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The Implicit Keller Box method for the one dimensional time fractional diffusion equation

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Abstract

There are a number of physical situations that can be modeled by fractional partial differential equations. In this paper, we discuss a numerical scheme based on Keller's box method for one dimensional time fractional diffusion equation with boundary values which are functions. The fractional derivative term is replaced by the Grünwald-Letnikov formula. The stability is analyzed by means of the Von Neumann method. An example is presented to show the feasibility and the accuracy of this method and a comparison between the approximate solution using this method and analytical solution is made. The results indicated that this scheme is unconditionally stable and is a feasible technique.

Mathematics Subject Classification: 34A08, 34K28, 34D20

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1 Introduction

Diffusion equations are partial differential equations which model the diffusive and thermodynamic phenomena and describe the spread of particles (ions, molecules, etc.) from the area of higher concentration to the area of lower concentration [1] .The random walk of these particles is called the Brownian motion.

Diffusion not described by normal diffusion in the long time limit has become known as anomalous (unnatural) [2]. Fractional diffusion equation is a generalization of the classic diffusion equation which models anomalous diffusive phenomena. Some particles of the diffusing species get stuck for long time thus giving rise to slow diffusion which is referred to as anomalous subdiffusion. Anomalous superdiffusion appear when the particles move faster than random diffusion [3]. In normal diffusion, the mean-square displacement $\left\langle x^2(t) \right\rangle$ of the diffusing particles is proportional to the time *t*.

$$\left\langle x^2(t) \right\rangle \sim Dt$$

where the diffusion coefficient *D* is constant [1]. Anomalous diffusion takes the nonlinear form [4]

$$\left\langle x^2(t) \right\rangle \sim D_a t^a$$

where D_a is the generalized diffusion coefficient and a is anomalous diffusion exponent [4]. Anomalous superdiffusion corresponds to the case where the mean-square displacement grows super linearly in time, whereas anomalous subdiffusion leads to sub linear growth [2].

Many problems in science and engineering have been solved using partial differential equations which involve fractional order in their derivatives. An excellent overview of recent developments is found in [5]. They have constructed a box type scheme for fractional subdiffusion and combined the order reduction method with L_1 discretization of the fractional derivative term. The stability analysis has been considered using energy method. In a different approach, Yuste [6] has developed weighted average methods which are linear combinations of the explicit and implicit Euler schemes. Karatay et al. [7] have studied an implicit method with Caputo formula for treating the fractional derivative. They have taken a different way to represent the system in matrix form; this system has been solved with respect to t instead of x. The stability analysis was discussed in this article. Implicit method approach and Caputo formula have also been used for homogeneous time fractional subdiffusion in [8]. However, they have used a simple quadrature formula to obtain a discrete approximation to the fractional term. One possible method was discussed by Khaled and Momani [9] is a decomposition method when $0 < \alpha \leq 2$ which leads to fractional diffusion-wave equation. Khan et al. [10] have proposed the same method for solving time fractional fourth-order equation with variable coefficients. Some papers focused on fractional subdiffusion problems in two dimensions [11] with Caputo definition. Tadjeran and Meerschaert [12] have considered methods based on finite difference method for solving fractional diffusion equation in 2D; they have used Grünwald-Letnikov definition to discretize the fractional derivatives in space. Furthermore, Grünwald-Letnikov formula can also be used for fractional partial differential equations with two-sides in space. These have been considered by Meerschaert and Tadjeran [13]; Pang and Sun[14] ; Su et al. [15]; Sweilam et al. [16]

In this paper, we construct a finite difference numerical scheme based on Keller box method for solving one dimensional time fractional diffusion equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < \mathrm{x} < 1, 0 < \mathrm{t} < \mathrm{T}, \tag{1}$$

with the initial condition

$$u(x,0) = 0; \quad 0 \leq x \leq 1, \tag{2}$$

and the boundary conditions

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t);\quad 0<x<1,\ \ 0<t<T, \tag{3}$$

where f, g_0 and g_1 are known functions, while the function u is the unknown function to be determined. We consider the case when $0<\alpha<1$ as we wish to study a kind of diffusion model known as anomalous subdiffusion equation.

There are many definitions of fractional derivative ${}_0D_t^{\alpha}$ of a function f with respect to time t. We shall take $\Delta x = 1/M$ in the x direction with
$xi = i\Delta x$, $i = 0,1, \ldots, M$, $tn = n\Delta t$ and $n = 0, 1\ldots N$.
The most common definition is Riemann-Liouville operator definition [17],

$${}_0D_t^{\alpha} f(t)=\frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha}}\right], 0 < \alpha < 1, \tag{4}$$

where $\Gamma(\cdot)$ is a Gamma function.

An alternative way of representing the fractional derivatives is by the Grünwald-Letnikov formula defined as [18],

$${}_0D_t^{\alpha} f(t)=\frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{t}{h}\right]}\omega_k^{(\alpha)} f(t-kh)+O(h),\quad t\ge 0. \tag{5}$$

The coefficients $\omega_k^{(\alpha)}$ are the first coefficients of the Taylor series expansions of the function $(1-z)^{\alpha}$, where

$$\omega_0^{(\alpha)}=1, \qquad \omega_k^{(\alpha)}=(1-\frac{\alpha+1}{k})\omega_{k-1}^{(\alpha)} \quad .$$

2 The Implicit Keller Box Method For The Time Fractional Diffusion Equation

Keller Box method is one of the important techniques for solving the parabolic flow equation, especially the boundary layer equations [19]. This scheme is implicit with second order accuracy in both space and time and allows the step size of time and space to be arbitrary (nonuniform). This makes it efficient and appropriate for the solution of parabolic partial differential equations. The disadvantage of the method is that the computational effort per time step is expensive due to its step which has to replace the higher derivative by first derivatives, so that the second-order diffusion equation can be written as a system of two first-order equations [20].

From the equation (1) when $\alpha = 1$ we have

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{6}$$

let

$$\frac{\partial u(x,t)}{\partial x} = u'(x,t) = v(x,t), \tag{7}$$

so that

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial v(x,t)}{\partial x} + f(x,t). \tag{8}$$

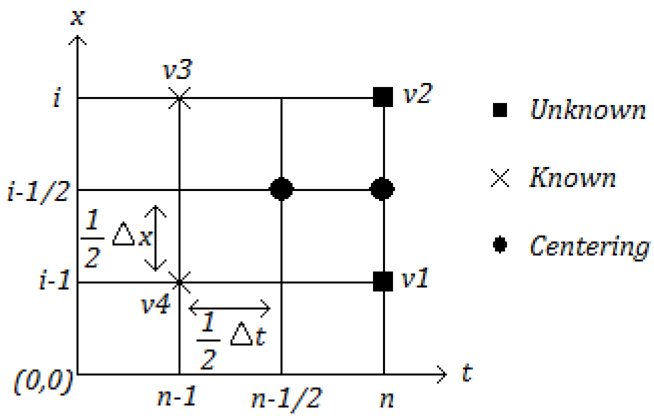

Figure 1: Finite difference grid points for the Keller Box scheme

We use the central difference in space in equation (7) in (n , i-1/2) of the segment v_1v_2 and the central difference in space and time in equation (8) in (n-1/2 , i-1/2) of the rectangle $v_1v_2v_3v_4$ as shown in Figure 1. This yields

$$\frac{u_i^n - u_{i-1}^n}{\Delta x_i} + O(\Delta x^2) = v_{i-1/2}^n, \tag{9}$$

and

$$\frac{u_{i-1/2}^n - u_{i-1/2}^{n-1}}{\Delta t_n} + O(\Delta t^2) = \frac{v_i^{n-1/2} - v_{i-1}^{n-1/2}}{\Delta x_i} + O(\Delta x^2) + f_{i-1/2}^{n-1/2}. \tag{10}$$

Using the expressions

$$t_{n-1/2} = \tfrac{1}{2}(t_n + t_{n-1}) \,\text{and}\, x_{i-1/2} = \tfrac{1}{2}(x_i + x_{i-1}) \tag{11}$$

yields

$$\frac{u_i^n - u_{i-1}^n}{\Delta x_i} + O(\Delta x^2) = \frac{v_i^n + v_{i-1}^n}{2}, \tag{12}$$

and

$$\tfrac{1}{2}\left(\frac{u_i^n + u_{i-1}^n - u_i^{n-1} - u_{i-1}^{n-1}}{\Delta t_n}\right) + O(\Delta t^2) = \tfrac{1}{2}\left(\frac{v_i^n + v_i^{n-1} - v_{i-1}^n - v_{i-1}^{n-1}}{\Delta x_i}\right) + O(\Delta x^2) + f_{i-1/2}^{n-1/2}. \tag{13}$$

Rearranging equations (12) and (13) and neglecting the truncation errors term we obtain

$$u_i^n - u_{i-1}^n - \frac{\Delta x_i}{2}\left(v_i^n + v_{i-1}^n\right) = 0, \tag{14}$$

$$u_i^n + u_{i-1}^n + \frac{\Delta t_n}{\Delta x_i}\left(v_{i-1}^n - v_i^n\right) = u_i^{n-1} + u_{i-1}^{n-1} + \frac{\Delta t_n}{\Delta x_i}\left(v_i^{n-1} - v_{i-1}^{n-1}\right) + 2\Delta t_n f_{i-1/2}^{n-1/2}. \tag{15}$$

This system of equations can be represented in a tridiagonal matrix form and can be solved with the use of the Thomas algorithm.

To obtain the fractional Keller Box scheme for the anomalous subdiffusion equation (1), the second-order partial differential equation (1) is first reduced to a system of two first-order partial differential equations as shown:

$$\frac{\partial u(x,t)}{\partial x} = u'(x,t) = v(x,t), \tag{16}$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial v(x,t)}{\partial x} + f(x,t), \tag{17}$$

using only central differences at (n , i-1/2) in equation (16) and at (n-1/2 , i-1/2) in equation (17) and discretizing the time fraction in equation (17) using the standard Grünwald-Letnikov formula which is defined in equation (5). We obtain

$$\frac{u_i^n - u_{i-1}^n}{\Delta x_i} + O(\Delta x^2) = v_{i-1/2}^n, \tag{18}$$

$$\frac{\partial^\alpha u(x_i,t_n)}{\partial t^\alpha} = \frac{1}{\Delta t_n^\alpha}\sum_{k=0}^{n}\omega_k^{(\alpha)} u_{i-1/2}^{n-k} + O(\Delta t) = \frac{v_i^{n-1/2} - v_{i-1}^{n-1/2}}{\Delta x_i} + O(\Delta x^2) + f_{i-1/2}^{n-1/2}. \tag{19}$$

Using the expressions

$$t_{n-1/2} = \tfrac{1}{2}(t_n + t_{n-1}) \text{ and } x_{i-1/2} = \tfrac{1}{2}(x_i + x_{i-1}) \tag{20}$$

yields

$$\frac{u_i^n - u_{i-1}^n}{\Delta x_i} + O(\Delta x^2) = \frac{v_i^n + v_{i-1}^n}{2}, \tag{21}$$

and

$$\begin{aligned}\frac{1}{2\Delta t_n^\alpha}&\left(\sum_{k=0}^{n}\omega_k^{(\alpha)} u_i^{n-k} + \sum_{k=0}^{n}\omega_k^{(\alpha)} u_{i-1}^{n-k}\right) + O(\Delta t)\\ &= \frac{1}{2\Delta x_i}\left(v_i^n + v_i^{n-1} - v_{i-1}^n - v_{i-1}^{n-1}\right) + O(\Delta x^2) + f_{i-1/2}^{n-1/2}.\end{aligned} \tag{22}$$

Rearranging equations (21), (22) and neglecting the truncation errors term we obtain

$$u_i^n - u_{i-1}^n - \frac{\Delta x_i}{2}\left(v_i^n + v_{i-1}^n\right) = 0, \tag{23}$$

$$\begin{aligned}u_i^n + u_{i-1}^n + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(v_{i-1}^n - v_i^n\right) &= -\omega_k^{(\alpha)}\left(u_i^{n-1} + u_{i-1}^{n-1}\right) + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(v_i^{n-1} - v_{i-1}^{n-1}\right) - \sum_{k=2}^{n}\omega_k^{(\alpha)} u_i^{n-k}\\ &\quad - \sum_{k=2}^{n}\omega_k^{(\alpha)} u_{i-1}^{n-k} + 2\Delta t_n^\alpha f_{i-1/2}^{n-1/2}.\end{aligned} \tag{24}$$

Note that if $\alpha = 1$, then we get the standard Keller Box method for the classical diffusion equation, where the weights $\omega_k^{(1)}$ vanish for $k > 1$.

2.1 Stability Of Fractional Implicit Keller Box Method

In this section we will apply Von Neumann method to study the stability of the fractional implicit Keller's box scheme. We shall follow the approach of Chen et al. [21] in their finite difference study of a fractional reaction-subdiffusion equation. Let U_i^n and V_i^n be the approximate solutions of the equations (23) and (24), and for simplicity let us write this system with no source. Then equations (23) and (24) become

$$U_i^n - U_{i-1}^n - \frac{\Delta x_i}{2}\left(V_i^n + V_{i-1}^n\right) = 0, \tag{25}$$

$$\begin{aligned} U_i^n + U_{i-1}^n + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(V_{i-1}^n - V_i^n\right) &= \alpha\left(U_i^{n-1} + U_{i-1}^{n-1}\right) + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(V_i^{n-1} - V_{i-1}^{n-1}\right) \\ &\quad - \sum_{k=2}^{n} \omega_k^\alpha U_i^{n-k} - \sum_{k=2}^{n} \omega_k^\alpha U_{i-1}^{n-k}. \end{aligned} \tag{26}$$

Lemma 2.1 The coefficients $\omega_k^{(\alpha)}\ (k = 0,1,\ldots)$ satisfy

(1) $\omega_0^{(\alpha)} = 1;\quad \omega_1^{(\alpha)} = -\alpha;\quad \omega_k^{(\alpha)} < 0,\quad k = 1,2,\ldots;$

(2) $\sum_{k=0}^{\infty} \omega_k^{(\alpha)} = 1;\quad \forall n \in N^+,\quad -\sum_{k=1}^{n} \omega_k^{(\alpha)} < 1.$

The errors are defined

$$\varepsilon_i^n = u_i^n - U_i^n, \qquad z_i^n = u_i^n - V_i^n,\quad n = 0,1,\ldots,N;\quad i = 0,1,\ldots,M. \tag{27}$$

The errors ε_i^n and z_i^n satisfy the finite difference equations in (25) and (26) and we obtain

$$\varepsilon_i^n - \varepsilon_{i-1}^n - \frac{\Delta x_i}{2}\left(z_i^n + z_{i-1}^n\right) = 0, \tag{28}$$

$$\varepsilon_i^n + \varepsilon_{i-1}^n + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(z_{i-1}^n - z_i^n\right) = \alpha\left(\varepsilon_i^{n-1} + \varepsilon_{i-1}^{n-1}\right) + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(z_i^{n-1} - z_{i-1}^{n-1}\right) - \sum_{k=2}^{n}\omega_k^\alpha \varepsilon_i^{n-k} - \sum_{k=2}^{n}\omega_k^\alpha \varepsilon_{i-1}^{n-k}. \tag{29}$$

$$\varepsilon_0^n = \varepsilon_M^n = 0, \quad z_0^n = z_M^n = 0, \; n = 1,2,\ldots,N. \tag{30}$$

Let us represent the errors functions $\varepsilon(i\Delta x) = \varepsilon_i$, $z(i\Delta x) = z_i$, $i = 0,1,\ldots,M$ as Fourier series [22]

$$\varepsilon_i = \sum_{m=0}^{M} \mathrm{A}_m e^{\sqrt{-1}qi\Delta x}, \quad z_i = \sum_{m=0}^{M} B_m e^{\sqrt{-1}qi\Delta x}, \quad i = 0,1,\ldots M, \tag{31}$$

where q is a real number. To study the propagation of errors let us omit the summation and constant A_m and B_m taking only a single term $e^{\sqrt{-1}qi\,\Delta x}$. Suppose the solution of equations (28), (29) and (30) in the form

$$\varepsilon_i^n = e^{\sqrt{-1}qx}e^{\beta t} = e^{\sqrt{-1}qi\Delta x}e^{\beta n\Delta t} = \xi_n e^{\sqrt{-1}qi\Delta x}, \tag{32}$$

and

$$z_i^n = \zeta_n e^{\sqrt{-1}qi\Delta x}, \tag{33}$$

where β is a complex number. It can be easily seen that at $n = 0$ the solutions reduce to $e^{\sqrt{-1}qi\,\Delta x}$, $\xi_0 = 1$ and $\zeta_0 = 1$.

Substituting equations (32) and (33) into equations (28) and (29) giving

$$\xi_n e^{\sqrt{-1}qi\Delta x} - \xi_n e^{\sqrt{-1}q(i-1)\Delta x} - \frac{\Delta x_i}{2}\left(\zeta_n e^{\sqrt{-1}qi\Delta x} + \zeta_n e^{\sqrt{-1}q(i-1)\Delta x}\right) = 0, \tag{34}$$

$$\begin{aligned}\xi_n e^{\sqrt{-1}qi\Delta x} + \xi_n e^{\sqrt{-1}q(i-1)\Delta x} + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(\zeta_n e^{\sqrt{-1}q(i-1)\Delta x} - \zeta_n e^{\sqrt{-1}qi\Delta x}\right) \\ = \alpha\left(\xi_{n-1} e^{\sqrt{-1}qi\Delta x} + \xi_{n-1} e^{\sqrt{-1}q(i-1)\Delta x}\right) + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(\zeta_{n-1} e^{\sqrt{-1}qi\Delta x} - \zeta_{n-1} e^{\sqrt{-1}q(i-1)\Delta x}\right) \\ - \sum_{k=2}^{n}\omega_k^\alpha \xi_{n-k} e^{\sqrt{-1}qi\Delta x} - \sum_{k=2}^{n}\omega_k^\alpha \xi_{n-k} e^{\sqrt{-1}q(i-1)\Delta x}.\end{aligned} \tag{35}$$

Simplifying equations (34) and (35) once we get

$$\xi_n = \frac{\Delta x_i}{2}\left(\frac{1+e^{-\sqrt{-1}q\Delta x}}{1-e^{-\sqrt{-1}q\Delta x}}\right)\zeta_n, \tag{36}$$

$$\xi_n - \frac{\Delta t_n^\alpha}{\Delta x_i}\left(\frac{1-e^{-\sqrt{-1}q\Delta x}}{1+e^{-\sqrt{-1}q\Delta x}}\right)\zeta_n = \alpha\xi_{n-1} + \frac{\Delta t_n^\alpha}{\Delta x_i}\left(\frac{1-e^{-\sqrt{-1}q\Delta x}}{1+e^{-\sqrt{-1}q\Delta x}}\right)\zeta_{n-1} - \sum_{k=2}^{n}\omega_k^\alpha \xi_{n-k}. \tag{37}$$

Substituting equation (36) into (37) we obtain

$$\xi_n - 2S\left(\frac{1-e^{-\sqrt{-1}q\Delta x}}{1+e^{-\sqrt{-1}q\Delta x}}\right)^2 \xi_n = \alpha\xi_{n-1} + 2S\left(\frac{1-e^{-\sqrt{-1}q\Delta x}}{1+e^{-\sqrt{-1}q\Delta x}}\right)^2 \xi_{n-1} - \sum_{k=2}^{n}\omega_k^\alpha \xi_{n-k}, \tag{38}$$

where $S = \frac{\Delta t_n^\alpha}{\Delta x_i^2}$.

Using the identity

$$e^{\sqrt{-1}q\Delta x} = \cos(q\Delta x) + \sqrt{-1}\sin(q\Delta x)$$

and

$$e^{-\sqrt{-1}q\Delta x} = \cos(q\Delta x) - \sqrt{-1}\sin(q\Delta x)$$

we get

$$\xi_n = \frac{(\alpha-\mu)\xi_{k-1} - \sum_{k=2}^{n}\omega_k^\alpha \xi_{n-k}}{(1+\mu)}, \tag{39}$$

where

$$\mu = \frac{2S\sin^2(q\Delta x)}{\left(1+\cos(q\Delta x)\right)^2} \geq 0.$$

Proposition 2.2. We assume that ξ_n (n =1, 2,…, N) is the solution of equation (39). We have

$$|\xi_n| \leq |\xi_0|, \quad n = 1,2,\ldots,\mathrm{N}.$$

Proof. Following [21], the proof is by using the mathematical induction method. Begin with n = 1 from equation (39) to get

$$\left|\xi_1\right| \le \frac{\alpha-\mu}{1+\mu}\left|\xi_0\right| \le \left|\xi_0\right|,$$

where $\mu \ge 0$ and $0 < \alpha < 1$. Now suppose that

$$\left|\xi_m\right| \le \left|\xi_0\right|, \qquad m = 1, 2 \ldots n\text{-}1,$$

then from Lemma 2.1 and equation (34)

$$\begin{aligned}
\left|\xi_n\right| &\le \frac{\alpha-\mu}{1+\mu}\left|\xi_{n-1}\right| + \frac{1}{1+\mu}\sum_{k=2}^{n}\left|\omega_k^{(\alpha)}\right|\left|\xi_{n-k}\right| \le \left[\frac{\alpha-\mu}{1+\mu} + \frac{1}{1+\mu}\sum_{k=2}^{n}\left|\omega_k^{(\alpha)}\right|\right]\left|\xi_0\right| \\
&\le \left[\frac{\alpha-\mu}{1+\mu} + \frac{1}{1+\mu}\left(\sum_{k=1}^{n}\left|\omega_k^{(\alpha)}\right| - \left|\omega_k^{(\alpha)}\right|\right)\right]\left|\xi_0\right| \\
&= \left[\frac{\alpha-\mu}{1+\mu} + \frac{1}{1+\mu}\left(-\sum_{k=1}^{n}\omega_k^{(\alpha)} - \alpha\right)\right]\left|\xi_0\right| \le \left[\frac{\alpha-\mu}{1+\mu} + \frac{1}{1+\mu}\left(1-\alpha\right)\right]\left|\xi_0\right| \\
&= \frac{1-\mu}{1+\mu}\left|\xi_0\right|.
\end{aligned}$$

Hence $\left|\xi_n\right| \le \frac{1-\mu}{1+\mu}\left|\xi_0\right| \le \left|\xi_0\right|$ for all $\mu \ge 0$.

Therefore, according to Von Neumann's criterion for stability and the earlier proposition, the fractional implicit Keller's box method is unconditionally stable.

3 Numerical Experiment

Let us consider the equation from Takaci et al. [23]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{2e^x t^{2-\alpha}}{\Gamma(3-\alpha)} - t^2 e^x; \quad \alpha = 0.5, \tag{40}$$

with the initial condition

$$u(x,0) = 0, \tag{41}$$

and the boundary conditions

$$u(0,t) = t^2; \;\; u(1,t) = et^2; \quad 0 < x < 1, 0 < t < T. \tag{42}$$

The exact solution of the equation (40) is

$$u(x,t) = e^x t^2. \quad (43)$$

We perform the fractional implicit Keller Box method to solve the example in (40) for $0 < x < 1$ and $0 < t < T$ with Grünwald-Letnikov formula and compare its solution with the exact solution.

Table 1 illustrates the average of the relative errors of the fractional Keller Box method at different values of α . In this problem and for the parameter values considered, the fractional implicit scheme produced reasonable results and it seems to be more accurate without significant loss of computation efficiency.

Table 1: Relative errors of fractional Keller Box method at $\Delta t = 1.25\times 10^{-5}$ and $\Delta x = 0.1$ 0.5

x	$a = 0.25$	$a = 0.5$	$a = 0.75$
0.1	2.44378×10^{-2}	3.37166×10^{-2}	3.30780×10^{-2}
0.2	3.84922×10^{-2}	4.77930×10^{-2}	2.52667×10^{-2}
0.3	4.63447×10^{-2}	5.05938×10^{-2}	2.64988×10^{-2}
0.4	4.93784×10^{-2}	5.11759×10^{-2}	2.66240×10^{-2}
0.5	4.87165×10^{-2}	5.12358×10^{-2}	2.62797×10^{-2}
0.6	4.49973×10^{-2}	5.10351×10^{-2}	2.66942×10^{-2}
0.7	3.84555×10^{-2}	4.99032×10^{-2}	2.65544×10^{-2}
0.8	2.90101×10^{-2}	4.61062×10^{-2}	2.46707×10^{-2}
0.9	1.63226×10^{-2}	3.39645×10^{-2}	3.44693×10^{-2}
Average	3.73506×10^{-2}	4.65526×10^{-2}	2.77929×10^{-2}

Figure 2 shows the comparison of $u(x,t)$ values between the approximate solutions using the fractional Keller Box method and the analytical solutions at $a = 0.5$. Thus, the high agreement of the numerical results is clear and remarkable.

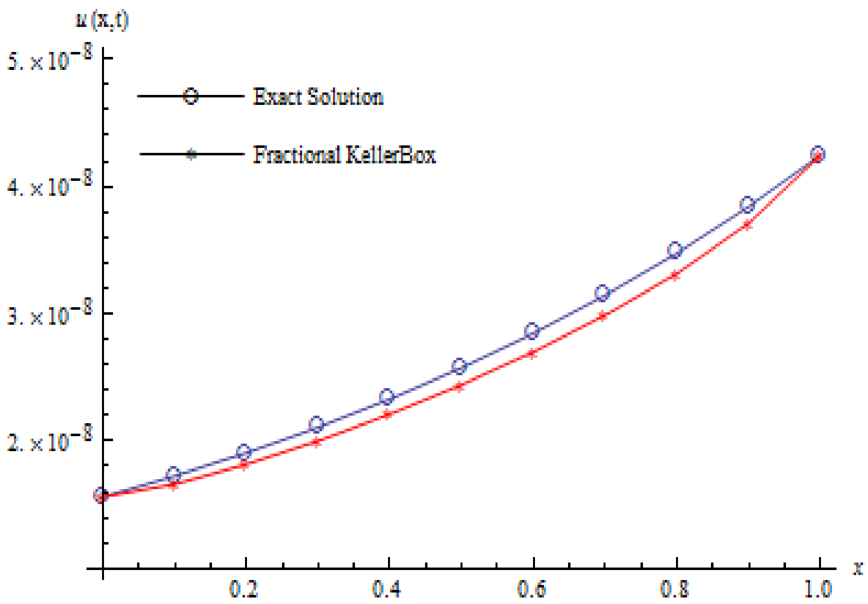

Figure 2: The graph represents the comparison of $u(x,t)$ values with fractional Keller Box method at α = 0.25, 0.5 and 0.75.

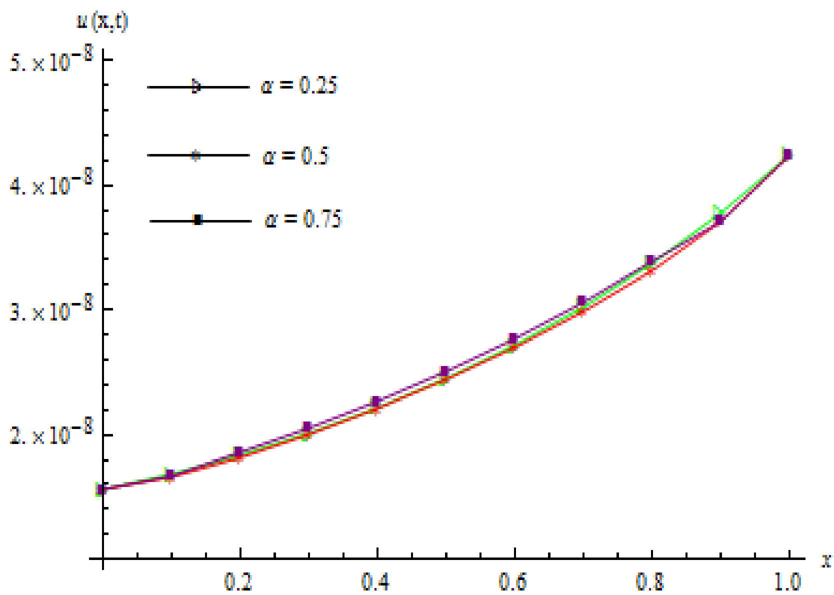

Figure 3: The approximate solution of $u(x,t)$ for the fractional Keller Box method at $\Delta t = 1.25 \times 10^{-5}$ and $\Delta x = 0.1$

4 Conclusion

In this paper, the implicit fractional Keller Box method was developed and used to solve one dimensional time fractional diffusion equation. The method replaces the higher derivatives by the first derivatives of the model problem. The scheme was applied to a problem for which the analytical solution is known and it was found that there was good agremeent between the numerical and analytical results and that the implicit fractional Keller Box method is a feasible scheme.

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References

[1] M.J. Saxton, A Biological Interpretation of Transient Anomalous Subdiffusion. I. Qualitative Model, *Biophysical Journal*, **92**(4), (2007), 1178-1191.

[2] M. Gandhi Mohan et al., *The Physics of Foraging*: *An Introduction to Random Searches and Biological Encounters*, Cambridge University Press, 2011.

[3] S. Das, *Functional Fractional Calculus*, 2th Ed., Verlag Berlin Heidelberg-Springer, 2011.

[4] E.L. Geneston, *Emergence of Complexity from Synchronization and Cooperation,* PhD dissertation, University of North Texas, United States, May 2008.

[5] X. Zhao and Z. Sun, A box-type scheme for fractional sub-diffusion equation with Neumann boundary conditions, *Journal of Computational Physics,* **230**(15), (2011), 6061-6074.

[6] S.B. Yuste, Weighted average finite difference methods for fractional diffusion equations, *Journal of Computational Physics*, **216(**1), (2006), 264-274.

[7] I. Karatay, S.R Bayramoglu and A. Sahin, Implicit difference approximation for the time fractional heat equation with the nonlocal condition, *Applied Numerical Mathematics*, **61**(12), (2011), 1281-1288.

[8] A. Diego Murio, Implicit finite difference approximation for time fractional diffusion equations, *Computers and Mathematics with Applications*, **56**, (2008), 1138-1145.

[9] K. Al-Khaled and S. Momani, An approximate solution for a fractional diffusion-wave equation using the decomposition method, *Applied Mathematics and Computation*, **165**, (2005), 473-483.

[10] N.A. Khan et al., Numerical study of time-fractional fourth-order differential equations with variable coefficients, *Journal of King Saud University-Science*, **23**(1), (2011), 91-98.

[11] H. Brunner, L. Ling and M. Yamamoto, Numerical simulations of 2D fractional subdiffusion problems, *Journal of Computational Physics*, **229**(18), (2010), 6613-6622.

[12] C. Tadjeran and M. M. Meerschaert, A second-order accurate numerical method for the two-dimensional fractional diffusion equation, *Journal of Computational Physics*, **220**(2), (2007), 813-823.

[13] M.M. Meerschaert and C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations, *Applied Numerical Mathematics,* **56**, (2006), 80-90.

[14] H.K. Pang and H.W. Sun, Multigrid method for fractional diffusion equations, *Journal of Computational Physics*, **231**(2), (2012), 693-703.

[15] L. Su, W. Wang and H. Wang, A characteristic difference method for the transient fractional convection-diffusion equations, *Applied Numerical Mathematics*, **61**(8), (2011), 946-960.

[16] N.H. Sweilam, M.M. Khader and A.M. Nagy, Numerical solution of two-sided space-fractional wave equation using finite difference method, *Journal of Computational and Applied Mathematics*, **235**(8), (2011), 2832-2841.

[17] R. Klages, G. Radons and I. M. Sokolov, *Anomalous Transport-Foundations and Applications*, First edition, WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2008.

[18] S.B. Yuste and L. Acedo, An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations, *SIAM Journal of Numerical Analysis*, **42**(5), (2004), 1862-1874.

[19] B. Lakshminarayana, *Fluid dynamics and heat transfer of turbomachinery*, John Wiley & Sons, Inc., 1996.

[20] J.C. Tannehill, D.A. Anderson and R.H. Pletcher, *Computational fluid mechanics and heat transfer,* Second Edition, Taylor & Francis, 1997.

[21] C. Chen, F. Liu and K. Burrage, Finite difference methods and a Fourier analysis for the fractional reaction–subdiffusion equation, *Applied Mathematics and Computation*, **198**, (2008), 754-769.

[22] M.H. Holmes, *Introduction to Numerical Methods in Differential Equations*, Springer, 2007.

[23] D. Takaci, A. Takaci and M. Strboja, On the character of operational solutions of the time-fractional diffusion equation, *Nonlinear Analysis*, **72**(5), (2010), 2367-2374.