi [29]). *Let* E *be a real smooth and uniformly convex Banach space, and let* $R > 0$. *Then, there exists a strictly increasing, continuous, and convex function* $g : [0,2R] \to \mathbb{R}$ *such that* $g(0) = 0$ *and* $g(\|x-y\|) \leq \|x\|^2 - 2\langle x, jy\rangle + \|y\|^2$ *for all* $x, y \in B_R$.

The following conditions are required on the bifunction $G : K \times K \to \mathbb{R}$ for solving equilibrium problems with respect to G:

(A1) $G(x,x) = 0$ for all $x \in K$;

(A2) G is monotone; that is, $G(x,y) + G(y,x) \leq 0$ for all $x, y \in K$;

(A3) for all $x, y, z \in K$, $\limsup_{t \to 0^+} G(tz + (1-t)x, y) \leq G(x,y)$;

(A4) for all $x \in K$, $G(x,\cdot)$ is convex and lower semicontinuous.

Lemma 8 (see Blum and Oettli [30]). *Let* E *be a real smooth, strictly convex, and reflexive Banach space. Let* $G : K \times K \to \mathbb{R}$ *be a bifunction satisfying (A1)–(A4), and let* $x \in E$, $r > 0$. *Then, there exists* $z \in K$ *such that*

$$G(z,y) + \frac{1}{r}\langle y-z, j(z-x)\rangle \geq 0, \quad \forall y \in K. \tag{14}$$

Lemma 9. *Let* K *be a closed convex nonempty subset of a real uniformly smooth and strictly convex Banach space* E. *Let* $G : K \times K \to \mathbb{R}$ *be a bifunction satisfying (A1)–(A4). For* $r > 0$ *and* $x \in E$, *define a map* $T_r : E \to K$ *by*

$$T_r x = \left\{ z \in K : G(z,y) + \frac{1}{r}\langle y-z, j(z-x)\rangle \geq 0, \forall y \in K \right\}. \tag{15}$$

Then, the following hold:

(i) T_r *is single-valued;*

(ii) $Fix(T_r) = EP(G)$;

(iii) *if* T_r *is firmly nonexpansive-type, that is, for* $x, y \in E$,

$$\left\langle T_r x - T_r y, j(T_r x - T_r y)\right\rangle \leq \left\langle x-y, j(T_r x - T_r y)\right\rangle, \tag{16}$$

then $EP(G)$ *is closed and convex.*

Proof. (i) Let $z_1, z_2 \in T_r$, then

$$\begin{aligned} G(z_1,z_2) + \frac{1}{r}\langle z_2 - z_1, j(z_1 - z_2)\rangle &\geq 0, \\ G(z_2,z_1) + \frac{1}{r}\langle z_1 - z_2, j(z_2 - z_1)\rangle &\geq 0. \end{aligned} \tag{17}$$

Adding these inequalities and using (A2), we get

$$\langle z_2 - z_1, j(z_1 - z_2)\rangle \geq 0, \tag{18}$$

which implies $z_1 = z_2$. Consider

(ii)

$$\begin{aligned} z \in F(T_r) &\Longleftrightarrow z = T_r z \\ &\Longleftrightarrow G(z,y) + \frac{1}{r}\langle y-z, j(z-z)\rangle \geq 0, \quad \forall y \in K, \\ &\Longleftrightarrow G(z,y) \geq 0, \quad \forall y \in K, \\ &\Longleftrightarrow z \in EP(G). \end{aligned} \tag{19}$$

(iii) EP(G) is closed and convex follows from (ii) and the fact that every firmly nonexpansive map is nonexpansive and the fixed point set of nonexpansive map is closed and convex. □

Let E be a real q-uniformly smooth Banach space, and for some $\lambda > 0$, let $I : K \to K$ and $A : K \to E$ be the identity and κ-inverse strongly accretive mappings, respectively. Then, for the map $(I - \lambda A) : K \to E$, we have the following estimates:

$$\begin{aligned} &\|(I-\lambda A)x - (I-\lambda A)y\|^q \\ &\quad = \|x - y - \lambda(Ax - Ay)\|^q \\ &\quad \leq \|x-y\|^q - q\lambda\langle Ax - Ay, j(x-y)\rangle + d_q\lambda^q\|Ax-Ay\|^q \\ &\quad \leq \|x-y\|^q - \lambda\left(q\kappa - d_q\lambda^{q-1}\right)\|Ax-Ay\|^q. \end{aligned} \tag{20}$$

If λ is chosen such that $0 \leq \lambda \leq (q\kappa/d_q)^{1/(q-1)}$, we then have

$$\|(I-\lambda A)x - (I-\lambda A)y\| \leq \|x-y\|, \tag{21}$$

and so $(I - \lambda A)$ become a nonexpansive mapping of K into E.

For $L_p(1 < p < \infty)$ spaces, we have the following relation: if $\lambda \in (0, 2\kappa/(p-1))$,

$$\begin{aligned} &\|(I-\lambda A)x-(I-\lambda A)y\|^2 \\ &\quad = \|x-y-\lambda(Ax-Ay)\|^2 \\ &\quad \leq \|x-y\|^2 - 2\lambda\langle Ax-Ay, j(x-y)\rangle \\ &\qquad + (p-1)\lambda^2\|Ax-Ay\|^2 \\ &\quad \leq \|x-y\|^2 - \lambda(2\kappa-(p-1)\lambda)\|Ax-Ay\|^2 \\ &\quad \leq \|x-y\|^2. \end{aligned} \tag{22}$$

Also if E is a Hilbert space and we choose $\lambda \in (0, 2\kappa)$, then $(I - \lambda A)$ is nonexpansive.

3. Path Convergence Theorems

In the sequel, we assume for each $t \in (0, 1)$ that the sequence $\{\sigma_{i,t}\}_i$ satisfies $\sum_{i\geq 1}\sigma_{i,t} = 1-t$ and the sequences $\{\alpha_n\}, \{\sigma_{i,n}\}_i \subset (0, 1)$, satisfy $\sum_{i\geq 1}\sigma_{i,n} = 1 - \alpha_n$.

For a countable family of nonexpansive mappings $\{T_i\}$ of E, we denote a set $\mathcal{N}_{\mathcal{J}} := \{i \in \mathbb{N} : T_i \neq I\}$ (I being the identity mapping on E).

Let K be a nonempty closed and convex subset of a real q-uniformly smooth Banach space E and P_K a nonexpansive projection of E onto K. For some real number $\kappa > 0$, let $A : K \to E$ be a κ-inverse strongly accretive mapping. For some real numbers $\delta \in (0, 1)$, $\lambda \in (0, (q\kappa/d_q)^{1/(q-1)})$, and $r > 0$ arbitrarily chosen but fixed and for each $t \in (0, 1)$, define a map $T_t : E \to E$ by $u \in E$, arbitrary and fixed

$$\begin{aligned} &G(y,\eta) + \frac{1}{r}\langle \eta - y, j(y-x)\rangle \geq 0, \quad \forall \eta \in K, \\ &T_t x = tu + (1-\delta)(1-t)x + \delta\sum_{i\geq 1}\sigma_{i,t}T_iP_K(y-\lambda Ay), \\ &\qquad \forall x \in E. \end{aligned} \tag{23}$$

Then, T_t is a strict contraction on E.

For $x^1, x^2 \in E$, $y^1 = T_r x^1$, $y^2 = T_r x^2$, we have

$$\begin{aligned} \|T_t x^1 - T_t x^2\| \leq\ & (1-t)(1-\delta)\|x^1-x^2\| \\ & + \delta\sum_{i\geq 1}\sigma_{i,t}\|T_iP_K(y^1-\lambda Ay^1) \\ & \qquad -T_iP_K(y^2-\lambda Ay^2)\| \\ =\ & (1-t)(1-\delta)\|x^1-x^2\| \\ & + \delta\sum_{i\geq 1}\sigma_{i,t}\|T_iP_K(I-\lambda A)T_r x^1 \\ & \qquad -T_iP_K(I-\lambda A)T_r x^2\| \\ \leq\ & (1-t)\|x^1-x^2\|. \end{aligned} \tag{24}$$

Thus, for each $t \in (0, 1)$, there exists a unique $z_t \in E$ such that

$$\begin{aligned} &G(y,\eta) + \frac{1}{r}\langle \eta - y, j(y-z_t)\rangle \geq 0, \quad \forall \eta \in K, \\ &z_t = tu + (1-\delta)(1-t)z_t + \delta\sum_{i\geq 1}\sigma_{i,t}T_iP_K(y-\lambda Ay). \end{aligned} \tag{25}$$

Lemma 10. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a closed, convex, and nonempty subset of E. For $t \in (0, 1)$, let $\{z_t\}$ be a net satisfying (25), and assume that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_iP_K(I-\lambda A)T_r) \neq \emptyset$. Then, $\{z_t\}$ is bounded and admits at most one accumulation point in $\mathcal{F}$ as $t \to 0$.*

Proof. Let $x^* \in \mathcal{F}$. Then, using (25), we have

$$\begin{aligned} &\|z_t - x^*\|^2 \\ &\quad = \langle t(u-x^*) + (1-t)(1-\delta)(z_t-x^*) \\ &\qquad + \delta\sum_{i\geq 1}\sigma_{i,t}(T_iP_K(T_r z_t - \lambda AT_r z_t) - x^*), \\ &\qquad j(z_t-x^*)\rangle \leq t\langle u-x^*, j(z_t-x^*)\rangle \\ &\qquad + (1-t)(1-\delta)\|z_t-x^*\|^2 + \delta\sum_{i\geq 1}\sigma_{i,t}\|z_t-x^*\|^2 \\ &\quad = t\langle u-x^*, j(z_t-x^*)\rangle + (1-t)\|z_t-x^*\|^2, \end{aligned} \tag{26}$$

which implies

$$\|z_t - x^*\| \leq \|u - x^*\|. \tag{27}$$

Thus, $\{z_t\}$ is bounded.

Now, assume for the sake of contradiction that x' and x^* are two distinct accumulation points of $\{z_t\}$ in $\mathcal{F}$; then, there exists a subnet $\{z_{t_s}\}$ of $\{z_t\}$ such that $z_{t_s} \to x'$ as $s \to \infty$, and so we have the following estimates:

$$\begin{aligned} &\|z_{t_s} - x^*\|^2 \\ &\quad = \Big\langle t_s(u-x^*) + (1-t_s)(1-\delta)(z_{t_s}-x^*) \\ &\qquad + \delta\sum_{i\geq 1}\sigma_{i,t}(T_iP_K(z_{t_s}-\lambda Az_{t_s}) - x^*), j(z_{t_s}-x^*)\Big\rangle \\ &\quad \leq t_s\langle u-x^*, j(z_{t_s}-x^*)\rangle + (1-t_s)\|z_{t_s}-x^*\|^2 \end{aligned} \tag{28}$$

so that

$$\|z_{t_s} - x^*\|^2 \leq \langle u - x^*, j(z_{t_s} - x^*)\rangle, \tag{29}$$

and since $z_{t_s} \to x'$ as $s \to \infty$, we get from (29)

$$\|x' - x^*\|^2 \leq \langle u - x^*, j(x' - x^*)\rangle. \tag{30}$$

Applying similar argument to x^* as an accumulation point of $\{z_t\}$ in $\mathcal{F}$, we also get

$$\left\|x^* - x'\right\|^2 \le \left\langle u - x', j\left(x^* - x'\right)\right\rangle. \tag{31}$$

Adding these last two inequalities, we get

$$2\left\|x^* - x'\right\|^2 \le \left\|x^* - x'\right\|^2, \tag{32}$$

a contradiction, and thus $x' = x^*$. This completes the proof. □

Lemma 11. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a closed, convex, and nonempty subset of E. Let $r \in (0,1)$ be fixed and $\{t_n\} \subset (0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty}(t_n/\sigma_{i,n}) = 0$ for all $i \in \mathcal{N}_{\mathcal{J}}$. Let $\{z_{t_n}\}$ be a sequence satisfying (25), and let $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_iP_K(I-\lambda A)T_r) \ne \emptyset$. Then, $\lim_{n\to\infty}\|z_{t_n} - (T_iP_K(I-\lambda A))T_r z_{t_n}\| = 0$, for all $i \in \mathbb{N}$.*

Proof. For $i \in \mathbb{N}$ and $x^* \in \mathcal{F}$, we have the following estimates (using Lemma 7 establishing the existence of g):

$$\begin{aligned}
&g\left(\left\|\left(T_iP_K(I-\lambda A)\right)T_r z_{t_n} - z_{t_n}\right\|\right)\\
&\quad = g\left(\left\|\left[x^* - \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}\right] - \left[x^* - z_{t_n}\right]\right\|\right)\\
&\quad \le \left\|x^* - \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}\right\|^2\\
&\qquad - 2\left\langle x^* - \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}, j\left(x^* - z_{t_n}\right)\right\rangle\\
&\qquad + \left\|x^* - z_{t_n}\right\|^2 \le \left\|x^* - z_{t_n}\right\|^2 - 2\\
&\qquad \times\left\langle x^* - z_{t_n} + z_{t_n} - \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}, j\left(x^* - z_{t_n}\right)\right\rangle\\
&\qquad + \left\|x^* - z_{t_n}\right\|^2\\
&\quad = 2\left\langle z_{t_n} - \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}, j\left(z_{t_n} - x^*\right)\right\rangle.
\end{aligned} \tag{33}$$

Using (25), we have

$$\begin{aligned}
&\left\langle z_{t_n} - x^*, j\left(z_{t_n} - x^*\right)\right\rangle\\
&\quad = t_n\left\langle u - x^*, j\left(z_{t_n} - x^*\right)\right\rangle\\
&\qquad + (1-t_n)(1-\delta)\left\langle z_{t_n} - x^*, j\left(z_{t_n} - x^*\right)\right\rangle + \delta\sum_{i\ge1}\sigma_{i,n}\\
&\qquad \times\left\langle \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n} - z_{t_n} + z_{t_n} - x^*, j\left(z_{t_n} - x^*\right)\right\rangle\\
&\quad = t_n\left\langle u - x^*, j\left(z_{t_n} - x^*\right)\right\rangle\\
&\qquad + \delta\sum_{i\ge1}\sigma_{i,n}\left\langle \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}\right.\\
&\qquad\qquad \left. - z_{t_n}, j\left(z_{t_n} - x^*\right)\right\rangle\\
&\qquad + (1-t_n)\left\langle z_{t_n} - x^*, j\left(z_{t_n} - x^*\right)\right\rangle
\end{aligned} \tag{34}$$

which implies

$$\begin{aligned}
&\delta\sum_{i\ge1}\sigma_{i,n}\left\langle z_{t_n} - \left(T_iP_K(I-\lambda A)\right)T_r z_{t_n}, j\left(z_{t_n} - x^*\right)\right\rangle\\
&\quad = t_n\left\langle u - z_{t_n}, j\left(z_{t_n} - x^*\right)\right\rangle.
\end{aligned} \tag{35}$$

Using this and (33), we get

$$\begin{aligned}
&\frac{\delta}{2}\sum_{i\ge1}\sigma_{i,n} g\left(\left\|\left(T_iP_K(I-\lambda A)\right)T_r z_{t_n} - z_{t_n}\right\|\right)\\
&\quad \le t_n\left\langle u - z_{t_n}, j\left(z_{t_n} - x^*\right)\right\rangle.
\end{aligned} \tag{36}$$

Thus,

$$\begin{aligned}
&\frac{\delta}{2} g\left(\left\|\left(T_iP_K(I-\lambda A)\right)T_r z_{t_n} - z_{t_n}\right\|\right)\\
&\quad \le \frac{t_n}{\sigma_{i,n}}\left\langle u - z_{t_n}, j\left(z_{t_n} - x^*\right)\right\rangle, \quad \forall i \in \mathbb{N}.
\end{aligned} \tag{37}$$

Since $\{z_{t_n}\}$ is bounded and $t_n/\sigma_{i,n} \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} g(\|(T_iP_K(I-\lambda A))T_r z_{t_n} - z_{t_n}\|) = 0$ for all $i \in \mathbb{N}$. By property of g, $\lim_{n\to\infty}\|(T_iP_K(I-\lambda A))T_r z_{t_n} - z_{t_n}\| = 0$ for all $i \in \mathbb{N}$. This completes the proof. □

Theorem 12. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a closed, convex, and nonempty subset of E. Let $r \in (0,1)$ be fixed and $\{t_n\} \subset (0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty}(t_n/\sigma_{i,n}) = 0$ for all $i \in \mathcal{N}_{\mathcal{J}}$. Let $\{z_{t_n}\}$ be a sequence satisfying (25), and let $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_iP_K(I-\lambda A)T_r) \ne \emptyset$. If the duality mapping j of E is weakly sequentially continuous, then $\{z_{t_n}\}$ converges strongly to an element in $\mathcal{F}$.*

Proof. Since $\{z_{t_n}\}$ is bounded, there exists a subsequence say $\{z_{t_{n_k}}\}$ of $\{z_{t_n}\}$ that converges weakly to some point $z \in K$. Using demiclosedness property of $[I - (T_iP_K(I-\lambda A))T_r]$ at 0 for $i \in \mathbb{N}$, and the fact that $\lim_{k\to\infty}\|(T_iP_K(I-\lambda A))T_r z_{t_{n_k}} - z_{t_{n_k}}\| = 0$, we get that z is a point in $\mathcal{F}$. We also observe from (33) that

$$\begin{aligned}
&\left\|z_{t_{n_k}} - z\right\|^2\\
&\quad = \left\langle t_{n_k}(u - z) + \left(1 - t_{n_k}\right)(1-\delta)\left(z_{t_{n_k}} - z\right)\right.\\
&\qquad \left. + \delta\sum_{i\ge1}\sigma_{i,n_k}\left(\left(T_iP_K(I-\lambda A)\right)T_r z_{t_{n_k}} - z\right), j\left(z_{t_{n_k}} - z\right)\right\rangle\\
&\quad \le t_{n_k}\left\langle u - z, j\left(z_{t_{n_k}} - z\right)\right\rangle + \left(1 - t_{n_k}\right)(1-\delta)\left\|z_{t_{n_k}} - z\right\|^2\\
&\qquad + \delta\sum_{i\ge1}\sigma_{i,n_k}\left\|z_{t_{n_k}} - z\right\|^2\\
&\quad = t_{n_k}\left\langle u - z, j\left(z_{t_{n_k}} - z\right)\right\rangle + \left(1 - t_{n_k}\right)\left\|z_{t_{n_k}} - z\right\|^2
\end{aligned} \tag{38}$$

which implies

$$\left\|z_{t_{n_k}} - z\right\|^2 \le \left\langle u - z, j\left(z_{t_{n_k}} - z\right)\right\rangle. \tag{39}$$

Since j admits weak sequential continuity, the last inequality implies that the subsequence $\{z_{t_{n_k}}\}$ converges strongly to z, and since $\{z_{t_n}\}$ admits unique accumulation point in $\mathscr{F}$, then it converges strongly to z. This completes the proof. □

The following corollary follows from Theorem 12.

Corollary 13. *Let E be a real L_p space, $(1 < p < \infty)$. Let K, $\{t_n\}$, $\mathscr{F}$, and $\{z_{t_n}\}$ be as in Theorem 12. Then, $\{z_{t_n}\}$ converges strongly to an element of $\mathscr{F}$.*

Theorem 14. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a closed, convex, and nonempty subset of E. Let $r \in (0,1)$ be fixed and $\{t_n\}$ a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty}(t_n/\sigma_{i,n}) = 0$ for all $i \in \mathscr{N}_{\mathscr{F}}$. Let $\{z_{t_n}\}$ be a sequence satisfying (25), and let $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i P_K(I - \lambda A)T_r) \neq \emptyset$. If for at least one i in $\mathbb{N}$, $T_i P_K(I - \lambda A)T_r$ is demicompact, then $\{z_{t_n}\}$ converges strongly to an element of $\mathscr{F}$.*

Proof. For some fixed $j_0 \in \mathbb{N}$, let $T_{j_0} P_K(I - \lambda A)T_r$ be demicompact. Since $\lim_{n\to\infty}\|T_{j_0} P_K(I - \lambda A)T_r z_{t_n} - z_{t_n}\| = 0$, there exists a subsequence say $\{z_{t_{n_k}}\}$ of $\{z_{t_n}\}$ that converges strongly to some point $z \in E$. By continuity of $T_i P_K(I-\lambda A)T_r$ for all $i \in \mathbb{N}$, we have that $z \in \mathscr{F}$. But the sequence $\{z_{t_n}\}$ admits unique accumulation point in $\mathscr{F}$; so, it converges strongly to z. □

The following corollaries follow from Theorem 14.

Corollary 15. *Let E be a real L_p space, $(1 < p < \infty)$. Let K, $\{t_n\}$, $\mathscr{F}$, and $\{z_{t_n}\}$ be as in Theorem 14. If for at least one $i \in \mathbb{N}$, the map $T_i P_K(I - \lambda A)T_r$ is demicompact, then $\{z_{t_n}\}$ converges strongly to an element of $\mathscr{F}$.*

Corollary 16. *Let K be a closed, convex, and nonempty subset of a real Hilbert space H. Let $\{t_n\}$, $\mathscr{F}$, and $\{z_{t_n}\}$ be as in Theorem 14. Then, $\{z_{t_n}\}$ converges strongly to an element of $\mathscr{F}$.*

4. Iterative Convergence Theorem

We now state and prove the following theorem.

Theorem 17. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a closed, convex, and nonempty subset of E. For some $\kappa > 0$, let $T_i : K \to Ei \in \mathbb{N}$ and $A : K \to E$ be a family of nonexpansive maps and a κ-inverse strongly accretive map, respectively. Let $G : K \times K \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let P_K be a nonexpansive projection of E onto K. For some fixed real numbers $r, \delta \in (0,1)$ and $\lambda \in (0, (q\kappa/d_q)^{1/(q-1)})$, define a sequence $\{x_n\}$ iteratively by $x_1, u \in E$ and $n \in \mathbb{N}$ as*

$$\begin{aligned} &G(y_n, \eta) + \frac{1}{r}\langle \eta - y_n, j(y_n - x_n)\rangle \ge 0, \quad \forall \eta \in K, \\ &x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta\sum_{i\ge1}\sigma_{i,n} T_i P_K \\ &\quad \times (y_n - \lambda A y_n), \end{aligned} \tag{40}$$

where $\{\alpha_n\}, \{\sigma_{i,n}\} \subset (0,1)$ are sequences satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\lim_{n\to\infty}\sum_{i\ge1}|\sigma_{i,n+1} - \sigma_{i,n}| = 0$.

Let $F := [\bigcap_{i=1}^{\infty} F(T_i)] \bigcap \mathrm{EP}(G) \bigcap \mathrm{VI}(K, A) \neq \emptyset$. If either the duality map j of E is weakly sequentially continuous or for at least one $i \in \mathbb{N}$, $T_i P_K(I - \lambda A)T_r$ is demicompact, then $\{x_n\}$ converges strongly to some element in F.

Proof. Let $x^* \in \mathscr{F}$ then, we claim that $\|x_n - x^*\| \le \max\{\|u - x^*\|, \|x_1 - x^*\|\}$ for all $n \ge 1$. It is clear that the claim is true for $n = 1$. Assume that it is true for $n = k$ for some $k \ge 1, k \in \mathbb{N}$. Then,

$$\begin{aligned} &\|x_{k+1} - x^*\| \\ &\quad \le \alpha_k\|u - x^*\| + (1-\alpha_k)(1-\delta)\|x_k - x^*\| \\ &\qquad + \delta\sum_{i\ge1}\sigma_{i,k}\|T_i P_K(I - \lambda A)T_r x_k - T_i P_K(I - \lambda A)T_r x^*\| \\ &\quad \le \alpha_k\|u - x^*\| + (1-\alpha_k)\|x_k - x^*\| \\ &\quad \le \max\{\|u - x^*\|, \|x_1 - x^*\|\}. \end{aligned} \tag{41}$$

Hence, the result, and so $\{x_n\}$ is bounded. Furthermore, $\{y_n\}$, $\{T_i P_K(y_n - \lambda A y_n)\}$, and $\{A y_n\}$ are each bounded.

We now show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Note that $y_n = T_r x_n$, $y_{n+1} = T_r x_{n+1}$, so that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|T_r x_{n+1} - T_r x_n\| \\ &\le \|x_{n+1} - x_n\|. \end{aligned} \tag{42}$$

Define two sequences $\{\beta_n\}$ and $\{w_n\}$ by $\beta_n := (1-\delta)\alpha_n + \delta$ and $w_n := (x_{n+1} - x_n + \beta_n x_n)/\beta_n$. Then,

$$w_n = \frac{\alpha_n u + \delta\sum_{i\ge1}\sigma_{i,n} T_i P_K(y_n - \lambda A y_n)}{\beta_n}. \tag{43}$$

Observe that $\{w_n\}$ is bounded and that

$$\begin{aligned}\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| &\le \left|\frac{\alpha_{n+1}}{\beta_{n+1}}-\frac{\alpha_n}{\beta_n}\right|\|u\|+\left|\frac{\delta(1-\alpha_{n+1})}{\beta_{n+1}}-1\right|\|x_{n+1}-x_n\| \\ &+\frac{\delta M}{\beta_{n+1}}\sum_{i\ge1}|\sigma_{i,n+1}-\sigma_{i,n}| \\ &+\frac{\delta M}{\beta_{n+1}\beta_n}|\beta_n-\beta_{n+1}|,\end{aligned} \tag{44}$$

for some positive real number M. This implies

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\le 0, \tag{45}$$

and by Lemma 4, $\lim_{n\to\infty}\|w_n-x_n\|=0$. Hence,

$$\|x_{n+1}-x_n\|=\beta_n\|w_n-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{46}$$

From (42) and (46), we have

$$\|y_{n+1}-y_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{47}$$

From (40), we have $x_{n+1}-x_n=\alpha_n(u-x_n)+\delta\sum_{i\ge1}\sigma_{i,n}[T_iP_K(y_n-\lambda Ay_n)-x_n]$ which implies

$$\begin{aligned}\delta\left\|\sum_{i\ge1}\sigma_{i,n}[T_iP_K(y_n-\lambda Ay_n)-x_n]\right\| \\ \le\|x_{n+1}-x_n\|+\alpha_n\|u-x_n\|,\end{aligned} \tag{48}$$

and thus $\lim_{n\to\infty}\|\sum_{i\ge1}\sigma_{i,n}[T_iP_K(y_n-\lambda Ay_n)-x_n]\|=0$. Let $\{t_n\}$ be a real sequence in $(0,1)$ satisfying the following conditions:

$$\begin{aligned}&\lim_{n\to\infty}t_n=0, \qquad \sum_{i\ge1}\sigma_{i,n}=(1-t_n), \\ &\lim_{n\to\infty}\frac{\|\sum_{i\ge1}\sigma_{i,n}[T_iP_K(y_n-\lambda Ay_n)-x_n]\|}{t_n}=0.\end{aligned} \tag{49}$$

Let $z_{t_n}\in K$ be the unique fixed point satisfying (25) for each $n\in\mathbb{N}$, and let $z_{t_n}\to z\in\mathcal{F}$ as $n\to\infty$. Using (25) and Lemma 2, we have the following estimates:

$$\begin{aligned}\|z_{t_n}-x_n\|^2 &\le\Big\|(1-\delta)(1-t_n)(z_{t_n}-x_n) \\ &\quad+\delta\sum_{i\ge1}\sigma_{i,n}\big(T_iP_K(I-\lambda A)T_rz_{t_n}-T_iP_K(I-\lambda A)T_rx_n \\ &\quad+T_iP_K(I-\lambda A)T_rx_n-x_n\big)\Big\|^2+2t_n\langle u-x_n,j(z_{t_n}-x_n)\rangle \\ &\le\Big[(1-\delta)(1-t_n)\|z_{t_n}-x_n\|+\delta(1-t_n)\|z_{t_n}-x_n\| \\ &\quad+\Big\|\delta\sum_{i\ge1}\sigma_{i,n}[T_iP_K(I-\lambda A)T_rx_n-x_n]\Big\|\Big]^2 \\ &\quad+2t_n\langle u-x_n,j(z_{t_n}-x_n)\rangle \\ &=\Big[(1-t_n)\|z_{t_n}-x_n\| \\ &\quad+\Big\|\delta\sum_{i\ge1}\sigma_{i,n}[T_iP_K(I-\lambda A)T_rx_n-x_n]\Big\|\Big]^2 \\ &\quad+2t_n\langle u-x_n,j(z_{t_n}-x_n)\rangle.\end{aligned} \tag{50}$$

This implies

$$\begin{aligned}\langle u-z_{t_n},j(x_n-z_{t_n})\rangle &\le\frac{t_n}{2}\|z_{t_n}-x_n\|^2+(1-t_n)\|z_{t_n}-x_n\| \\ &\quad\times\left(\frac{\delta\|\sum_{i\ge1}\sigma_{i,n}[T_iP_K(I-\lambda A)T_rx_n-x_n]\|}{t_n}\right) \\ &\quad+\frac{\delta^2\|\sum_{i\ge1}\sigma_{i,n}[T_iP_K(I-\lambda A)T_rx_n-x_n]\|^2}{2t_n},\end{aligned} \tag{51}$$

and, hence,

$$\limsup_{n\to\infty}\langle u-z_{t_n},j(x_n-z_{t_n})\rangle\le 0. \tag{52}$$

Moreover,

$$\begin{aligned}\langle u-z_{t_n},j(x_n-z_{t_n})\rangle &=\langle u-z,j(x_n-z)\rangle \\ &\quad+\langle u-z,j(x_n-z_{t_n})-j(x_n-z)\rangle \\ &\quad+\langle z-z_{t_n},j(x_n-z_{t_n})\rangle,\end{aligned} \tag{53}$$

and since j is norm-to-*weak** uniformly continuous on bounded sets, we have

$$\limsup_{n\to\infty} \langle u - z, j(x_n - z)\rangle \le 0. \quad (54)$$

From the recursion formula (40) and Lemma 2, we have the following:

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \Big\|\alpha_n (u - z) + (1 - \alpha_n)(1 - \delta)(x_n - z) \\
&\quad + \delta \sum_{i\ge1} \sigma_{i,n} [T_i P_K (I - \lambda A) T_r x_n - T_i P_K (I - \lambda A) T_r z]\Big\|^2 \\
&\le \Big\|(1 - \alpha_n)(1 - \delta)(x_n - z) \\
&\quad + \delta \sum_{i\ge1} \sigma_{i,n} [T_i P_K (I - \lambda A) T_r x_n - T_i P_K (I - \lambda A) T_r z]\Big\|^2 \\
&\quad + 2\alpha_n \langle u - z, j(x_{n+1} - z)\rangle \le (1 - \alpha_n)\|x_n - z\|^2 \\
&\quad + 2\alpha_n \langle u - z, j(x_{n+1} - z)\rangle,
\end{aligned} \quad (55)$$

and by Lemma 5, we get that $\{x_n\}$ converges strongly to $z \in \mathcal{F}$.

To complete the proof, we show that $z \in \mathrm{EP(G)} \cap \mathrm{VI}(K, A)$.

We start by showing that $z \in \mathrm{EP}(G)$.

Let $x^* \in F$; then,

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|T_r x_n - T_r x^*\|^2 \\
&\le \langle T_r x_n - T_r x^*, j(x_n - x^*)\rangle = \langle y_n - x^*, j(x_n - x^*)\rangle \\
&\le \frac{1}{2}\left[\|x_n - x^*\|^2 + \|y_n - x^*\|^2 - g(\|x_n - y_n\|)\right];
\end{aligned} \quad (56)$$

thus,

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - g(\|x_n - y_n\|). \quad (57)$$

Using (40) and (57), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \Big\|\alpha_n (u - x^*) + (1 - \alpha_n)(1 - \delta)(x_n - x^*) \\
&\quad + \delta \sum_{i\ge1} \sigma_{i,n} [T_i P_K (I - \lambda A) y_n - T_i P_K (I - \lambda A) x^*]\Big\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + (1 - \alpha_n)(1 - \delta)\|x_n - x^*\|^2 \\
&\quad + \delta(1 - \alpha_n)\|y_n - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + (1 - \alpha_n)(1 - \delta)\|x_n - x^*\|^2 \\
&\quad + \delta(1 - \alpha_n)\left[\|x_n - x^*\|^2 - g(\|x_n - y_n\|)\right].
\end{aligned} \quad (58)$$

This implies

$$\begin{aligned}
&\delta(1 - \alpha_n) g(\|x_n - y_n\|) \\
&\quad \le \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - \|x_{n+1} - x^*\|^2 \le \alpha_n \|u - x^*\|^2 \\
&\quad + \|x_n - x_{n+1}\| [\|x_n - x^*\| + \|x_{n+1} - x^*\|],
\end{aligned} \quad (59)$$

and thus $\lim_{n\to\infty}\delta(1 - \alpha_n)g(\|x_n - y_n\|) = 0$. Using property of g, we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \quad (60)$$

From (60), we have $y_n \to z$ and $j(x_n - y_n) \to 0$ as $n \to \infty$. Since $y_n = T_r x_n$, we have

$$G(y_n, \eta) + \frac{1}{r}\langle \eta - y_n, j(y_n - x_n)\rangle \ge 0, \quad \forall \eta \in K, \quad (61)$$

It follows from (A2) that

$$\left\langle \eta - y_n, \frac{j(y_n - x_n)}{r} \right\rangle \ge G(\eta, y_n), \quad (62)$$

and so using (A4), we have $G(\eta, z) \le 0$ for all $\eta \in K$. For real number t, $0 < t \le 1$, and $\eta \in K$, let $\eta_t = t\eta + (1 - t)z$. Clearly, $\eta_t \in K$. So, using (A1) and (A4), we have

$$0 = G(\eta_t, \eta_t) \le tG(\eta_t, \eta) + (1 - t)G(\eta_t, z) \le tG(\eta_t, \eta). \quad (63)$$

This implies $G(\eta_t, \eta) \ge 0$, and using this and (A3), we have that $G(z, \eta) \ge 0$ for all $\eta \in K$; hence, $z \in \mathrm{EP}(G)$.

Next, we show that $z \in \mathrm{VI}(K, B)$.

Let $x^* \in F$ and $b_n := P_K(y_n - \lambda A y_n)$; then,

$$\begin{aligned}
\|b_n - x^*\|^q &= \|P_K(y_n - \lambda A y_n) - P_K(x^* - \lambda A x^*)\|^q \\
&\le \|y_n - x^* - \lambda(A y_n - A x^*)\|^q \\
&\le \|y_n - x^*\|^q + \lambda\left(d_q \lambda^{q-1} - q\kappa\right)\|A y_n - A x^*\|^q.
\end{aligned} \quad (64)$$

Using recursion formula (40), we have the following estimates:

$$\begin{aligned}
&\|x_{n+1}-x^*\|^q \\
&= \Big\|\alpha_n(u-x^*)+(1-\alpha_n)(1-\delta)(x_n-x^*) \\
&\quad +\delta\sum_{i\geq 1}\sigma_{i,n}[T_ib_n-x^*]\Big\|^q \leq \alpha_n\|u-x^*\|^q \\
&\quad +(1-\alpha_n)(1-\delta)\|x_n-x^*\|^q \\
&\quad +(1-\alpha_n)\delta\|b_n-x^*\|^q \leq \alpha_n\|u-x^*\|^q \\
&\quad +(1-\alpha_n)(1-\delta)\|x_n-x^*\|^q+(1-\alpha_n)\delta \\
&\quad \times\left[\|x_n-x^*\|^q+\lambda\left(d_q\lambda^{q-1}-q\kappa\right)\|Ay_n-Ax^*\|^q\right] \\
&\leq \alpha_n\|u-x^*\|^q+\|x_n-x^*\|^q+(1-\alpha_n)\delta\lambda\left(d_q\lambda^{q-1}-q\kappa\right) \\
&\quad \times\|Ay_n-Ax^*\|^q,
\end{aligned} \tag{65}$$

which implies, by Mean Value Theorem, that

$$\begin{aligned}
&-(1-\alpha_n)\delta\lambda\left(d_q\lambda^{q-1}-q\kappa\right)\|Ay_n-Ax^*\|^q \\
&\quad\leq \alpha_n\|u-x^*\|^q+\|x_n-x^*\|^q-\|x_{n+1}-x^*\|^q \\
&\quad\leq \alpha_n\|u-x^*\|^q+q\sigma_n^{q-1}\left|\|x_n-x^*\|-\|x_{n+1}-x^*\|\right| \\
&\quad\leq \alpha_n\|u-x^*\|^q+q\sigma_n^{q-1}\|x_n-x_{n+1}\|,
\end{aligned} \tag{66}$$

where σ_n is some nonnegative real number between $\|x_n-x^*\|$ and $\|x_{n+1}-x^*\|$ for each n. Since $\{x_n\}$ is bounded, $\alpha_n \to 0$, and $\|x_n-x_{n+1}\| \to 0$ as $n \to \infty$, we have $\|Ay_n-Ax^*\| \to 0$ as $n \to \infty$.

We also have the following:

$$\begin{aligned}
&\|b_n-x^*\|^2 \\
&\quad=\|P_K(y_n-\lambda Ay_n)-P_K(x^*-\lambda Ax^*)\|^2 \\
&\quad\leq \langle y_n-\lambda Ay_n-(x^*-\lambda Ax^*), j(b_n-x^*)\rangle \\
&\quad= \langle y_n-x^*, j(b_n-x^*)\rangle-\lambda\langle Ay_n-Ax^*, j(b_n-x^*)\rangle \\
&\quad\leq \frac{1}{2}\left[\|b_n-x^*\|^2+\|y_n-x^*\|^2-g(\|b_n-y_n\|)\right] \\
&\qquad-\lambda\langle Ay_n-Ax^*, j(b_n-x^*)\rangle
\end{aligned} \tag{67}$$

so that

$$\begin{aligned}
\|b_n-x^*\|^2 &\leq \|x_n-x^*\|^2-g(\|b_n-y_n\|) \\
&\quad-2\lambda\langle Ay_n-Ax^*, j(b_n-x^*)\rangle.
\end{aligned} \tag{68}$$

We then have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad=\Big\|\alpha_n(u-x^*)+(1-\alpha_n)(1-\delta)(x_n-x^*) \\
&\qquad+\delta\sum_{i\geq 1}\sigma_{i,n}[T_ib_n-x^*]\Big\|^2 \leq \alpha_n\|u-x^*\|^2 \\
&\qquad+(1-\alpha_n)(1-\delta)\|x_n-x^*\|^2+(1-\alpha_n)\delta\|b_n-x^*\|^2 \\
&\quad\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)(1-\delta)\|x_n-x^*\|^2 \\
&\qquad+(1-\alpha_n)\delta\left[\|x_n-x^*\|^2-g(\|b_n-x_n\|)\right. \\
&\qquad\left.-2\lambda\langle Ay_n-Ax^*, j(b_n-x^*)\rangle\right] \\
&\quad=\alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-(1-\alpha_n) \\
&\qquad\times\delta g(\|b_n-y_n\|)-2\delta\lambda(1-\alpha_n)\langle Ay_n-Ax^*, j(b_n-x^*)\rangle,
\end{aligned} \tag{69}$$

and so

$$\begin{aligned}
&(1-\alpha_n)\delta g(\|b_n-y_n\|) \\
&\quad\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\qquad-2\delta\lambda(1-\alpha_n)\langle Ay_n-Ax^*, j(b_n-x^*)\rangle.
\end{aligned} \tag{70}$$

As $\alpha_n \to 0$, $\|x_n-x_{n+1}\| \to 0$ and $\|Ax_n-Ax^*\| \to 0$ as $n \to \infty$, we get

$$\lim_{n\to\infty}(1-\alpha_n)\delta g(\|b_n-y_n\|)=0, \tag{71}$$

which implies

$$\lim_{n\to\infty}\|b_n-y_n\|=0. \tag{72}$$

Let

$$Mv=\begin{cases} Av+N_Kv, & v\in K, \\ \emptyset, & v\notin K. \end{cases} \tag{73}$$

Then, M is maximal accretive. Let $\mathrm{GF}(M)$ denote the graph of M.

Let $(v,w)\in \mathrm{GF}(M)$. Since $w-Av\in N_Kv$ and $b_n\in K$, we have $\langle w-Av, j_q(v-b_n)\rangle \geq 0$ by definition of N_Kv. Also, as $b_n=P_K(y_n-\lambda Ay_n)$ (using property of the projection P_K), we have

$$\left\langle b_n-(y_n-\lambda Ay_n), j_q(v-b_n)\right\rangle \geq 0 \tag{74}$$

and, hence,

$$\left\langle \frac{b_n-y_n}{\lambda}+Ay_n, j_q(v-b_n)\right\rangle \geq 0. \tag{75}$$

Using this, we obtain the following estimates:

$$\begin{aligned}\left\langle w, j_q\left(v-b_n\right)\right\rangle &\geq \left\langle Av, j_q\left(v-b_n\right)\right\rangle \\ &\geq \left\langle Av, j_q\left(v-b_n\right)\right\rangle \\ &\quad - \left\langle \frac{b_n-y_n}{\lambda} + Ay_n, j_q\left(v-b_n\right)\right\rangle \\ &= \left\langle Av - \frac{b_n-y_n}{\lambda} - Ay_n, j_q\left(v-b_n\right)\right\rangle \\ &= \left\langle Av - Ab_n, j_q\left(v-b_n\right)\right\rangle \\ &\quad + \left\langle Ab_n - Ay_n, j_q\left(v-b_n\right)\right\rangle \\ &\quad - \left\langle \frac{b_n-y_n}{\lambda}, j_q\left(v-b_n\right)\right\rangle \\ &\geq \left\langle Ab_n - Ay_n, j_q\left(v-b_n\right)\right\rangle \\ &\quad - \left\langle \frac{b_n-y_n}{\lambda}, j_q\left(v-b_n\right)\right\rangle,\end{aligned} \tag{76}$$

which implies $\langle w, j_q(v-z)\rangle \geq 0$ (letting $n \to \infty$).

Since M is maximal accretive, we obtained that $z \in M^{-1}(0)$, and, hence, $z \in \mathrm{VI}(K, A)$. This completes the proof. □

The following corollaries follow from Theorem 17.

Corollary 18. *Let $E = L_p$ space $(1 < p < \infty)$. Let K, P_K, δ, A, r, and T_i, $i = 1, 2, \ldots$ be as in Theorem 17. Let $\lambda \in (0, (2\kappa/(p-1)))$, and define sequences $\{x_n\}$ and $\{y_n\}$ by (40). Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to some element in F.*

Corollary 19. *Let $E = L_p$ space $(1 < p < \infty)$. Let K, P_K, δ, A, r, and T_i, $i = 1, 2, \ldots$ be as in Theorem 17. Let $\lambda \in (0, (2\kappa/(p-1)))$, and define sequences $\{x_n\}$ and $\{y_n\}$ by (40). If for at least one i in $\mathbb{N}$, $T_i P_K(I - \lambda A)T_r$ is demicompact, then the sequences $\{x_n\}$ and $\{y_n\}$ both converge strongly to some element in F.*

Corollary 20. *Let $E = H$ be a real Hilbert space. Let K, P_K, δ, A, r, and T_i, $i = 1, 2, \ldots$ be as in Theorem 17. Let $\lambda \in (0, 2\kappa)$, and define sequences $\{x_n\}$ and $\{y_n\}$ by (40). Then, the sequences $\{x_n\}$ and $\{y_n\}$ both converge strongly to some element in F.*

Remark 21. Prototypes of the sequences $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ in our theorems are the following:

$$\alpha_n := \frac{1}{n+1}, \qquad \sigma_{i,n} := \frac{n}{2^i(n+1)}, \quad \forall i \in \mathbb{N}. \tag{77}$$

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