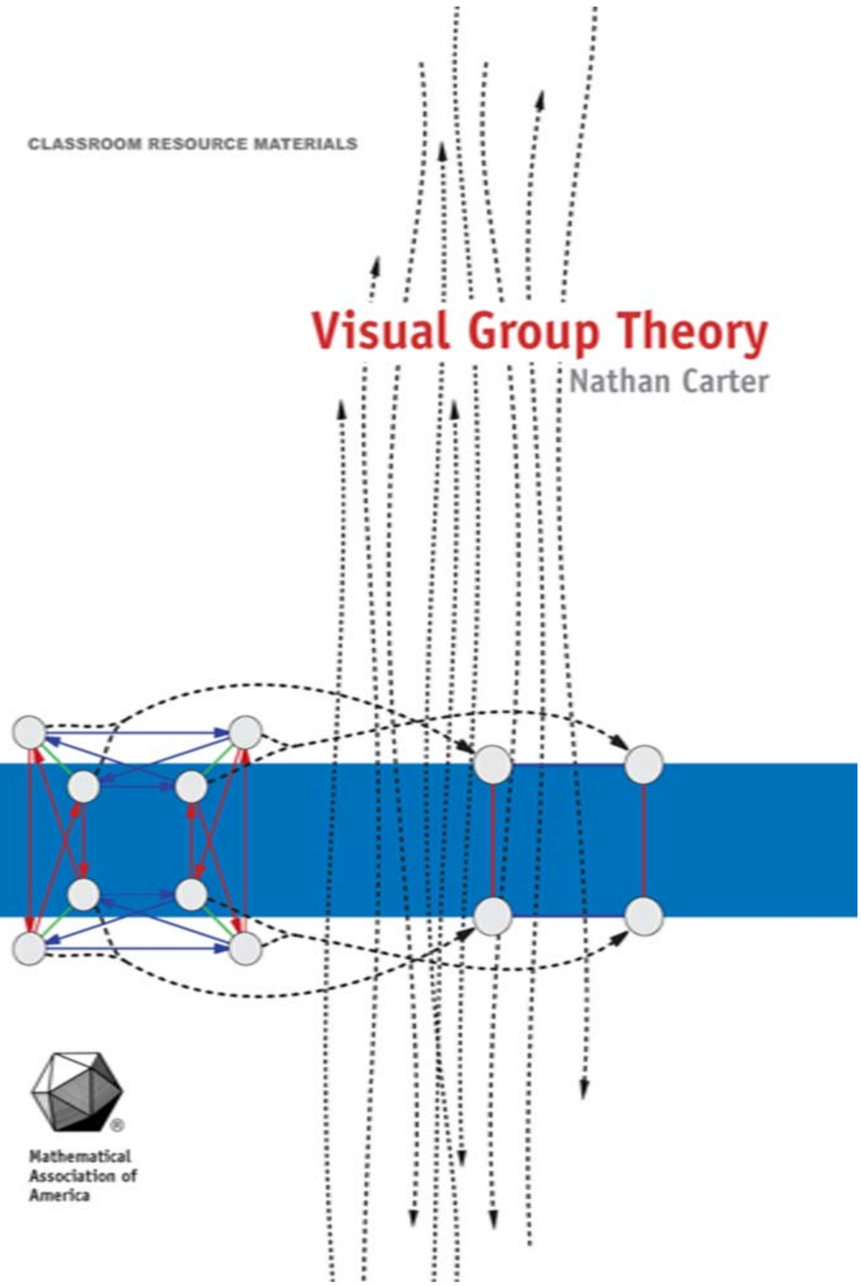
CLASSROOM RESOURCE MATERIALS

Visual Group Theory

Nathan Carter



Mathematical
Association of
America

Visual Group Theory

Library of Congress Catalog Card Number 2009923532

Print edition ISBN 978-0-88385-757-1

Electronic edition ISBN 978-1-61444-102-1

Printed in the United States of America

Current Printing (last digit):
10 9 8 7 6 5 4 3 2 1

Visual Group Theory

Nathan C. Carter
Bentley University

Published and Distributed by
The Mathematical Association of America

CLASSROOM RESOURCE MATERIALS

Classroom Resource Materials is intended to provide supplementary classroom material for students—laboratory exercises, projects, historical information, textbooks with unusual approaches for presenting mathematical ideas, career information, etc.

101 Careers in Mathematics, 2nd edition edited by Andrew Sterrett

Archimedes: What Did He Do Besides Cry Eureka?, Sherman Stein

Calculus Mysteries and Thrillers, R. Grant Woods

Conjecture and Proof, Miklós Laczkovich

Creative Mathematics, H. S. Wall

Environmental Mathematics in the Classroom, edited by B. A. Fusaro and P. C. Kenschaft

Exploratory Examples for Real Analysis, Joanne E. Snow and Kirk E. Weller

Geometry From Africa: Mathematical and Educational Explorations, Paulus Gerdes

Historical Modules for the Teaching and Learning of Mathematics (CD), edited by Victor Katz and Karen Dee Michalowicz

Identification Numbers and Check Digit Schemes, Joseph Kirtland

Interdisciplinary Lively Application Projects, edited by Chris Arney

Inverse Problems: Activities for Undergraduates, Charles W. Groetsch

Laboratory Experiences in Group Theory, Ellen Maycock Parker

Learn from the Masters, Frank Swetz, John Fauvel, Otto Bekken, Bengt Johansson, and Victor Katz

Math Made Visual: Creating Images for Understanding Mathematics, Claudi Alsina and Roger B. Nelsen

Ordinary Differential Equations: A Brief Eclectic Tour, David A. Sánchez

Oval Track and Other Permutation Puzzles, John O. Kiltinen

A Primer of Abstract Mathematics, Robert B. Ash

Proofs Without Words, Roger B. Nelsen

Proofs Without Words II, Roger B. Nelsen

She Does Math!, edited by Marla Parker

Solve This: Math Activities for Students and Clubs, James S. Tanton

Student Manual for Mathematics for Business Decisions Part 1: Probability and Simulation, David Williamson, Marilou Mendel, Julie Tarr, and Deborah Yoklic

Student Manual for Mathematics for Business Decisions Part 2: Calculus and Optimization, David Williamson, Marilou Mendel, Julie Tarr, and Deborah Yoklic

Teaching Statistics Using Baseball, Jim Albert

Visual Group Theory, Nathan C. Carter

Writing Projects for Mathematics Courses: Crushed Clowns, Cars, and Coffee to Go, Annalisa Crannell, Gavin LaRose, Thomas Ratliff, Elyn Rykken

MAA Service Center
P.O. Box 91112
Washington, DC 20090-1112
1-800-331-1MAA FAX: 1-301-206-9789

Acknowledgments

I am grateful to God for life and breath and mathematics, as well as my ability to write it, draw it, and enjoy it. I am grateful to my family, especially Lydia, who put up with my absence during times I needed to work on this manuscript.

Many thanks to Doug Hofstadter, who showed me the power of group theory visualization and supported my work on it in several ways, including much good advice. Many thanks to Charlie Hadlock, who kept me away from not-so-good advice (such as "Don't write a book before you get tenure") and who guided many of the early steps of this project. I also thank Don Albers, the MAA, and Bentley University (particularly Rick Cleary and Kate Davy), all of whom encouraged, supported, and took a chance on a first-time author.

I must also thank those whose mathematical work helped me in mine. One of my early exposures to group theory visualization techniques was Magnus and Grossman's *Groups and Their Graphs.* In writing this book I made frequent reference to the excellent texts by Michael Artin, John Fraleigh, Charles Hadlock, and Thomas Hungerford. The powerful and well-designed software *Asymptote* by John Bowman helped me make the more than 300 figures herein. Thank you!

Lastly, there are many people who read early chapters and drafts and provided much helpful feedback. These include Doug Hofstadter, Jon Zivan, an anonymous referee, my parents, and my Fall 2006 Discrete Mathematics class, especially Kathryn Ogorzalek. Thank you for helping me see my work through your eyes, and improve it.

Preface

If you are interested in learning about group theory in a relaxed, intuitive way, then this book is for you. I say learning *about* group theory because this book does not aim to cover group theory comprehensively. Herein you will find clear, illustrated exposition about the basics of the subject, which will give you a solid foundation of intuitions, images, and examples on which you can build with further study.

This book is ideal for a student beginning a first course in group theory. It can be used in place of a traditional textbook, or as a supplement to one, but its aim is quite different than that of a traditional text. Most textbooks present the theory of groups using theorems, proofs, and examples. Their exercises teach you how to make conjectures about groups and prove or refute them. This book, however, teaches you to *know* groups. You will see them, experiment with them, and understand their significance. The mental library of images and intuitions you gain from reading this book will enable you to appreciate far better the facts and proofs in a traditional text.

This book is also appropriate for recreational reading. If you want an overview of the theory of groups, or to learn key principles without going as deep as some upper-level undergraduate mathematics courses, you can read this book by itself. Only a typical high school mathematics education is assumed, but you should have a willingness to think analytically.

My work on this book stems from *Group Explorer*, a software package I wrote that creates illustrations for finite groups, and allows the user to interact and experiment with them. Many of the illustrations in this text were generated with the help of *Group Explorer*, and the investigations possible in *Group Explorer* can help you with some of this book's exercises.

You do not need *Group Explorer* to benefit from this book; very few exercises specifically direct you to *Group Explorer*. But I recommend taking full advantage of hands-on, interactive learning experiences when they're available; the more involved we are, the more we tend to learn. *Group Explorer* is free software, available for all major operating systems. You can find it online at

`http://groupexplorer.sourceforge.net`.

Contents

Overview

I highlight here three essential aspects of this book's nonstandard approach to group theory, and briefly discuss its organization.

First and foremost, images and visual examples are the heart of this book. There are more than 300 images, an average of more than one per page. The most used visualization tool is Cayley diagrams (defined in Chapter 2) because they represent group structure clearly and faithfully. But multiplication tables and objects with symmetry also appear regularly, and to a lesser extent cycle graphs, Hasse diagrams, action diagrams, homomorphism diagrams, and more. As you can tell by flipping through the pages, visualization is the name of the game.

Second, I focus more on finite groups than on infinite ones. This is partially because they are easier to diagram, but more so because they give a solid foundation of intuition for group theory in general. Understanding finite groups well makes the infinite a natural generalization. This approach sacrifices little, if anything, because so much remains to study in the realm of the finite. I cover the most common infinite groups, and each chapter's exercises includes some on infinite groups.

Lastly, this book approaches groups from the opposite direction of many traditional textbooks. The usual paradigm defines a group as a set with a binary operation, and later proves Cayley's Theorem, that every group is a collection of permutations (or you could say that every group acts on some set, most notably itself). The traditional definition does not appear in this book until Chapter 4; I define groups in Chapter 1 as collections of actions, and later prove that they can also be thought of as sets with binary operations. This nonstandard paradigm facilitates my introduction in Chapter 2 of Cayley diagrams, which depict groups as collections of actions.

The book's structure is linear, to be read in order; later sections usually depend on earlier ones. There are two exceptions. Chapter 5, which gives broad exposure to finite groups, is helpful but not strictly necessary for what follows. You could skip most of it (except the definition of abelian in Section 5.2) and turn back to it later as needed. The other exception is that Chapter 10 depends only slightly on Chapter 9; Cauchy's Theorem from Section 9.2 is used in Section 10.7, and the remainder of Chapter 9 may be useful in a few of the exercises in Chapter 10.

Chapter 10, on Galois theory, aims to show the power of group theory and some of its historical roots. It includes an introduction to fields, but several theorems are stated without proof. It gives enough understanding to see how group theory ties in, and points the reader elsewhere for more details on fields. The beautiful and historic result of the unsolvability of the quintic is the focus and endpoint of that final chapter.

1

What is a group?

1.1 A famous toy

In 1974, Ernö Rubik of Budapest, Hungary unleashed his fascinating invention called Rubik's Cube. It infiltrated popular culture, appearing in feature films, inspiring competitions, and captivating children and geniuses alike. Mathematics journals carried research articles analyzing the cube and its patterns. Those unable to solve the cube could learn solutions from any of dozens of books.

A new Rubik's Cube comes out of the box looking like Figure 1.1. Each face of the cube contains nine smaller faces of smaller cubes, with the colors arranged to agree. You begin playing with the cube by rotating its faces to mix up the colors. Figure 1.2 shows how two different rotations in succession begin to disorder the cube. After playing with a cube idly for a few minutes, an innocent customer finds that the colors are completely shuffled, and there is no obvious way of returning to the original, pristine state. The challenge of Rubik's Cube is to restore a scrambled cube to its original state.

Rubik's Cube has a unique flavor among puzzles and games, and much of this flavor comes from the mathematical patterns inherent in the cube itself. At first the cube may

Figure 1.1. An untouched Rubik's Cube

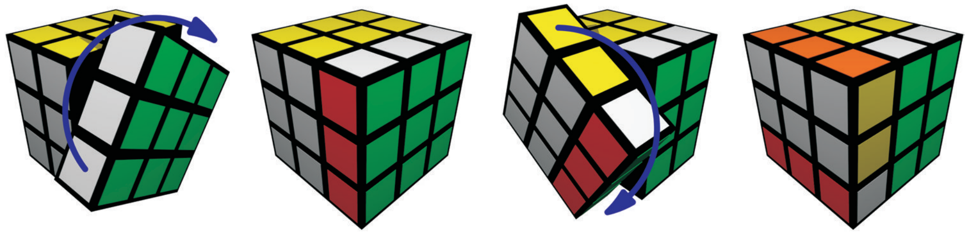


Figure 1.2. The leftmost cube shows the green face rotating 90 degrees clockwise; the next cube shows the result of that move. The third cube shows the white face rotating 90 degrees clockwise; the final cube shows the result of that move.

not seem very mathematical, given that it doesn't seem to require any skill with numbers, equations, or quantities in order to play with it, or even to solve it. But that is because the mathematics covered in this book, group theory, is not primarily about numbers; it is about patterns. Group theory studies symmetry, which can be found not only in the cube itself, but also in the patterns of its movements.

Marketing slogans like "Easy to learn, a challenge to master" adorned several puzzles invented by Ernö Rubik. This tagline certainly describes the cube, and we can get an excellent start to our group theory studies by examining why this is so. In this chapter we will examine those aspects of the cube that make it easy to learn and in Chapter 2 we will look at what makes it a challenge to master.

1.2 Considering the cube

To explore why Rubik's Cube is easy to learn, let's make several observations.

In part, the cube is easy to play with because users don't need to learn a complex list of rules. In contrast, players wanting to play chess must first learn the different rules for each piece. Furthermore, some of the available moves may become unavailable to the player as the game progresses. Rubik's Cube does not have this type of intricacy; we might even say it has only six moves—rotating any one of the six faces 90 degrees clockwise.[1] By combining these six moves, players can explore the full (enormous) gamut of cube configurations. This accessibility leads us to our first observation about the cube.

Observation 1.1. There is a predefined list of moves that never changes.

Another agreeable aspect of Rubik's Cube is that it is somewhat forgiving of mistakes. If a player rotates a face and immediately regrets the move, no great harm has been done. The player can simply rotate the face the other way to undo the mistake. Let us make this our second observation.

Observation 1.2. Every move is reversible.

Another helpful aspect of the cube is a bit less noticeable, but becomes clearer in contrast with other games: Rubik's Cube is not a game of chance. A tennis player may have every intention of hitting a great shot, but fail to do so because their body does not execute their wishes. A poker player may have an excellent strategy, but not win because of the cards they were dealt. Unpredictability and chance give a certain flavor to a game, and

[1]Although a player can also rotate faces 90 degrees counterclockwise, that movement is the same as three sequential 90 degree turns clockwise.

Rubik's Cube is devoid of that flavor. The turning of a face of the cube has a predictable outcome, depending on neither skill nor luck. An action whose outcome can be determined in advance, free from influences that are random or uncertain, is called deterministic.

Observation 1.3. Every move is deterministic.

But we must be fair. On their own, these observations seem to imply that the cube has no complexity. To give an accurate account, we must admit that the seemingly limitless combinations of moves make the cube challenging. Thus we balance our first three observations with one more.

Observation 1.4. Moves can be combined in any sequence.

Note the impact of Observation 1.2 on Observation 1.4. If a player were so meticulous as to write down carefully every move that he or she did, then even a long sequence of moves could be carefully reversed one at a time.

1.3 The study of symmetry

We could add many other observations to the four above. We could mention the colors of the faces, the physical construction of the cube, the number of moves it typically takes to arrive at a solution, or any other aspect of the puzzle or the experience of playing with it. But we will concentrate on the four above because they highlight those aspects of the cube that are most relevant to this book. Group theory is the study of symmetry, and we will see how Observations 1.1 through 1.4 describe the symmetry in Rubik's Cube.

All cubes are symmetrical in that they have all sides the same, all angles the same, and all edges the same. But Rubik's Cube has the added intricacy of its moving parts. The set of possible motions and configurations of the smaller cubes that make up the whole also has a great deal of symmetry. But to clearly explain this, we need to know what "symmetry" means.

This brings us to the gateway of this book, because group theory is the branch of mathematics that answers the question, "What is symmetry?" The first three chapters of this book give a careful answer to that question by introducing groups and showing how they describe symmetry. That introduction is already underway—Rubik's Cube is the first example we will see of objects and patterns that exhibit symmetry. As we analyze them, we will be learning group theory. Our explorations in future chapters will lead us to other practical examples of symmetry, including ones as diverse as molecular crystals and dancing.

But we will not limit ourselves only to seeing examples of symmetrical objects. Our investigations will use many other tools as well; the purpose of this book is to focus on tools that are *visual.* This includes common group theory tools such as the multiplication table as well as less common ones such as Cayley diagrams and cycle graphs. I will introduce each of these in chapters to come.

Beginning in Chapter 2, you will see examples of objects that have symmetry. Some of them are physical objects, some of them are actions or behaviors of physical objects, and some of them are purely imaginary situations. But what they have in common is that Observations 1.1 through 1.4 apply to all of them. Group theory studies the mathematical consequences of those observations, and therefore can help answer interesting questions about symmetrical objects. For instance, how many different configurations does Rubik's

Cube have? Although we will not do the computation of that number in this book, interested readers can refer to [21].

1.4 Rules of a group

Let's therefore return to those observations on which group theory is founded. But instead of considering them as descriptions of Rubik's Cube, let's rephrase them as rules that will define the boundaries of our study. This is how mathematical subject areas are typically introduced. A set of rules is laid down, then mathematicians proceed to study things that obey those rules. The mathematical term for such rules is *axioms,* but I will call them rules as long as our discussion remains informal.

Laying down such rules has some noteworthy advantages. First, they make the boundaries of our study clear. Things that fit the rules introduced below are part of the study of group theory, and things that don't are not. Second, by agreeing on the rules in advance as a mathematical community, we can ensure that we all have the same ideas when we discuss the subject. In other words, codifying our ideas into rules helps us be sure we are speaking the same language and minimizes communication problems. Third, we can use the rules as a basis from which to make logical deductions, and thereby unearth new facts (in our case, facts about symmetry) that we may not have anticipated from the rules alone. The computation mentioned above about Rubik's Cube depends on just such facts. You will do some of these deductions yourself in the exercises for this chapter.

So let us rephrase the *observations* about Rubik's Cube into *rules*. As mentioned above, think of these rules as the requirements that something must meet to belong in our study of symmetry.

Rule 1.5. There is a predefined list of actions that never changes.

Rule 1.6. Every action is reversible.

Rule 1.7. Every action is deterministic.

Rule 1.8. Any sequence of consecutive actions is also an action.

My rephrasing involved making only two changes. First, I changed the word *move* to the word *action* to make it sound less like a game. In the Rubik's Cube context, the former wording was appropriate, but we do not want to restrict our attention only to games. Second, when writing Rule 1.8, I rephrased Observation 1.4 to make the terminology clearer; not only is every sequence of actions possible, but we will go so far as to call every sequence an action in its own right. This does not mean that such an action has to appear on the list required by Rule 1.5, but rather that the (usually short) list in Rule 1.5 is a starting point from which we can create new actions using Rule 1.8, that is, by chaining actions together into sequences.

In the case of Rubik's Cube, although I list only six basic actions, you can form many actions by combining these six. As a simple example, combining two 90-degree rotations of the front face in sequence creates a new action, a 180-degree rotation of the front face. More complicated examples can be created from sequences of three or four or more different basic actions chained together. The standard way to say this is that Rule 1.5 gives us actions that generate all the others, and are therefore called ***generators.***

We are now ready for our first definition, which marks your passage into the study of group theory. This is not the usual mathematical definition of a group, but it expresses the same idea as the usual one, as we will see later. For now, let's call this our unofficial definition.

Definition 1.9 (group, unofficially)**.** A ***group*** is a system or collection of actions satisfying Rules 1.5 through 1.8.

You may now be wondering what things besides Rubik's Cube fit this definition. What other things deserve to be called groups? As we will see in Chapters 2 and 3, anything in which symmetry arises will satisfy Rules 1.5 through 1.8, including important examples from science, art, and mathematics. But for now, the following exercises will allow you to play with the rules themselves a bit, and see a few simple contexts in which they apply. They will give you a firmer grasp on the meaning of the rules before we proceed.

1.5 Exercises

The following exercises are thought experiments to help you understand the concepts just discussed. With mathematics and similar scientific endeavors, the exercises usually require some thought, and therefore take more time than the reading itself. Although you can appreciate the material with a quick reading, you can know it intimately only with lengthier consideration. Therefore, do not be discouraged if the exercises take some time; this is normal.

Also, feel no shame looking up the answers to a few exercises in the Appendix to get the idea for how to complete them. Because these are not typical mathematical exercises, they may seem unfamiliar or strange. Therefore looking up an answer or two to get the hang of them is a reasonable strategy.

1.5.1 Satisfying the rules

Exercise 1.1. Place a penny and a nickel side by side on a table or desk. Consider just one action, the action of swapping the places of the two coins. Is this a group? (Check each of the four rules to see if they are true for this situation. Explain your conclusion.)

Exercise 1.2. Consider the same situation as in Exercise 1.1, but add a dime to the right of the other two coins. The only action is still that of swapping the places of the penny and nickel. Is this a group?

Exercise 1.3. Imagine that you have five marbles in your left pocket. Consider two actions, moving a marble from your left pocket into your right pocket and moving a marble from your right pocket into your left pocket. Is this a group?

Exercise 1.4. Three walls in your bedroom hold pieces of art, one hung on each wall. You are rearranging them to see which arrangement best suits your taste. You cannot use the fourth wall, because it has a window.

(a) Count the number of ways there are to rearrange the pictures, as long as only one is hung on each wall.

(b) Consider two actions: You may swap the art on the left wall with the art on the center wall, and you may swap the art on the center wall with the art on the right wall. Can these two actions alone generate all of the configurations you counted?

(c) Does part (b) describe a group? If not, what rule or rules were broken?

(d) Now add a new action, moving all art from the right wall to the center wall, even if this causes there to be more than one piece of art there. Is this new situation a group? If not, what rule or rules were broken?

1.5.2 Consequences of the rules

Exercise 1.5. Does Rule 1.8 imply that every group must contain infinitely many actions? Explain your reasoning carefully.

Exercise 1.6. For each of Exercises 1.1 through 1.4 that actually described a group, determine exactly how many actions there are in that group. That is, do not count only the generators, but all possible actions obtainable using Rule 1.8.

Exercise 1.7. Consider again the situation from Exercise 1.1.

(a) Only one action was given. By Rule 1.8, performing this action twice in a row is also a valid action. Describe it.

(b) Will every group have an action like this? Explain your reasoning carefully.

Exercise 1.8. For each of the following requirements (a) through (e), devise a group that meets that requirement. (Groups always satisfy Rules 1.5 through 1.8; the requirements below are additional ones just for this exercise.) If some groups you've already encountered fit these requirements, you may feel free to reuse them.

(a) The order in which actions are performed impacts the outcome.

(b) The order in which actions are performed does not impact the outcome.

(c) There are exactly three actions.

(d) There are exactly four actions.

(e) There are infinitely many actions.

Exercise 1.9. Can you devise a plan for creating a group with any given number of actions? (I do not mean a plan for creating a group with a specified number of generators, but rather a specified number of actions, once Rule 1.8 has been applied.)

1.5.3 Breaking the rules

Rules 1.6 through 1.8 would not make much sense without Rule 1.5, because it introduces the notion of a list of actions. But for each of the other rules, we can ask what it would be like if that rule were not present.

Devising a context in which all but one of the four rules is true shows something significant—that the missing rule is not redundant. For instance, let's say someone suggested to you that Rule 1.8 could be dropped, because the combination of the other three rules implies that Rule 1.8 must be true. By finding a context in which the other three rules are true, but Rule 1.8 is false, you can prove that the combination of Rules 1.5, 1.6, and 1.7 does not imply the truth of Rule 1.8. Thus by doing Exercises 1.10 through 1.12, you will have shown that all the rules are necessary; that is, none are redundant.

Exercise 1.10. Devise a situation that satisfies all of the four rules except Rule 1.6.

Exercise 1.11. Devise a situation that satisfies all of the four rules except Rule 1.7.

Exercise 1.12. Devise a situation that satisfies all of the four rules except Rule 1.8.

1.5.4 Groups of numbers

Since group theory is a mathematical subject, it should not surprise us that numbers can be formed into groups, in various ways.

Exercise 1.13. Pick any whole number and consider this set of actions: adding any whole number to the one you chose. This is an infinite set of actions; we might name them things like "add 1" and "add −17." Is it a group? If so, how small a set of generators can you find?

Exercise 1.14.

(a) What would be the answer to the previous exercise if we allowed only even numbers?

(b) What would be the answer to the previous exercise if we allowed only whole numbers from 0 to 10?

(c) What would be the answer to the previous exercise if we allowed all whole numbers, but changed the set of actions to *multiplying* by any whole number? (I do not ask you to consider those actions as generators, but as the complete set of actions.)

(d) What would be the answer to the previous exercise if we allowed only the numbers 1 and −1, and just two actions, multiplying by either 1 or −1?

2

What do groups look like?

2.1 Mapmaking

Chapter 1 introduced group theory by examining those properties of Rubik's Cube that make it attractive to beginners. Let us now investigate those aspects of the cube that make it difficult to solve.

When you have in your hands an unsolved Rubik's Cube, and you do not know a method for solving it, experimentally twisting the faces quickly begins to seem pointless. You are wandering aimlessly through the multitude of possible configurations of the cube. Somewhere in this enormous wilderness of jumbled cube configurations is the one oasis you want to find—the solved cube. But without any idea of where you are, where the oasis is, or what direction you're walking as you make moves, the situation is (almost) completely hopeless.

What would be very helpful to someone lost in the endless reaches of Rubik's wilderness is a map—a reference that could say, "You are here," and "The solution is over here," and show you the path to get there. Such a map would eliminate all your navigational problems. Anyone could solve the cube quickly with such a map (although it may not be much fun to do so).

Many books on Rubik's Cube teach solution techniques, and some academic papers use group theory to provide detailed analyses of the cube [21]. My purpose is not to duplicate their efforts here by providing you with a map of Rubik's wilderness. Instead, I want to conduct a thought experiment about what a fully detailed map of that wilderness would be like. Such a map is not just a technique for getting to the solved state, but a complete map of *every* configuration of the cube and how those configurations relate to one another. Since all this information will obviously not fit on an ordinary-sized map, let's say we'll put it in a big book. Because this is only a thought experiment, let's call our book *The Big Book*, and consider what this book will need to contain.

Imagine that you have both a jumbled Rubik's Cube and a copy of *The Big Book*. *The Big Book* should first help you determine where you are in the wilderness. To make this possible, let's organize *The Big Book* to have one cube configuration on each page, and let's order the pages in a dictionary-like way. Just as all the words starting with "a" in the dictionary are together, perhaps all cubes with a red sticker on the leftmost square of their top face would be together. Within that group of cubes, another sticker's color would organize, like the second letter of a dictionary word, and so on. The details of such an organization are not important, because this is only a thought experiment, after all. Let's agree that we could come up with such an organization if necessary, enabling the readers of *The Big Book* to look up their jumbled cubes in the book (with a good dose of patience and care).

Now let's say you have found the page in *The Big Book* showing your cube's configuration. That page should provide some navigational help, the details of which are best shown by an example. Figure 2.1 shows a page from *The Big Book*. The page shows all faces of a certain cube configuration, front and back, and tells the reader how far that cube is from being solved. A table then provides navigational help; it tells the reader what each available move would accomplish. For instance, the first row of the table tells us that twisting the front face of the cube clockwise would result in the configuration on page $36,131,793,510,312,058,964$ of the book, which is closer to the solution than the configuration shown on the current page.

You can see how you could use such a book to solve your cube. Once you have found the page on which your cube is shown, simply make a move that puts you closer to being solved, and turn to the corresponding page. Then repeat the process from there. You always know how far you are from the solution, and you can always compare your cube against the picture on the page to be sure you haven't made a navigational error.

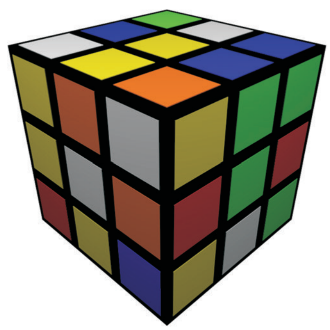
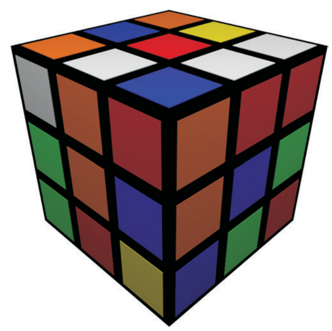
The Big Book has one significant problem, which you may have already noticed from Figure 2.1. Rubik's Cube has more than 4×10^{19} potential configurations, which would make a prohibitively large book. We might suggest putting more than one cube state per page, but in fact even if we could fit one cube state in a square inch, the amount of paper needed to print the book would cover the surface of the earth many times over. Storing the book electronically won't help either; even using a very efficient encoding scheme, no computer in existence at the time of this writing has sufficient storage to contain it.[1] So *The Big Book* is a thought experiment only, but the remainder of this chapter will show how valuable a thought experiment it has been.

The most important thing for *The Big Book* to teach us is that a map of Rubik's wilderness is really a map of a particular group. Let me explain. The moves in Rubik's Cube form a group because they satisfy Definition 1.9. (In fact, those moves were our first example of a group, motivating that very definition.) The rules in Definition 1.9 refer only to the cube's moves, and combinations thereof. Because *The Big Book* contains complete data on the moves in Rubik's Cube and how they combine, it is a map of *the group constituted by those combinations*.

So the mapmaking ideas introduced by our discussion about *The Big Book* do not need to be abandoned simply because the group exhibited by Rubik's Cube is too large. We can use the same ideas to map out any group, and in the next section we do just that.

[1]To store only the data (from which to later reconstruct text and images) would require approximately 10^{21} bytes, or one billion terabytes.

Page 12, 574, 839, 257, 438, 957, 431

Cube front	Cube back
	

You are **15** steps from the solution.

Face	Direction	Destination page	Progress
Front	Clockwise	36, 131, 793, 510, 312, 058, 964	Closer to solved
Front	Counterclockwise	12, 374, 790, 983, 135, 543, 959	Farther from solved
Back	Clockwise	26, 852, 265, 690, 987, 257, 727	Closer to solved
Back	Counterclockwise	41, 528, 397, 002, 624, 663, 056	Farther from solved
Left	Clockwise	6, 250, 961, 334, 888, 779, 935	Closer to solved
Left	Counterclockwise	10, 986, 196, 967, 552, 472, 974	Farther from solved
Right	Clockwise	26, 342, 598, 151, 967, 155, 423	Farther from solved
Right	Counterclockwise	40, 126, 637, 877, 673, 696, 987	Closer to solved
Top	Clockwise	35, 275, 154, 257, 268, 472, 234	Closer to solved
Top	Counterclockwise	33, 478, 478, 689, 143, 786, 858	Farther from solved
Bottom	Clockwise	20, 625, 256, 145, 628, 342, 363	Farther from solved
Bottom	Counterclockwise	7, 978, 947, 168, 773, 308, 005	Closer to solved

Figure 2.1. Sample page from *The Big Book*

2.2 A not-so-famous toy

Allow me to introduce you to a puzzle much like Rubik's Cube. This puzzle has been around much longer, but has a much less impressive name. It is called *the rectangle,* and I'm sure you've heard of it before! Even though nearly everyone's mathematical education includes rectangles at some early point, I provide a useful illustration in Figure 2.2, with the corners of the rectangle conveniently numbered. The first thing to notice about this puzzle is that it is much less complicated than Rubik's Cube, and so it might cooperate better with our mapmaking aspirations.

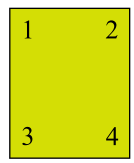


Figure 2.2. A rectangle with its corners numbered

But first, to fully benefit from the upcoming discussion of the rectangle, you should make one for yourself. Take an ordinary sheet of paper and label it with numbers in the corners as shown in Figure 2.2. Making your own rectangle may sound unnecessary, but we will be investigating flips and twists of this object in space, which are difficult to picture accurately with the mind's eye alone. So go ahead and grab a piece of paper and make your own personal (numbered) rectangle.

Here are the rules for the rectangle puzzle. Begin with the rectangle flat on a table, on a desk, or in your lap so that you can read the four numbers, as in Figure 2.2. Like the Rubik's Cube, the rectangle puzzle starts out in its solved state—the way you have it now. You will mix up the puzzle and your job will then be to return it to this original state.

The rectangle puzzle has two legal moves. You can flip the paper over horizontally and you can flip it over vertically, as shown in Figure 2.3.[2] In Chapter 3 I'll talk about

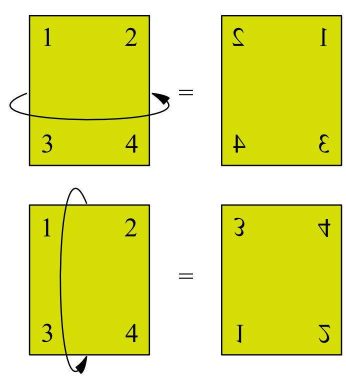


Figure 2.3. The top arrow illustrates a horizontal flip; the upper right rectangle shows the result of such a flip. The bottom arrow illustrates a vertical flip; the lower right rectangle shows the result of such a flip.

[2] Readers familiar with axes of rotation may consider the naming convention in Figure2.3 backwards. Because I have not yet introduced axes of rotation, I use a simpler naming convention. "Horizontal" and "vertical" describe the motion of the player's hands, which follow the arrows in Figure 2.3.

why these moves (and not others) make sense for the rectangle puzzle, but for now, let's just call them the rules of the game. As with Rubik's Cube, you may feel free to repeat and combine moves as much as you like. (In fact, you may have already noticed that the moves in the rectangle puzzle form a group.)

Take a moment now to mix up your rectangle puzzle, and then solve it. This should not take very long, but be sure you do not (accidentally) cheat! A natural mistake is to pick up the rectangle and inspect it a bit, rotating it freely and experimentally until it is solved. This move is rarely valid, and thus is not what you're supposed to be doing. Making such a move is analogous to disassembling Rubik's Cube and reassembling it solved. Remember that you are limited to the two moves in Figure 2.3 as you mix up the puzzle *and* as you solve it.

Let's take a moment to verify that the moves in the rectangle puzzle form a group, and to compare it to Rubik's Cube. Rule 1.5 requires a predefined list of moves, which we have given: horizontal and vertical flips. Rule 1.6 requires that each move be reversible. It holds true for the rectangle because in fact each move undoes itself. For example, if you have performed a horizontal flip and wish to return to the previous state, simply perform another horizontal flip. The same is true of vertical flips. Rule 1.7 requires that these moves be deterministic, free from randomness, and they are because they are completely within your control. Rule 1.8 requires that any possible combination of moves also be a valid move. This rule is satisfied because a horizontal or vertical flip puts the rectangle into a physical position from which either valid move is still possible. In contrast, if we were to add the move "rip up the rectangle and throw it away," that would bring about a dead end; no subsequent moves would be possible. Neither of our two moves has this problem; they always permit further moves and thus allow us to string them together in any sequence.

In the next section we will map out the rectangle puzzle's group. But first, notice how its physical construction contrasts with that of Rubik's Cube. The rectangle puzzle requires you to place the rectangle on a flat surface, to remember its original orientation, and to try to return to it. Rubik's Cube is not this way; you can toss it around the room, drop it in your sock drawer, and when you come back to it, its configuration will not have changed. The cube's moving parts internalize the puzzle, so that no external reference (to a table, or an original orientation) is required. Because the rectangle puzzle is a simple do-it-yourself rectangle made of paper, it has no moving parts and thus to use it as a puzzle required external landmarks (the table and the original orientation you remembered). We could design a puzzle that behaves identically to the rectangle puzzle, but which is self-contained; it would just not be as easy to make at home.

2.3 Mapping a group

Mapmaking begins with exploration. We need to know the lay of the land if we are to draw a useful map of it. Let's therefore explore the realm of the rectangle puzzle, and map it out as we go. We will need to ensure that our exploration is thorough—that we find all possible configurations of the rectangle puzzle—so we should conduct our search systematically.

Begin with the rectangle puzzle in its solved state (Figure 2.2). Our two moves (horizontal and vertical flips) are our only means for exploring. They are the group's generators, and our map will show *how* they generate the group.

Start exploring by performing a horizontal flip. Because our exploration has just begun, this of course leads us to a configuration we have not yet mapped. But it is a configuration that was originally introduced in Figure 2.3. Performing another horizontal flip returns the rectangle to its original, solved state. Therefore let's begin making our map with this information. Figure 2.4 shows a map of the terrain we have explored so far. I use a two-way arrow to mean that from either of the two configurations in the figure, a horizontal flip leads to the other configuration.

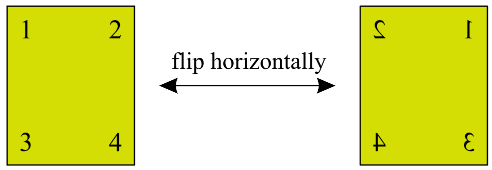


Figure 2.4. Partial map of the configurations of the rectangle puzzle, using only horizontal flips

These configurations are as far as we can go with horizontal flips alone. Keeping with the exploration metaphor, we can say that we have found two places in the rectangle realm. From each of these places, the map in Figure 2.4 tells us where a horizontal flip will take us, but there is (so far) no information about where vertical flips lead. Our map is not complete without such information, and so we must explore further.

Let's return to the rectangle in its solved state, and explore the results of vertical flips. Figure 2.3 already tells us what vertical flips do, but let's be thorough and explore those states as we add them to our map. From the solved state, a vertical flip leads to a new state we have not yet visited, and from there a vertical flip returns the rectangle to the solved state. We augment our map as shown in Figure 2.5.

Our map is still not complete, because we have not yet recorded where a vertical flip leads from the upper right configuration in Figure 2.5. To explore from that configuration, we first need to get our rectangle in that configuration. If you've been following along with your own rectangle, we left it in the solved state, and can get to the upper right configuration from Figure 2.5 by doing a horizontal flip—that's what our map tells us! After that horizontal flip, we perform a vertical flip to see where it leads. This move is the first exploration we've done whose outcome we could not predict from Figure 2.3. Perform this move yourself and see where it leads. Does it lead to a new location we must add to our map, or to a location we've already been? (You'll find the answer in Figure 2.6.)

Figure 2.6 contains four states of the rectangle, but the lower two have no "horizontal flip" arrows leading to or from them. Therefore our map is still incomplete. For instance, if your rectangle is like the lower left one in Figure 2.6, where does a horizontal flip lead you? The map does not say; we still have work to do.

Use the map to get your rectangle to look like the lower left rectangle in Figure 2.6, then perform a horizontal flip and see what configuration results. It is not a new configuration; it is the same one you discovered moments ago. Performing another horizontal flip will, of course, get you back to the lower left rectangle from Figure 2.6. The final map for the rectangle puzzle then looks like Figure 2.7. We can tell that our explorations are complete because there are no unanswered questions. From every location, it is clear from the map where every given move leads.

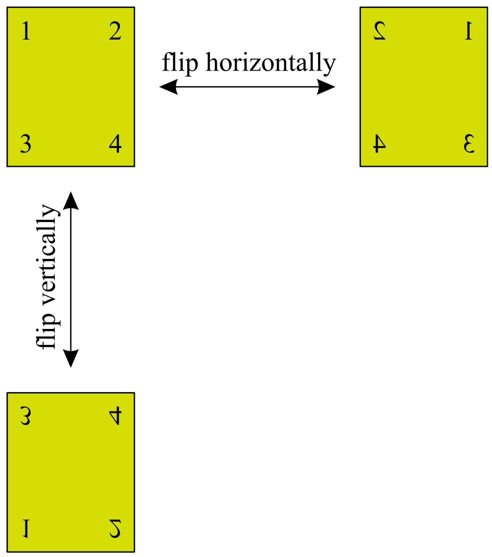


Figure 2.5. Partial map of the configurations of the rectangle puzzle, exploring from the solved state using one type of move at a time

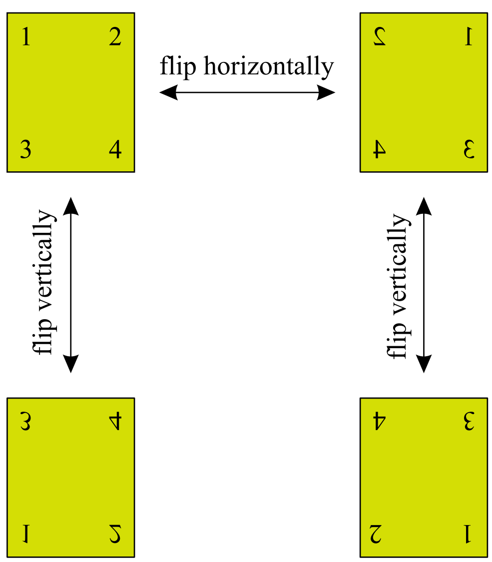


Figure 2.6. Partial map of the configurations of the rectangle puzzle, continuing to explore from Figure 2.5. The lower right state shown in this map does not appear in Figure 2.3.

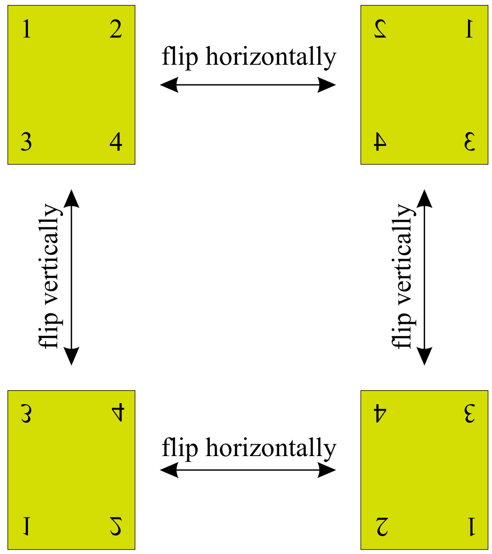


Figure 2.7. Full map of the configurations of the rectangle puzzle

You have just created your first map of a group! We have to admit that the map in Figure 2.7 is a bit unnecessary, because the rectangle puzzle is easy to solve without a map. But the map does serve to show us exactly the structure of the rectangle puzzle, and to let us see why it is easy. (For instance, from the map, you can see that alternating horizontal and vertical flips will walk you through every location in the rectangle realm.) The map we made is also an excellent first example of how to map a group, and allows us get our feet wet before we come upon more complicated groups.

2.4 Cayley diagrams

Maps like Figure 2.7 are called ***Cayley diagrams,*** after their inventor, Arthur Cayley, a nineteenth century British mathematician. We will use Cayley diagrams extensively throughout this book; they can be very potent visualization tools. It will help to begin by noting some important facts about the Cayley diagram we just made. These facts may seem obvious or uninteresting as far as they apply to the rectangle puzzle, because it is a puzzle that is so easy to solve. But we will be making Cayley diagrams for more complicated groups, and these facts will remain true.

The map in Figure 2.7 allows us to get from any place in the rectangle realm to any other without any guesswork. For instance, suppose you wanted to get from the lower right configuration to the solved configuration. From the map, you can see that there are two different (short) paths you could follow (up and then left or left and then up). To make use of the map, as you trace either of these paths on the map with your finger or your eyes, obey the instructions on that path using your rectangle. Going up from the lower right configuration, flip your rectangle vertically; then going left, flip it horizontally. Doing so successfully navigates to your desired destination; the map could help you plan and execute any such journey.

Recall also that we took pains to ensure that the map in Figure 2.7 is comprehensive. There is no location in the rectangle realm that does not appear on the map. Our construction of the map ensured this. We branched out from the starting position using each generator, and then branched out from each of those positions using each generator again. If the puzzle had been more complicated, we could have continued this process further, exploring farther and wider until we had found every location in the realm. We know our explorations are incomplete if our map fails to answer a question like "Where does a horizontal flip take me from this location?" That is, if there is a location on your map from which you have not yet explored where all moves lead, then your map is incomplete. When all such questions are answered, the map is complete.

Cayley diagrams have the two important properties just discussed: they clearly show all possible paths and they include every configuration. Just as the rectangle puzzle has a map, every other group also has a map with the same two properties. From now on, I will call such maps by their official name, Cayley diagrams. The most useful aspect of Cayley diagrams is that they give a clear picture of the structure of a group. Seeing the Cayley diagram of a group gives a much more immediate and complete idea of the group's size, complexity, and structure than a simple prose description can. This illustrative power is why we use them so frequently for learning about groups hereafter.

You can make a Cayley diagram for any group the way we made the one in Figure2.7. Beginning at any one configuration or situation, explore carefully using each generator, one at a time. Explore thoroughly and carefully, making a map as you go and labeling the transitions between states with the moves that cause them. Continue until your map contains no unanswered questions, as described above. Although Figures 2.4 through 2.7 are laid out nicely, the first time you draw a Cayley diagram it will probably be disorganized. As you explore a realm for the first time, the Cayley diagram that evolves is messy because you do not know in advance the simplest way to lay it out. Cayley diagrams created by exploration need to be reorganized into a more symmetric or aesthetically pleasing arrangement after the exploration is complete.

The exercises at the end of this chapter encourage you to make a few Cayley diagrams using this exploratory technique. You will appreciate the previous paragraph more after some personal involvement with this type of mapmaking. Feel free to jump ahead and do the first few exercises now, and then return to this point in your reading.

2.5 A touch more abstract

It is important to get to the heart of the mapmaking concepts that we have just learned. As Definition 1.9 makes clear, what is important about a group is the interactions of its actions, not the specific situation that gave rise to those actions. Let me illustrate this fact with an example. Consider two light switches side-by-side on a wall. You are allowed two actions, flipping the first switch and flipping the second switch. This collection of (two) actions generates a group; you can check the rules yourself. The map of this group is shown in Figure 2.8.

You will notice that it has the same *structure* as the map of the configurations of the rectangle puzzle from Figure 2.7. The four rectangle configurations have become four light switch configurations, and the arrows labeled "horizontal flip" and "vertical flip"

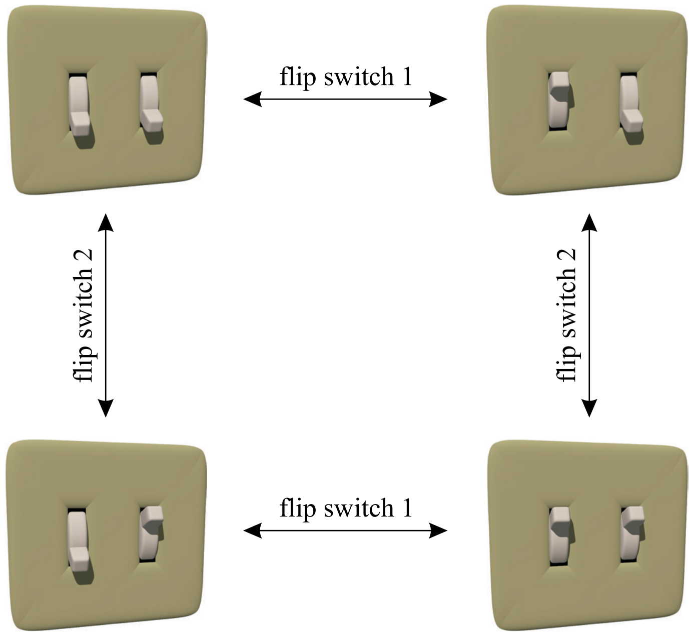


Figure 2.8. Full map of the configurations of the two-light switch group

have become arrows labeled "flip switch 1" and "flip switch 2" respectively. But the arrows connect the configurations in the same pattern as before, making a clear analogy between Figures 2.7 and 2.8.

So although these two groups are superficially different, they are structurally the same. The important lesson to learn here is that two different groups may have the same structure, and that the Cayley diagrams help us see this. Therefore, in order for us to study groups in the abstract, we wish to remove from our Cayley diagrams the details of the practical situation from which they were constructed. After all, a group is a mathematical structure, and mathematicians study groups as abstract (purely mathematical) objects. The rectangle puzzle and the light switches example just help us anchor our abstract study in something familiar.

So let's replace each rectangle in Figure 2.7 with something purely meaningless, a spot we will call a ***node.*** And we will replace the two different types of arrows (distinguished by their labels "horizontal flip" and "vertical flip") with different colored connectors that have no labels. (In fact, we can simplify even further by removing the arrowheads, since all arrows point in both directions anyway; I'll still refer to these headless connectors as "arrows.") The result is Figure 2.9, a Cayley diagram of a group, now shown pure and without any trappings of the example from which we learned it. You will note that the structure shown in Figure 2.9 is not only the heart of Figure 2.7, but also that of Figure 2.8.

This group in Figure 2.9 is called the Klein 4-group[3]. I chose it as our first group to visualize because it was simple enough for us to map quickly and easily, and yet still have some interesting structure. Figure 2.10 shows several other Cayley diagrams, to give you

[3]Also simply called the 4-group, and denoted V or V_4 for *vierergruppe*, "four-group" in German. It is named for the mathematician Felix Christian Klein.

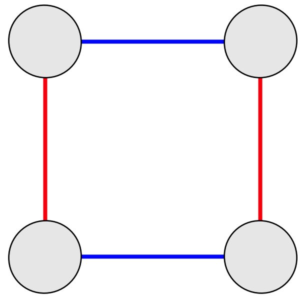


Figure 2.9. Cayley diagram of the Klein 4-group

a broader idea of what they tend to look like. As you can see, some are very simple, and others very complex. This diversity in the diagrams is indicative of a range of complexities in the underlying groups as well. You should not feel as if you must understand every part of Figure 2.10 already; it is present as an example of what is to come.

My interest in group theory visualization led me to write *Group Explorer*, a free software package that draws Cayley diagrams (and other illustrations you'll learn in later chapters). *Group Explorer* is an optional (but helpful) interactive companion to this book, and you can retrieve it from `http://groupexplorer.sourceforge.net`. It provides a list of groups, and extensive information about each one, including at least one Cayley diagram. *Group Explorer* creates Cayley diagrams using much the same algorithm we did—it uses the rules of the group to follow arrows, exploring the realm, and after it has found every location, it makes an attempt to arrange them presentably.

This chapter taught you your first technique for visualizing groups—the Cayley diagram. We will explore applications of this technique in chapters to come, and will use it extensively throughout our group theory studies. First, take some time to build your understanding of Cayley diagrams using the following exercises.

2.6 Exercises

2.6.1 Basics

Exercise 2.1. In the rectangle puzzle, what actions were the generators? What other actions are there besides the generators?

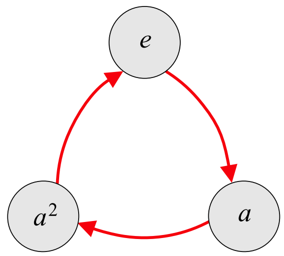
Exercise 2.2. In the light switch puzzle, what actions were the generators? What other actions are there besides the generators?

Exercise 2.3. Can an arrow in a Cayley diagram ever connect a node to itself?

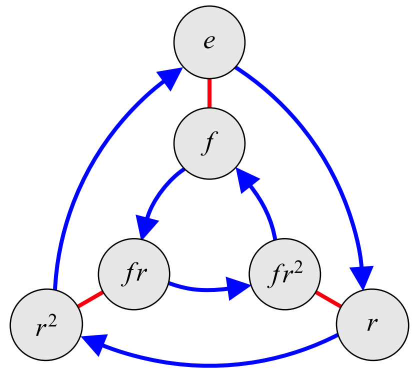
2.6.2 Mapmaking

Exercise 2.4. Exercise 1.1 of Chapter 1 defined a group. Create its Cayley diagram using the technique from this chapter. (Hint: This group is simpler than even those done so far; the diagram will be small.)

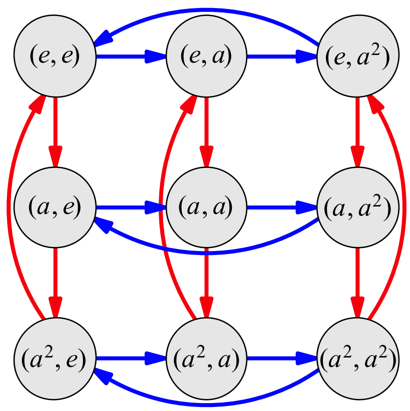
Exercise 2.5. Exercise 1.4 of Chapter 1 defined a group. Create its Cayley diagram using the technique from this chapter.



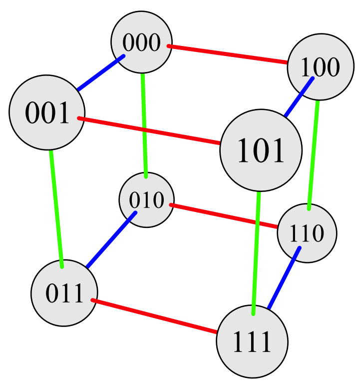
Cyclic group C_3 (or $\mathbb{Z}_3$)



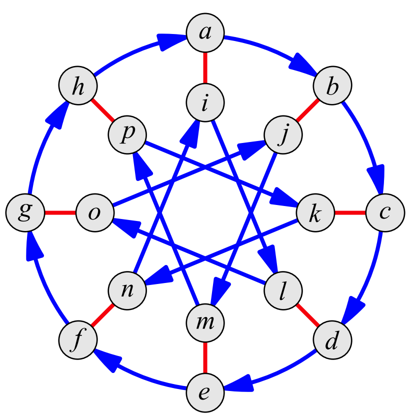
Symmetric group S_3



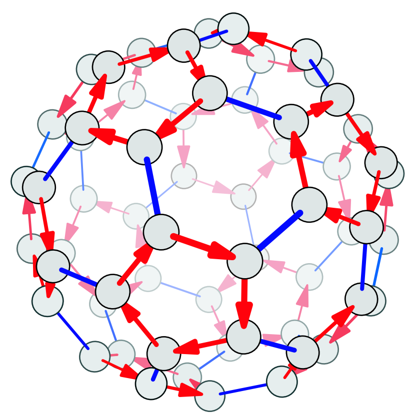
Direct product group $C_3 \times C_3$



Direct product group $C_2 \times C_2 \times C_2$



Quasihedral group with 16 elements



Alternating group A_5

Figure 2.10. Cayley diagrams of some small, finite groups

Exercise 2.6. Exercise 1.13 described an infinite group which can be generated with just one generator. Can you draw an infinite Cayley diagram for it? (Just draw a portion of the diagram that makes the infinite repeating pattern clear.)

How does that Cayley diagram compare to one for the group in Exercise 1.14 part (a)?

Exercise 2.7. Exercise 1.14 part (d) described a two-element group. Can you draw a Cayley diagram for it? Which arrow or arrows should you use and why?

Exercise 2.8. Section 2.2 introduced the rectangle puzzle. Imagine instead a *square* puzzle with its corners labeled the same way. Such a puzzle would allow a new move that was not possible with the rectangle puzzle; you could rotate a quarter-turn clockwise.

(a) Make the map of this group.

(b) Why is the quarter-turn move not "allowed" in the rectangle puzzle?

Exercise 2.9. Most groups can be generated many different ways, and each way gives rise to a corresponding way to connect a Cayley diagram with arrows. For example, consider the group V_4, which we met in the rectangle puzzle. Let's shorten the names of its actions to n, h, v and b, meaning (respectively) no action, horizontal flip, vertical flip, and both (a horizontal flip followed by a vertical flip).

We saw that h and v together generate V_4. But it is also true that h and b together would generate V_4, or v and b together. (You can verify these facts by exploring the rectangle realm using these generators on your own numbered rectangle.)

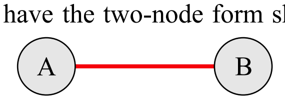
(a) Make a copy of Figure 2.9 and add to it a new type of arrow, representing the action b.

(b) Make a copy of your answer to part (a), with the arrows representing h removed. How does your diagram show that v and b are sufficient to generate V_4?

(c) Make a copy of your answer to part (a), with the arrows representing v removed. How does your diagram show that h and b are sufficient to generate V_4?

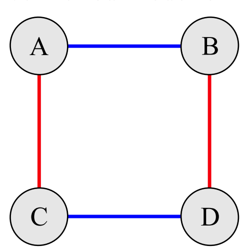
2.6.3 Going backwards

Exercise 2.10. If you've done all the exercises to this point, you've encountered two different Cayley diagrams that have the two-node form shown here.



Can you come up with another group whose Cayley diagram has this form?

Exercise 2.11. If you've done all the exercises to this point, you've encountered two different Cayley diagrams that have the four-node form shown here.



Can you come up with another group whose Cayley diagram has this form?

Exercise 2.12. We have not yet seen a group whose Cayley diagram has the three-node form called C_3, shown in the top left of Figure 2.10. Can you come up with a group whose Cayley diagram has that form?

2.6.4 Rules

Exercise 2.13. A group's generators have a special status in a Cayley diagram for the group. What is that special status?

Exercise 2.14. Chapter 1 required groups to satisfy Rule 1.5, which states, "There is a predefined list of actions that never changes." How does this rule impact the appearance of Cayley diagrams? (Or how would diagrams be different if this rule were not a requirement?)

Exercise 2.15. Chapter 1 required groups to satisfy Rule 1.6, which states, "Every action is reversible." What constraint does this place on the arrows in a Cayley diagram? Can you draw a diagram that does not fit this constraint? (That is, draw a diagram that almost deserves the name "Cayley diagram," except for that one rule violation.)

Exercise 2.16. Chapter 1 required groups to satisfy Rule 1.7, which states, "Every action is deterministic." What constraint does this place on the arrows in a Cayley diagram? Can you draw a diagram that does not fit this constraint? (That is, draw a diagram that almost deserves the name "Cayley diagram," except for that one rule violation.)

Exercise 2.17. Chapter 1 required groups to satisfy Rule 1.8, which states, "Any sequence of consecutive actions is also an action." How do we depend upon this fact when using a Cayley diagram as a map?

2.6.5 Shapes

Exercise 2.18. If we created an equilateral triangle puzzle, like the square puzzle in Exercise 2.8, what would the valid moves be? Map the group of such a puzzle.

Exercise 2.19. A regular n-gon is a polygon with n equal sides and n equal angles. You have already analyzed regular n-gons with $n = 3$ (equilateral triangle, Exercise 2.18) and $n = 4$ (square, Exercise 2.8).

(a) Based on what you know about the cases when $n = 3$ and $n = 4$, make a conjecture about how many actions will be in the group of the regular n-gon for any $n > 2$.

(b) Test your conjecture by making the map of the group for a regular pentagon ($n = 5$).

(c) Find the equilateral triangle group, the square group, and the regular pentagon group in *Group Explorer*'s library. (Hint: Use the search feature.)
   (i) Do your Cayley diagrams look like those in *Group Explorer*?
   (ii) Does your conjecture hold up against all the data you can find in *Group Explorer*?

(d) Write a paragraph giving as convincing an argument as you can in favor of your conjecture. Try to anticipate the counterarguments of a skeptical reader.

The groups you studied in this exercise are called *dihedral groups*. We will return to them in Exercise 4.9, and then formally study them in Chapter 5.