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Computational Complexity of Decoding Orthogonal Space-Time Block Codes

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***Abstract*—The computational complexity of optimum decoding for an orthogonal space-time block code $\mathcal{G}_N$ satisfying $\mathcal{G}_N^H \mathcal{G}_N = c(\sum_{k=1}^{K} |s_k|^2) I_N$ where c is a positive integer is quantified. Four equivalent techniques of optimum decoding which have the same basic computational complexity are specified. Modifications to the basic formulation in special cases are calculated and illustrated by means of examples. This paper corrects and extends [2],[3], and unifies them with the results from the literature. In addition, a number of results from the literature are extended to the case $c > 1$.**

***Index Terms*—OSTBC, maximum likelihood decoding, quadrature amplitude modulation (QAM), decoding QAM, square QAM.**

I. Introduction

In [4], an optimum Maximum Likelihood metric is introduced for Orthogonal Space-Time Block Codes (OSTBCs). A general description of this metric and specific forms for a number of space-time codes can be found in [5]. This metric is complicated and, in a straightforward implementation, its computational complexity would depend on the size of the signal constellation. By a close inspection, it can be observed that it can actually be simplified and made independent of the constellation size. Alternatively, the Maximum Likelihood formulation can be made differently and the simplified metric can be obtained via different formulations [6],[7]. In [2],[3], yet another formulation is provided. Although it is stated in [2],[3] that the formulation depends on the size of the signal constellation as $O(\sqrt{L})$ for square Quadrature Amplitude Modulation (QAM) with L signal points, in reality the detection can be performed using conventional quantization operation, independently of L. Therefore the computational complexity figures should be updated. However, the technique proposed in [2],[3], when properly implemented, happens to be one of the optimum decoding techniques for the decoding of OSTBCs. In this paper, we will unify all of the approaches cited above and calculate the computational complexity of the optimum decoding of an OSTBC. We will begin our discussion within the framework of [2],[3].

Consider the decoding of an OSTBC with N transmit and M receive antennas, and an interval of T symbols during which the channel is constant. The received signal is given by

$$Y = \mathcal{G}_N H + V \tag{1}$$

where $Y = [y_t^j]_{T\times M}$ is the received signal matrix of size $T \times M$ and whose entry y_t^j is the signal received at antenna j at time t, $t = 1, 2, \ldots, T$, $j = 1, 2 \ldots, M$; $V = [v_t^j]_{T\times M}$ is the noise matrix, and $\mathcal{G}_N = [g_t^i]_{T\times N}$ is the transmitted signal matrix whose entry g_t^i is the signal transmitted at antenna i at time t, $i = 1, 2, \ldots, N$. The matrix $H = [h_{i,j}]_{N\times M}$ is the channel coefficient matrix of size $N \times M$ whose entry $h_{i,j}$ is the channel coefficient from transmit antenna i to receive antenna j. The entries of the matrices H and V are independent, zero-mean, and circularly symmetric complex Gaussian random variables. $\mathcal{G}_N$ is an OSTBC with complex symbols s_k, $k = 1, 2, \ldots, K$ and therefore $\mathcal{G}_N^H \mathcal{G}_N = c(\sum_{k=1}^{K} |s_k|^2) I_N$ where c is a positive integer and I_N is the identity matrix of size N.

II. A Real-Valued Representation

Arrange the matrices Y, H, and V, each in one column vector by stacking their columns on top of one another

$$\begin{aligned} y &= \mathrm{vec}(Y) = (y_1^1, \ldots, y_T^M)^T, & (2)\\ h &= \mathrm{vec}(H) = (h_{1,1}, \ldots, h_{N,M})^T, & (3)\\ v &= \mathrm{vec}(V) = (v_1^1, \ldots, v_T^M)^T. & (4) \end{aligned}$$

Then one can write

$$y = \check{\mathcal{G}}_N h + v \tag{5}$$

where $\check{\mathcal{G}}_N = I_M \otimes \mathcal{G}_N$, with $\otimes$ denoting the Kronecker matrix multiplication. In [2],[3], a real-valued representation of (1) is obtained by decomposing the MT-dimensional complex problem defined by (5) to a $2MT$-dimensional real-valued problem and by applying the real-valued lattice representation defined in [8] to obtain

$$\check{y} = \check{H} x + \check{v} \tag{6}$$

where

$$\begin{aligned} \check{y} &= (\mathrm{Re}(y_1^1), \mathrm{Im}(y_1^1), \ldots, \mathrm{Re}(y_T^M), \mathrm{Im}(y_T^M))^T, & (7)\\ x &= (\mathrm{Re}(s_1), \mathrm{Im}(s_1), \ldots, \mathrm{Re}(s_K), \mathrm{Im}(s_K))^T, & (8)\\ \check{v} &= (\mathrm{Re}(v_1^1), \mathrm{Im}(v_1^1), \ldots, \mathrm{Re}(v_T^M), \mathrm{Im}(v_T^M))^T. & (9) \end{aligned}$$

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The real-valued fading coefficients of $\check{H}$ are defined using the complex fading coefficients $h_{i,j}$ from transmit antenna i to receive antenna j as $h_{2i-1+2(j-1)N} = \mathrm{Re}(h_{i,j})$ and $h_{2i+2(j-1)N} = \mathrm{Im}(h_{i,j})$ for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. Since $\mathcal{G}_N$ is an orthogonal matrix and due to the real-valued representation of the system using (6), it can be observed that the columns $\check{h}_i$ of $\check{H}$ are orthogonal to each other and their inner products with themselves are a constant [2],[3]

$$\check{H}^T \check{H} = \sigma I_{2K}. \tag{10}$$

By multiplying (6) by $\check{H}^T$ on the left, we have

$$\bar{\bar{y}} = \sigma x + \bar{\bar{v}} \tag{11}$$

where $\bar{\bar{y}} = \check{H}^T \check{y}$, and $\bar{\bar{v}} = \check{H}^T \check{y}$ is a zero-mean random vector. Due to (10), $\bar{\bar{v}}$ has independent and identically distributed Gaussian members. The Maximum Likelihood solution is found by minimizing

$$\|\bar{\bar{y}} - \sigma x\|_2^2 \tag{12}$$

or equivalently

$$\|\sigma^{-1}\bar{\bar{y}} - x\|_2^2 \tag{13}$$

over all combinations of $x \in \Omega^{2K}$. As a result, the joint detection problem of an OSTBC decouples into K symbol detection problems

$$\|\sigma^{-1}(\bar{\bar{y}}_{2k-1}, \bar{\bar{y}}_{2k}) - (x_{2k-1}, x_{2k})\|_2^2 \tag{14}$$

one per symbol $(x_{2k-1}, x_{2k}) \in \Omega^2$, where $k = 1, 2, \ldots, K$. Further, assuming that the signal constellation is separable as Ω^2 where $\Omega = \{\pm 1, \pm 3 \ldots, \pm(2L-1)\}$, and L is an integer, the Maximum Likelihood decoding problem can be further simplified to

$$\min_{x_k \in \Omega} |x_k - \hat{x}_k|^2 \tag{15}$$

where we denoted

$$\hat{x}_k = \sigma^{-1}\bar{\bar{y}}_k, \qquad k = 1, 2, \ldots, 2K, \tag{16}$$

which is a standard operation in conventional Quadrature Amplitude Modulation (QAM). In the sequel, we will compute the decoding complexity up to this quantization operation.

The decoding operation consists of the multiplication

$$\bar{\bar{y}} = \check{H}^T \check{y}, \tag{17}$$

the calculation of

$$\sigma = \check{h}_1^T \check{h}_1, \tag{18}$$

the inversion of σ, and the multiplications in (16).

In what follows, we will show that when $\mathcal{G}_N^H \mathcal{G}_N = c(\sum_{k=1}^K |s_k|^2) I_N$ where c is a positive integer, then $\sigma = c\|H\|^2$. The development will lead to the four equivalent optimal decoding techniques discussed in the next section.

Let $\bar{s}_k = \mathrm{Re}[s_k]$ and $\tilde{s}_k = \mathrm{Im}[s_k]$. Form two vectors, $\bar{s}$ and $\tilde{s}$, consisting of $\bar{s}_k$ and $\tilde{s}_k$, respectively

$$\bar{s} = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_K)^T, \quad \tilde{s} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_K)^T, \tag{19}$$

and form a vector s' that is the concatenation of $\bar{s}$ and $\tilde{s}$

$$s' = (\bar{s}^T, \tilde{s}^T)^T. \tag{20}$$

By rearranging the right hand side of (5), we can write

$$y = Fs' + v = F_a \bar{s} + F_b \tilde{s} + v \tag{21}$$

where $F = [F_a\ F_b]$ is an $MT \times 2K$ complex matrix and F_a and F_b are $MT \times K$ complex matrices whose entries consist of (linear combinations of) channel coefficients $h_{i,j}$. In [6], it was shown that when $\mathcal{G}_N^H \mathcal{G}_N = (\sum_{k=1}^K |s_k|^2) I_N$, then $\mathrm{Re}[F^H F] = \|H\|^2 I$. It is straightforward to extend this result so that when $\mathcal{G}_N^H \mathcal{G}_N = c(\sum_{k=1}^K |s_k|^2) I_N$, then

$$\mathrm{Re}[F^H F] = c\|H\|^2 I \tag{22}$$

where c is a positive integer. Let

$$\bar{y} = \mathrm{Re}[y], \quad \tilde{y} = \mathrm{Im}[y], \quad \bar{v} = \mathrm{Re}[v], \quad \tilde{v} = \mathrm{Im}[v], \tag{23}$$

and

$$\begin{aligned} \bar{F}_a &= \mathrm{Re}[F_a], & \tilde{F}_a &= \mathrm{Im}[F_a], \\ \bar{F}_b &= \mathrm{Re}[F_b], & \tilde{F}_b &= \mathrm{Im}[F_b]. \end{aligned} \tag{24}$$

Now define

$$y' = \begin{bmatrix} \bar{y} \\ \tilde{y} \end{bmatrix} \quad F' = \begin{bmatrix} \bar{F}_a & \bar{F}_b \\ \tilde{F}_a & \tilde{F}_b \end{bmatrix} \quad v' = \begin{bmatrix} \bar{v} \\ \tilde{v} \end{bmatrix} \tag{25}$$

so that we can write

$$y' = F's' + v' \tag{26}$$

which is actually the same expression as (6) except the vectors and matrices have their rows and columns permuted.

It can be shown that (22) implies

$$F'^T F' = c\|H\|^2 I. \tag{27}$$

Let P_y and P_s be $2MT \times 2MT$ and $2K \times 2K$, respectively, permutation matrices such that

$$\check{y} = P_y y', \quad x = P_s s'. \tag{28}$$

It follows that $P_y^T P_y = P_y P_y^T = I$ and $P_s^T P_s = P_s P_s^T = I$. We now have

$$\check{y} = P_y(F's' + v') = P_y F' P_s^T x + P_y v' = \check{H}x + \check{v}. \tag{29}$$

Therefore,

$$\check{H} = P_y F' P_s^T \tag{30}$$

which implies

$$\check{H}^T \check{H} = P_s F'^T P_y^T P_y F' P_s^T = c\|H\|^2 I. \tag{31}$$

As a result, $\sigma = c\|H\|^2$.

III. Four Equivalent Optimum Decoding Techniques for OSTBCs

For an OSTBC $\mathcal{G}_N$ satisfying $\mathcal{G}_N^H \mathcal{G}_N = c(\sum_{k=1}^K |s_k\|^2) I_N$ where c is a positive integer, the Maximum Likelihood solution is formulated in four equivalent ways with equal squared norm values

$$\|Y - \mathcal{G}_N H\|^2 = \|y - Fs'\|^2 = \|y' - F's'\|^2 = \|\check{y} - \check{H}x\|^2. \tag{32}$$

There are four solutions, all equal. The first solution is obtained by expanding $\|Y - \mathcal{G}_N H\|^2$ and is given by eq. (7.4.2)

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of [6] when $c = 1$[1]. When $c > 1$, it should be altered as

$$\hat{s}_k = \frac{1}{c\|H\|^2}[\mathrm{Re}\{\mathrm{Tr}(H^H A_k^H Y)\} - \hat{\imath} \cdot \mathrm{Im}\{\mathrm{Tr}(H^H B_k^H Y)\}] \tag{33}$$

for $k = 1, 2, \ldots, K$, where A_k and B_k are the matrices in the linear representation of $\mathcal{G}_N$ in terms of $\bar{s}_k$ and $\tilde{s}_k$ as

$$\mathcal{G}_N = \sum_{k=1}^{K} \bar{s}_k A_k + \hat{\imath}\tilde{s}_k B_k = \sum_{k=1}^{K} s_k \check{A}_k + s_k^* \check{B}_k, \tag{34}$$

$\hat{\imath} = \sqrt{-1}$, $A_k = \check{A}_k + \check{B}_k$, and $B_k = \check{A}_k - \check{B}_k$ [6]. Once $\{\hat{s}_k\}_{k=1}^K$ are calculated, the decoding problem can be solved by

$$\min_{\bar{s}_k \in \Omega} |\bar{s}_k - \mathrm{Re}[\hat{s}_k]|^2, \qquad \min_{\tilde{s}_k \in \Omega} |\tilde{s}_k - \mathrm{Im}[\hat{s}_k]|^2 \tag{35}$$

once for each $k = 1, 2, \ldots, K$. Similarly to (15), this is a standard quantization problem in QAM.

The second solution is obtained by expanding the second expression in (32) and is given by

$$\hat{s}' = \frac{\mathrm{Re}[F^H y]}{c\|H\|^2}. \tag{36}$$

This is given in [4. eq. (7.4.20)] for $c = 1$. The third solution corresponds to the minimization of the third expression in (32) and is given by

$$\hat{s}' = \frac{F'^T y'}{c\|H\|^2}. \tag{37}$$

The fourth solution is the one introduced in [2]. It is obtained by minimizing the fourth expression in (32) and is given by

$$\hat{x} = \frac{\check{H}^T \check{y}}{\sigma} = \frac{\check{H}^T \check{y}}{c\|H\|^2}. \tag{38}$$

Considering that

$$F_a = [\mathrm{vec}(A_1 H) \; \cdots \; \mathrm{vec}(A_K H)] \tag{39}$$

$$F_b = [\hat{\imath} \cdot \mathrm{vec}(B_1 H) \; \cdots \; \hat{\imath} \cdot \mathrm{vec}(B_K H)] \tag{40}$$

[4, eq. (7.1.7)], it can be verified that (33) and (36) are equal. The equality of (36) and (37) follows from (23)-(25). The equality of (37) and (38) follows from (28) and (30). Therefore, equations (33), (36)-(38) yield the same result, and when properly implemented, will have identical computational complexity.

Although these four techniques are equivalent, a straightforward implementation of (33) or (36) can actually result in larger complexity than (37) or (38). The proper implementation requires that in (33) or (36), the terms not needed due to elimination by the Tr[ ], Re[ ], and Im[ ] operators are not calculated.

Let's now compare these techniques with the minimization of the metric introduced in [4]. For a complex OSTBC, let [4],[5]

$$r_k = \sum_{t \in \eta(k)} \sum_{j=1}^{M} \mathrm{sgn}_t(k) \check{h}_{\epsilon(k),j} \check{y}_t^j(k) \tag{41}$$

where $\eta(k)$ is the set of rows of $\mathcal{G}_N$ in which s_k appears, $\epsilon_t(k)$ expresses the column position of s_k in the tth row, $\mathrm{sgn}_t(k)$ denotes the sign of s_k in the tth row,

$$\check{h}_{\epsilon_t(k),j} = \begin{cases} h_{\epsilon_t(k),j}^* & \text{if } s_k \text{ is in the } t\text{th row of } \mathcal{G}_N, \\ h_{\epsilon_t(k),j} & \text{if } s_k^* \text{ is in the } t\text{th row of } \mathcal{G}_N, \end{cases} \tag{42}$$

and

$$\check{y}_t^j(k) = \begin{cases} y_t^j & \text{if } s_k \text{ is in the } t\text{th row of } \mathcal{G}_N, \\ (y_t^j)^* & \text{if } s_k^* \text{ is in the } t\text{th row of } \mathcal{G}_N \end{cases} \tag{43}$$

for $k = 1, 2, \ldots, K$. A close inspection shows that r_k in (41)-(43) is equal to the numerator of (33).

The metric to be minimized for s_k is given as [4],[5]

$$|s_k - r_k|^2 + \left(c \sum_{i=1}^{N} \sum_{j=1}^{M} |h_{i,j}|^2 - 1 \right) |s_k|^2. \tag{44}$$

Implemented as it appears in (44), this metric has larger complexity than the metrics for four equivalent techniques described above. Furthermore, its complexity depends on the constellation size L due to the presence of the factor $|s_k|^2$. It can be simplified, however.

For minimization purposes, we can write (44) as

$$\begin{aligned} &|s_k|^2 - 2\mathrm{Re}[s_k^* r_k] + |r_k|^2 + c\|H\|^2 |s_k|^2 - |s_k|^2 \\ &= c\|H\|^2 \left(|s_k|^2 - \frac{2\mathrm{Re}[s_k^* r_k]}{c\|H\|^2} + \frac{|r_k|^2}{c^2\|H\|^4} \right) + \text{const.} \\ &= c\|H\|^2 \left| s_k - \frac{r_k}{c\|H\|^2} \right|^2 + \text{const.} \end{aligned} \tag{45}$$

where the first equality follows from the fact that the third term inside the paranthesis in (45) is independent of s_k. Because of our observation that r_k is the same as the numerator of (33), we have

$$\hat{s}_k = \frac{r_k}{c\|H\|^2} \qquad k = 1, 2, \ldots, K \tag{46}$$

and then this method becomes equivalent to our four equivalent techniques. We would like to note that observations equivalent to the expression in (33) were made in [9] and [10].

IV. Optimum Decoding Complexity of OSTBCs

Since the four decoding techniques (33), (36)-(38) are equivalent, we will calculate their computational complexity by using one of them. This can be done most simply by using (37) or (38). We will use (38) for this purpose.

First, assume $c = 1$. Note $\check{H}$ is a $2MT \times 2K$ matrix. The multiplication $\check{H}^T \check{y}$ takes $2MT \cdot 2K$ and calculation of $\sigma = \|H\|^2$ takes $2MN$ real multiplications, its inverse takes a real division, and $\sigma^{-1}\bar{\bar{y}}$ takes $2K$ real multiplications. Similarly, the multiplication $\check{H}^T \check{y}$ takes $2K \cdot (2MT - 1)$, and calculation of σ takes $2MN - 1$ real additions. Letting R_D, R_M and R_A be the number of real divisions, the number of real multiplications, and the number of real additions, the complexity of decoding the transmitted complex signal $(s_1, s_2, \ldots, s_K)$ with the technique described in (17),(18), and (16) is

$$\begin{aligned} \mathcal{C} = 1R_D, (4KMT + 2MN + 2K)R_M, \\ (4KMT + 2MN - 2K - 1)R_A. \end{aligned} \tag{47}$$

[1]The notation in [5] and [6] is the transposed form of the one adopted in this paper.

Note that the complexity does not depend on the constellation size L. If we take the complexity of a real division as equivalent to 4 real multiplications as in [2],[3], then the complexity is

$$\begin{aligned} \mathcal{C} = (4KMT + 2MN + 2K + 4)R_M, \\ (4KMT + 2MN - 2K - 1)R_A \end{aligned} \tag{48}$$

which is smaller than the complexity specified in [2],[3] and does not depend on L. In the rest of this paper, we will use this assumption. The conversion from this form to that in (47) can be made simply by adding a real division and reducing the number of real multiplications by 4.

When $c > 1$, the number of real multiplications to calculate σ increases by 1, however, in the examples it will be seen that the complexity of the calculation of $\check{H}^T\check{y}$ is reduced by a factor of c.

In what follows, we will calculate the exact complexity values for four examples. See [4],[5] for explicit metrics of the form (41)-(44) for these examples.

Example 1: Consider the Alamouti OSTBC with $N = K = T = 2$ and $M = 1$ where

$$\mathcal{G}_2 = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}. \tag{49}$$

The matrix $\check{H}$ can be calculated as

$$\check{H} = \begin{bmatrix} h_1 & -h_2 & h_3 & -h_4 \\ h_2 & h_1 & h_4 & h_3 \\ h_3 & h_4 & -h_1 & -h_2 \\ h_4 & -h_3 & -h_2 & h_1 \end{bmatrix}. \tag{50}$$

Note that the matrix $\check{H}$ is orthogonal and all of its columns have the same squared norm. One needs 16 real multiplications to calculate $\bar{\bar{y}} = \check{H}^T\check{y}$, 4 real multiplications to calculate $\sigma = \check{h}_1^T\check{h}_1$, 4 real multiplications to calculate σ^{-1}, and 4 real multiplications to calculate $\sigma^{-1}\bar{\bar{y}}$. There are $3 \cdot 4 = 12$ real additions to calculate $\check{H}^T\check{y}$ and 3 real additions to calculate σ. As a result, with this approach, decoding takes a total of 28 real multiplications and 15 real additions.

The complexity figures in (48) are 28 real multiplications and 15 real additions, which hold exactly.

Example 2: Consider the OSTBC with $M = 2$, $N = 3$, $T = 8$, and $K = 4$ given by [11]

$$\mathcal{G}_3 = \begin{bmatrix} s_1 & s_2 & s_3 \\ -s_2 & s_1 & -s_4 \\ -s_3 & s_4 & s_1 \\ -s_4 & -s_3 & s_2 \\ s_1^* & s_2^* & s_3^* \\ -s_2^* & s_1^* & -s_4^* \\ -s_3^* & s_4^* & s_1^* \\ -s_4^* & -s_3^* & s_2^* \end{bmatrix}. \tag{51}$$

For this $\mathcal{G}_N$, one has $\mathcal{G}_3^H\mathcal{G}_3 = 2\left(\sum_{k=1}^K |s_k|^2\right) I_3$. In [3], it has been shown that the 32×8 real-valued channel matrix $\check{H}$ is

$$\check{H} = \begin{bmatrix} h_1 & -h_2 & h_3 & -h_4 & h_5 & -h_6 & 0 & 0 \\ h_2 & h_1 & h_4 & h_3 & h_6 & h_5 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_7 & -h_8 & h_9 & -h_{10} & h_{11} & -h_{12} & 0 & 0 \\ h_8 & h_7 & h_{10} & h_9 & h_{12} & h_{11} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & h_{11} & h_{12} & -h_9 & -h_{10} & -h_7 & -h_8 \\ 0 & 0 & h_{12} & -h_{11} & -h_{10} & h_9 & -h_8 & h_7 \end{bmatrix} \tag{52}$$

where h_i, $i = 1, 2, \ldots, 11$ and h_j, $j = 2, 4, \ldots, 12$ are the real and imaginary parts, respectively, of $h_{1,1}$, $h_{2,1}$, $h_{3,1}$, $h_{1,2}$, $h_{2,2}$, $h_{3,2}$. The matrix $\check{H}^T$ is 8×32 where each row has 8 zeros, while each of the remaining 24 symbols has one of $h_1, h_2, \ldots, h_{12}$, repeated twice. Let's first ignore the repetition of h_i in a row. Then, the calculation of $\check{H}^T\check{y}$ takes $8 \cdot 24 = 192$ real multiplications. The calculation of $\sigma = \check{h}_1^T\check{h}_1 = 2\sum_{k=1}^{12} h_i^2$ takes $12 + 1 = 13$ real multiplications, In addition, one needs 4 real multiplications to calculate σ^{-1}, and 8 real multiplications to calculate $\sigma^{-1}\bar{\bar{y}}$. To calculate $\check{H}^T\check{y}$, one needs $8 \cdot 23 = 184$ real additions, and to calculate σ, one needs 11 real additions. As a result, with this approach, one needs a total of 217 real multiplications and 195 real additions to decode.

For this example, (48) specifies 300 real multiplications and 279 real additions. The reduction is due to the elements with zero values in $\check{H}$.

It is important to make the observation that the repeated values of h_i in the columns of $\check{H}$, or equivalently $h_{m,n}^*$ in the rows of $H^H A_k^H$ or $H^H B_k^H$, have a substantial impact on complexity. Due to the repetition of h_i, by grouping the two values of $\check{y}_j$ that it multiplies, it takes $8 \cdot 12 = 96$ real multiplications to compute $\check{H}^T\check{y}$, not $8 \cdot 24 = 192$. The summations for each row of $\check{H}^T\check{y}$ will now be carried out in two steps, first 12 pairs of additions per each h_i, and then after multiplication by h_i, addition of 12 real numbers. This takes $12 + 11 = 23$ real additions, with no change from the way the calculation was made without grouping. With this change, the complexity of decoding becomes 121 real multiplications and 195 real additions, a huge reduction from 300 real multiplications and 279 real additions.

Example 3: We will now consider the code $\mathcal{G}_4$ from [11]. The parameters for this code are $N = K = 4$, $M = 1$, and $T = 8$. It is given as

$$\mathcal{G}_4 = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4 & -s_3 & s_2 & s_1 \\ s_1^* & s_2^* & s_3^* & s_4^* \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & s_4^* & s_1^* & -s_2^* \\ -s_4^* & -s_3^* & s_2^* & s_1^* \end{bmatrix}. \tag{53}$$

Similarly to $\mathcal{G}_3$ of Example 2, this code has the property that $\mathcal{G}_4^H\mathcal{G}_4 = 2(\sum_{k=1}^K |s_k|^2)I_4$. The $\check{H}$ matrix is 16×8 and can

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be calculated as

$$\check{H}=\begin{bmatrix} h_1 & -h_2 & h_3 & -h_4 & h_5 & -h_6 & h_7 & h_8 \\ h_2 & h_1 & h_4 & h_3 & h_6 & h_5 & h_8 & h_7 \\ h_3 & -h_4 & -h_1 & h_2 & h_7 & -h_8 & -h_5 & h_6 \\ h_4 & h_3 & -h_2 & -h_1 & h_8 & h_7 & -h_6 & -h_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_5 & h_6 & -h_7 & h_8 & -h_1 & -h_2 & h_3 & h_4 \\ h_6 & -h_5 & -h_8 & h_7 & -h_2 & h_1 & h_4 & -h_3 \end{bmatrix}. \quad (54)$$

This matrix consists entirely of nonzero entries. Each entry in a column equals $\pm h_i$ for some $i \in \{1, 2, \ldots, 8\}$, every h_i appearing twice in a column. Ignoring this repetition for now, calculation of $\check{H}^T \check{y}$ takes $8 \cdot 16 = 128$ real multiplications. Calculation of σ takes 9 real multiplications, its inverse 4 real multiplications, and the calculation of $\sigma^{-1}\bar{\bar{y}}$ takes 8 real multiplications. Calculation of $\check{H}^T \check{y}$ takes $8 \cdot 15 = 120$ real additions, and calculation of σ takes 7 real additions. As a result, with this approach, to decode, one needs 149 real multiplications and 127 real additions.

For this example, equation (48) specifies 156 real multiplications and 135 real additions. The reduction is due to the fact that one row of $\check{H}^T$ has each h_i appearing twice. This reduces the number of multiplications and summations to calculate σ by about a factor of 2.

However, because each h_i appears twice in every row of $\check{H}^T$, the number of multiplications can actually be reduced substantially. As discussed in Example 2, we can reduce the number of multiplications to calculate $\check{H}^T \check{y}$ by grouping the two multipliers of each h_i by summing them prior to multiplication by h_i, $i = 1, 2, \ldots, 8$. As seen in Example 2, this does not alter the number of real additions. With this simple change, the number of real multiplications to decode becomes 85 and the number of real additions to decode remains at 127.

Example 4: It is instructive to consider the code $\mathcal{H}_3$ given in [11] with $N = 3$, $K = 3$, $T = 4$ which we will consider for $M = 1$ where

$$\mathcal{H}_3 = \begin{bmatrix} s_1 & s_2 & s_3/\sqrt{2} \\ -s_2^* & s_1^* & s_3/\sqrt{2} \\ s_3^*/\sqrt{2} & s_3^*/\sqrt{2} & (-s_1 - s_1^* + s_2 - s_2^*)/2 \\ s_3^*/\sqrt{2} & -s_3^*/\sqrt{2} & (s_2 + s_2^* + s_1 - s_1^*)/2 \end{bmatrix}. \quad (55)$$

For this code, $\mathcal{H}_3^H \mathcal{H}_3 = (\sum_{k=1}^{3} |s_k|^2) I_3$ is satisfied. In this case, the matrix $\check{H}$ can be calculated as

$$\check{H}_{1-4} = \begin{bmatrix} h_1 & -h_2 & h_3 & -h_4 \\ h_2 & h_1 & h_4 & h_3 \\ h_3 & h_4 & -h_1 & -h_2 \\ h_4 & -h_3 & -h_2 & h_1 \\ -h_5 & 0 & 0 & -h_6 \\ -h_6 & 0 & 0 & h_5 \\ 0 & h_6 & h_5 & 0 \\ 0 & -h_5 & h_6 & 0 \end{bmatrix}, \quad (56)$$

$$\check{H}_{5-6} = \begin{bmatrix} h_5/\sqrt{2} & -h_6/\sqrt{2} \\ h_6/\sqrt{2} & h_5/\sqrt{2} \\ h_5/\sqrt{2} & -h_6/\sqrt{2} \\ h_6/\sqrt{2} & h_5/\sqrt{2} \\ (h_1+h_3)/\sqrt{2} & (h_2+h_4)/\sqrt{2} \\ (h_2+h_4)/\sqrt{2} & -(h_1+h_3)/\sqrt{2} \\ (h_1-h_3)/\sqrt{2} & (h_2-h_4)/\sqrt{2} \\ (h_2-h_4)/\sqrt{2} & (-h_1+h_3)/\sqrt{2} \end{bmatrix} \quad (57)$$

where due to space limitations we showed the first four columns of $\check{H}$ as $\check{H}_{1-4}$ and the last two columns of $\check{H}$ as $\check{H}_{5-6}$. It can be verified that every column $\check{h}_i$ of $\check{H}$ has the property that $\check{h}_i^T \check{h}_i = \sigma = \|H\|^2 = \sum_{k=1}^{6} h_k^2$ for $i = 1, 2, \ldots, 6$. In this case, the number of real multiplications to calculate $\check{H}^T \check{y}$ requires more caution than the previous examples. For the first four rows of $\check{H}^T$, this number is 6 real multiplications per row. For the last two rows, due to combining, e.g., h_1 and h_3 in $(h_1+h_3)/\sqrt{2}$ in the fifth element of $\check{h}_5$, and the commonality of h_5 and h_6 for the first and third, and second and fourth, respectively, elements of $\check{h}_5$, and one single multiplier $1/\sqrt{2}$ for the whole column, the number of real multiplications needed is 7. As a result, calculation of $\check{H}^T \check{y}$ takes 38 real multiplications. Calculation of σ takes 6 real multiplications. One needs 4 real multiplications to calculate σ^{-1}, and 6 real multiplications to calculate $\sigma^{-1}\bar{\bar{y}}$. First four rows of $\check{H}^T \check{y}$ require 5 real additions each. Last two rows of $\check{H}^T \check{y}$ require $4+7 = 11$ real additions each. This is a total of 42 real additions to calculate $\check{H}^T \check{y}$. Calculation of σ requires 5 real additions. Overall, with this approach one needs 54 real multiplications and 47 real additions to decode.

For this example, (48) specifies 66 real multiplications and 49 real additions. The reduction is due to the presence of the zero entries in $\check{H}$. On the other hand, the presence of the factor $1/\sqrt{2}$ in the last two rows of $\check{H}^T$ adds two real multiplications to the total number of real multiplications.

V. Conclusion

Equation (47) yields the computational complexity of decoding an OSTBC when its $\check{H}$ matrix consists only of nonzero entries in the form of h_i when $c = 1$. It should be updated as specified in the paragraph following (48) when $c > 1$. The presence of zero values within $\check{H}$ reduces the computational complexity. In the examples its effect has been a reduction in the number of real multiplications to calculate $\check{H}^T \check{y}$ by a factor equal to the ratio of the rows of A_k and B_k that consist only of zero values to the total number of all rows in A_k and B_k for $k = 1, 2 \ldots, K$, with a similar reduction in the number of real additions to calculate $\check{H}^T \check{y}$. With the modifications outlined above, (47) specifies the computational complexity of decoding the majority of OSTBCs. In some cases, the contents of the $\check{H}$ matrix can have linear combinations of h_i values, which result in minor changes in computational complexity as specified by this formulation, as shown in Example 4. Finally, note that $L = 2$ is a special case where the signal belongs to one of the four quadrants, calculation of and division by $c\|H\|^2$ are not needed and the computational complexity will be correspondingly lower.

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