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An exact Method for Stochastic Maximal Covering Problem of Preventive Healthcare Facilities

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Abstract

Effective preventive healthcare services have a significant role in reducing fatality and medical expenses in all human societies and the level of accessibility of customers to these services can be considered as a measure of their efficiency and effectiveness. The main purpose of this paper is to develop a service network design model of preventive healthcare facilities with the principal objective of maximizing participation in the offered services. While considering utility constraints and incorporating demand elasticity of customers due to travel distance and congestion delays, optimal number, locations and capacities of facilities as well as customer assignment o facilities are determined. First, the primary nonlinear integer program is transformed, and then the linearized model is solved by developing an exact algorithm. Computational results show that large-sized instances can be solved in a reasonable amount of time. An illustrative case study of network of hospitals in Shiraz, Iran, is used to demonstrate the model and the managerial insights are discussed.

Keywords: Preventive healthcare, service system design, elastic demand, utility, accessibility, nonlinear integer program

1- Introduction

The success of preventive healthcare programs, as a special field of healthcare services, requires careful and comprehensive considerations which may differs from treatment-related programs and should be studied from different points of view. Denoon (2009) explained that, diseases easily preventable by adult vaccines kill more Americans each year than car wrecks, Breast cancer, or AIDS and due to surveys by the Centers for Disease Control and Prevention (CDC) and the National Foundation for Infection Diseases (NFID), relatively few in the U.S know much about these diseases and far too few adults get vaccinated yet. Flu, Hepatitis B, Meningitis, Shingles and Tetanus are some of Vaccine-Preventable diseases which taxing the families as well as economy. It means that, although human communities suffer from huge expenses related to preventable diseases, acceptable participation does not occur, since the people are not aware of serious consequences as well as prevention and control of such diseases. Actually, as it mentioned in Zhang et al. (2010), in contrast with sick people who need urgent medical attention, the potential clientele of preventive healthcare often do not feel the necessity to receive these services and may not participate in preventive programs offered in their region. Therefore, besides culture change programs like public advertising, accessibility of people should be facilitated to increase the participation in preventive healthcare programs. Indeed, accessibility of centers where preventive healthcare services offered has a critical role on their effectiveness and efficiency. In this regard, the network of preventive healthcare facilities should be designed properly. Nowadays, there are hospitals in population zones and such Places can be considered as potential locations for offering preventive healthcare programs.

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In this paper, a service network design problem is studied with the primary objective of maximizing the accessibility of customers to preventive healthcare facilities. Actually, enough clinics with proper service capacity at convenient locations are needed in order to offer such services. The purpose of these clinics is to avoid complications of preventable diseases and also control of them. The optimal number of facilities, their locations and capacities as well as customer assignment o facilities is determined in this problem. The travel distance is deterministic and the assumption of demand elasticity, due to travel distance and service time at the facilities, resulted in congestion delays. In addition, the utility-related constraints represent the user-choice environment of preventive healthcare. So the customer decision to participate in the program is made based on the difference between their willingness to participate on one hand and the travel distance and waiting costs on the other hand. More details on this will be given in the problem definition.

The contributions of this paper can be stated as:

1. Mathematical transformation of the problem: Since we model a user-equilibrium problem that is nonlinear, first, the model is transformed and linearized, and then an exact algorithm to find the optimal solution is applied.
2. Utility Constraints as discussed in “Horizontal Differentiation” by Hotelling (1929), are incorporated in the problem which ensure that customer assignment just happens if the utility of customers is not negative, comparing to the concept of “Coverage”, no fixed coverage radius is assumed in our problem.
3. An exact and relatively simple approach is proposed to find the optimal solution of the problem.

The remainder of the paper is organized as follows. Section 2 provides a review of the relevant literature. In section 3, the problem formulation is described, followed by its non-linear MIP model. Section 4 presents the transformation and linearization approach for the non-linear MIP model. To solve the model, we present an exact solution algorithm in section 5. Computational results with an illustrative case study are reported in section 6. Conclusions and future research directions are provided in the final section. Proofs appear in the Appendices.

2- Literature review

Covering-type problems have long been studied. Reviews of Schilling et al. (1993) and Farahani et al. (2012) extensively presented the covering problems in facility location. Pereira et al., (2015) developed a hybrid algorithm combined a metaheuristic and an exact method to solve the probabilistic maximal covering location-allocation problem where the objective function is similar to ours but has differences in constraints and the proposed solution approach. On the other hand, we consider the concept of participation in our work, which is analogous to the concept of Coverage (Zhang et al., 2009), but it differs in that the accessibility to facilities is reduced as the distance increases which is not incorporated in the concept of coverage. A network design problem considering service level constraint and coverage radius is studied by Ghezavati et al. (2009) such that, there may be customers who cannot receive services.

Stochastic demand and congestion delays are considered in the model of Marianov and Serra (1998) and Marianov and Rios (2000) through constraining the waiting at the facilities. In some papers elasticity of demand is not considered explicitly, but penalties are assigned to congestion delays; Models presented in Wang et al. (2002) and Berman and Drezner (2006, 2007) aim at minimizing the total time- travel and waiting time at the facilities. Aboolian et al., (2012) developed a profit-maximizing network design model which incorporates demand elasticity and congestion in the facilities. A class of location-allocation problems with immobile servers, stochastic demand and congestion aims at minimizing the total cost consists of the fixed cost of opening facilities with sufficient capacities, the travel cost and queuing delay cost studied by Vidyarthi and Jayaswal (2014) assuming multiple parallel servers with a given single capacity level.

The review by Daskin and Dean (2004) studied the healthcare facility location problems without considering preventive programs. Other papers Berman and Krass (2002) and Marianov and Serra (2002), also studied healthcare facility design problems. Shavandi and Mahlooji (2007) developed hierarchical location-allocation models for congested systems in healthcare environment allowing partial coverage of demand nodes and approximate determination of parameters. To the best of our knowledge, the first paper studied the network design problem of preventive healthcare facilities was presented by Verter and Lapierre (2002). Furthermore, Zhang et al (2009) and Zhang et al (2010) studied a location model with elastic demand and congestion delays in preventive healthcare environment.

The most relevant papers to our work are Aboolian et al. (2016) and Zhang et al. (2010). However, the differences to comparing to our paper can be mentioned as: First, Zhang et al. (2010) aim at optimizing the number of servers at each facility, whereas our work optimizes the service rate at each facility. Second, the problem, in Zhang et al. (2010) is formulated as a bi-level problem and we proposed an exact (single-level) approach to find the optimal solution. Third, none of these two papers consider "Utility-related" constraints in their problem formulation.

3- Problem definition

Suppose that $N = 1,...,n$ represents the set of customer nodes in the intended network and each node's demand happens according to a Poisson's process with homogeneous rate $\lambda_i \geq 0$, such that the maximum demand rate that can be generated by node $i \in N$ is indicated by $\lambda^{max}{}_i \geq 0$. Also, let $M = 1,..., m$, $M \subset N$ is the set of potential facility locations. We assume that t_{ij} shows the travel distance between nodes $i,j \in M \cup N$. The fraction of the population of node $i \in N$ that requests service from facility $j \in M$ is denoted by y_{ij}. Then, the demand rate of node i, λ_i:

$$\lambda i = \sum_{j \in M} \lambda^{max}{}_i \; yij \tag{1}$$

And the aggregate demand arrival rate at facility j, denoted by Λ_j:

$$\Lambda j = \sum_{i \in N} \lambda^{max}{}_i \; yij \tag{2}$$

Table 1. Notations Description

Indices and Parameters	Description
i	index for customer nodes, $i \in N$
j	index for potential facility nodes, $j \in M$
tij	Travel distance between customer node i and facility j
$\lambda^{max}{}_i$	maximum demand rate that can be generated by node i
V	Willingness to pay (participate)
W^{max}	maximum waiting time at each facility
C^{max}	Maximum available capacity
Variables	**Description**
x_j	Binary variables taking the value 1 the facility at node j is open and 0 otherwise
yij	Continuous variables in [0,1] which is the fraction of the population of node $i \in N$ who request service from facility $j \in M$
μ_j	Nonnegative Continuous capacity allocation variables

As stated in the introduction, our main aim was to maximize the total number of people who would take advantage of the service. So, the objective function can be written as follows:

$$Z(x,y,\mu) = \sum_{i \in N} \sum_{j \in M} \lambda^{max}{}_i \; yij \tag{3}$$

We assume that the service at each facility j is exponentially distributed with service rate $\mu_j \geq 0$. We will consider a single-server Markovian queue, *M/M/*1 queue. Then, the expected waiting time, W_j, can be computed as follows:

$$W_j = W(\Lambda_j,\mu_j) = 1/(\mu_j - \Lambda_j) \qquad \Lambda_j < \mu_j \tag{4}$$

Also, we assume that a facility is chosen by a customer if it has a positive *utility*, which is defined as follows:

$$U_{ij} = V - t_{ij} - 1/(\mu_j - \Lambda_j) \qquad i \in N, j \in M \tag{5}$$

Where, due to horizontal differentiation model given in Hotelling (1929), U_{ij} denotes the *Utility* of customer at node i when receiving service from facility j and V is the *Willingness to pay (participate)* which represents the customers' valuation of service and assumed to be homogeneous for all of them.
Due to the above explanations, it is obvious that customers choose the facility- to receive service from- such that the corresponding utility will be maximized which, eventually, is a user-equilibrium problem where, at equilibrium, no customer wants to change his/her choice. Also, we want to incorporate it in the model in a way that when customer i doesn't receive service from facility j, U_{ij} needs not to be positive. In addition, more than one facility can be chosen by a customer, only if the relevant utilities are identical.
Let W^{max} be the maximum waiting time at each facility such that $W_j \leq W^{max}$ for $j \in M$. From (3) we have:

$$\mu_j - \sum_{i\in N} \lambda_i^{max} y_{ij} - \frac{x_j}{w^{max}} \geq 0 \tag{6}$$

Table 1, summarizes the notations and the problem can be formulated as follows:
Formulation1:

$$Max Z(x,y,\mu) = \sum_{i\in N}\sum_{j\in M} \lambda^{max}_i \; yij \tag{7}$$

Subject to

$$\sum_{j\in M} y_{ij} \leq 1 \quad i \in N \tag{8}$$

$$yij \leq xj \qquad i \in N, j \in M \tag{9}$$

$$\mu_j - \sum_{i\in N} \lambda_i^{max} y_{ij} - \frac{x_j}{w^{max}} \geq 0, \qquad j \in M \tag{10}$$

$$\sum_{j\in M} \mu_j \leq C^{max} \tag{11}$$

$$V - t_{ij} - \frac{1}{\mu_j - \sum_{i\in N} \lambda_i^{max} y_{ij}} \geq 0, \qquad i \in N, j \in M \tag{12}$$

$$y_{ij} \cdot U_{ij} \geq 0, \qquad i \in N, j \in M \tag{13}$$

$$xj \cdot xj' \cdot Uij = xj \cdot xj' \cdot Uij' \qquad for \qquad yij, yij' > 0 \tag{14}$$

$$Uij \geq Uij', \qquad if \qquad yij > 0, yij' = 0 \tag{15}$$

$$y_{ij} \geq 0, x_j \in \{0,1\}, \mu_j \geq 0, i \in N, j \in M \tag{16}$$

The objective function (6) aims at maximizing the total number of people who participate in the public service. Constraints (7) ensure that the total demand from customers at node i to all facilities cannot exceed one. Constraints (8) stipulate that service can be received from only open facilities. Constraints (9) limits the waiting time at each facility to W^{max}. Constraint (10) makes sure that the most total available service capacity C^{max} is distributed to the open facilities. Constraints (11) guarantee that customer's utility is nonnegative in case of assigning a customer to a facility. Actually, they are not necessary in the current formulation but will be useful later. Constraints (12) ensure that no assignment from customer at node i to facility j will be occurred in case of negative utilities. Constraints (13) guarantee that, at equilibrium, a customer can be assigned to more than one facility just if all the corresponding utilities are identical. Constraints (14) ensure that the assignment with greatest utility is chosen.

Since U_{ij} is a function of W_j, constraints (11)-(14) are nonlinear. Therefore, the problem is nonlinear and since the location decision variables are binary, the problem is difficult to solve. Thus, the focus of the study is on, first, transforming and linearizing the nonlinear integer model as much as possible and then developing an exact algorithm to obtain the optimal solution.

4- Model transformation and linearization

This section aims at transforming the nonlinear model introduced in the previous section into a linear model in order to solve it as a Mixed Integer Programming Model. First, we show that how the following lemma affects the mathematical formulation of the problem:

Lemma 1. *In the problem of jointly finding optimal location X*, optimal server allocation* $\mu*$ *and optimal customer allocation Y*,* $Z^*(X^*,\mu^*,Y^*)$, *there exists an optimal solution such that* $y_{ij^*}>0$, *for all* $i \in N, j \in M$, *at most for one j.*

Proof appears in Appendix 1.

Lemma1 demonstrates that there exists an optimal solution in which the demand of each population zone is either not covered or the whole covered demand assigns to one facility. Thus, since there is not any tradeoff between facilities to be assigned to a population zone, the equilibrium constraints (13-14) aren't necessary any more. Therefore, by applying lemma 1, the primary model can be rewritten as follows:

Formulation 2

$$MaxZ(x,y,\mu) = \sum_{i\in N}\sum_{j\in M}\lambda^{max}_i y_{ij} \quad (17)$$

Subject to

$$\sum_{j\in M} y_{ij} \leq 1 \qquad i \in N \quad (18)$$

$$y_{ij} \leq x_j \qquad i \in N, j \in M \quad (19)$$

$$\mu_j - \sum_{i\in N}\lambda_i^{max} y_{ij} - \frac{x_j}{w^{max}} \geq 0, \qquad j \in M \quad (20)$$

$$\sum_{j\in M} \mu_j \leq C^{max} \quad (21)$$

$$V - t_{ij} - \frac{1}{\mu_j - \sum_{i\in N}\lambda_i^{max} y_{ij}} \geq 0, \qquad i \in N, j \in M \quad (22)$$

$$Uij \geq Uij', \qquad if \qquad yij > 0, yij' = 0 \quad (23)$$

$$y_{ij} \geq 0, x_j \in \{0,1\}, \mu_j \geq 0, i \in N, j \in M \quad (24)$$

This is not completely linearized, because the constraint (23) remains nonlinear.

Let's consider a similar mathematical formulation:

Formulation 3:

$$MaxZ(x,y,\mu) = \sum_{i\in N}\sum_{j\in M} \lambda^{max}_i yij \tag{25}$$

Subject to

$$\sum_{j\in M} y_{ij} \leq 1 \quad i \in N \tag{26}$$

$$y_{ij} \leq x_j \qquad i \in N, j \in M \tag{27}$$

$$\mu_j - \sum_{i\in N} \lambda_i^{max} y_{ij} - \frac{x_j}{w^{max}} \geq 0, \qquad j \in M \tag{28}$$

$$\sum_{j\in M} \mu_j \leq C^{max} \tag{29}$$

$$V - t_{ij} - \frac{1}{\mu_j - \sum_{i\in N} \lambda_i^{max} y_{ij}} \geq 0, \qquad i \in N, j \in M \tag{30}$$

$$y_{ij} \geq 0, x_j \in \{0,1\}, \mu_j \geq 0, i \in N, j \in M \tag{31}$$

Next, we claim that by applying the following rule (Rule1) to the optimal solution of the last model (Formulation 3), we will arrive at a solution that is optimal for the model (Formulation 2), too.

Rule 1. *If $Z^*(X^*,Y^*,\mu^*)$ is an optimal solution in the problem of jointly finding optimal location X*, optimal server allocation μ* and optimal customer allocation Y*, Then, there exists an optimal solution in which the facilities are assigned to the customers such that the greatest Utilities are occured.*

Mathematically, let $J_{Open} = \{j|x_j > 0, j \in M\}$,

$$\forall j \in JOpen, yij1 > 0,$$

If $Uij1 < \max_{j\in J_{Open}} Uij$

Then

Let

$$j* = arg\{\max_{j\in J_{Open}} U_{ij}\} \tag{32}$$

Suppose "n" denotes the new values of the variables where "p" presents the previous values.

$$y^n_{ij*} = y^p_{ij1}, \qquad y^n_{ij1} = 0 \tag{33}$$

$$\mu^n_{j*} = \mu^p_{j*} + \lambda^{max}_k y^n_{kj*}, \qquad \mu^n_{j1} = \mu^p_{j1} - \lambda^{max}_k y^n_{kj*} \tag{34}$$

In Appendix 2, we will show that after this substitution in the objective function and constraints (when needed) we will arrive at the optimal solution.

5- Exact solution algorithm

In this section, we develop an efficient algorithm to solve the primary problem (7). The algorithm below is based on transforming the nonlinear model into a linear one and solving the obtained Mixed Integer

Programming model and test the solution, whether the ignored constraint set is satisfied or not, and finally adjusts the optimal solution if is needed. The algorithm proceeds as follows.

Algorithm1

Step1. Transform the problem formulation (7) using lemma1 and ignore Constraint (15).

Step2.Solve the problem (25). This formulation can be viewed as a mixed integer programming model.

Step3. In the optimal solution of (25), if $\forall j \in M$ constraint (15) is satisfied, Report the optimal solution as the optimal solution of (7).Else, if there exists $j \in M$ such that constraint (15) is not satisfied, apply Rule 1.

Step4. Repeat step3.

In step 1, the problem formulation (7) is transformed using Lemma1 and also the nonlinear constraint is ignored, resulting in a Mixed Integer Programming model, which is solved to optimality in step 2. In step 3, the satisfaction of ignored constraint (15) is checked and the optimal solution obtained in step 2 is adjusted if needed.

6- Computational results

The purpose of this section is to analyze the performance of the proposed algorithm. Also, an illustrative case study is presented and analyzed.

6-1-The efficiency of the algorithm

We designed a series of computational experiments and applied the proposed exact algorithm to them. The properties of the experiments mentioned below.

The number of potential facilities (m) was set to 10, 20, 30 and 40, and the number of population zones (n) was set to 100, 200,300 and 400. So, there were sixteen problem sets. We generated 10 instances in each set with the following properties: The maximum demand rate at each zone, $\lambda^{max}{}_i = 1$, the travel times were randomly generated in the interval [0, 5] hours and the maximum waiting time, $W^{max} = 100$. For problem instances of different sizes, total service capacity $C^{max} = \frac{n}{2}$ where total number of population zones denoted by n. All runs were performed on a machine with AMD FX-7600 Radeon R7 with 2.7 GHz CPU and 8 GB of RAM, running Windows 8.

Table 2. Average CPU times (sec)

n m	10	20	30	40
100	0.10	0.23	0.53	0.86
200	0.28	0.72	1.77	2.35
300	0.58	1.62	3.66	7.8
400	1.18	3.91	10.22	57.28

Table 3. Average CPU times (sec)

Location	Region	No. of hospitals	Postal code	C^{max}=30 Facility Located (1=yes)	Service rate assigned	Demand served	% utilization	C^{max}=45 Facility Located (1=yes)	Service rate assigned	Demand served	% utilization
1	34	1	7134					1	22.0125	22	99.94
2	35	1	7135					1	3.0125	3	99.58
3	36	1	7136					1	3.0125	3	99.58
4	43	2	7143	1	1.025	1	97.56	1	2.025	2	98.76
5	45	1	7145	1	12.8	12.78	99.90	1	1.0125	1	98.76
6	46	1	7146	1	1.0125	1	98.76				
7	53	1	7153	1	3.0125	3	99.58	1	2.0125	2	99.37
8	63	1	7163	1	3.0125	3	99.58	1	2.0125	2	99.37
9	64	1	7164					1	3.0125	3	99.58
10	73	1	7173	1	1.0125	1	98.76	1	2.8	2.79	99.64
11	87	1	7187	1	1.0125	1	98.76	1	2.0125	2	99.37
12	93	2	7193	1	2.025	2	98.76	1	1.025	1	97.56
13	94	3	7194	1	5.0375	5	99.25	1	1.0375	1	96.38
Total				9	30	29.78	98.99	12	45	44.79	91.39

The program was coded in and solved using Matlab R2014a and a time limit of 3,600 seconds (1 hour) was considered for applying the exact algorithm to each instance.
We evaluated the performance of the proposed algorithm in terms of CPU time, for which we set the limit as 1 hour per instance.
Average CPU times for all problem sets are summarized in Table 2. As it can be seen from Table 2, as the number of facilities increase, the average CPU times increase accordingly.

6-2- An illustrative case study

Control and prevention of preventable diseases is one of major medical concerns of government and related decision makers have conducted several comprehensive programs to coordinate the activities in this field. In this case study, a hypothetical program of locating a set of preventive medicine clinics in Shiraz, Iran is discussed. This study is based on the network of 17 public hospitals in Shiraz, Iran in which presenting the mentioned services is conceivable. To represent the total population network, we consider a 47-node network. Each node represents a region defined by the first 4 digits of the postal code. We placed the nodes at the centroid of each region and consider a link between the nodes if the corresponding regions have a common boundary. Using GIS data, the shortest distance between all node pairs is calculated. There are three hospitals in one region: 7194, two hospitals in two regions: 7143 and 7193 and a single hospital in ten regions. Totally, it is assumed that there are 13 hospital sites as the potential locations to establish the clinics. The total number of residential in the area represented in our model is 1.8 million. Now, by above explanations, the government needs to decide: (i) which existing hospitals should be housing a new clinic to offer preventive medical cares, (x_j), (ii) the allocation of service capacity at each clinic (μ_j) and (iii) The participation of people in the offered services, (y_{ij}).

A comparison between two scenarios with different total system capacity C^{max} is depicted in Table 3. Scenario 1: C^{max}=30 and Scenario 2: C^{max}=45. Under Scenario 1, 9 clinics are established with an average utilization of 98.99 %, whereas under Scenario 2, 12 clinics are open (all sites except 7146), with an average utilization of 91.39 %. The results show that having more than one hospital in the regions: 7143, 7193 and

7194 did not result in higher service rates than other the sites with just one clinic. It can be drawn from very high service capacity which allocated to clinic 7134 in Scenario 2 and to clinic 7145 in Scenario 1 that, having a clinic with reduced waiting time can result in better accessibility of people rather than reducing travel times.

7- Concluding remarks and future research

In this paper, we discussed the problem of designing preventive healthcare facility networks. The location of facilities, assignments of customers to facilities and capacity decisions are determined, and the objective is to maximize accessibility of customers to facilities while utility constraints are satisfied. A method to linearize the nonlinear model is described. Especially, an exact approach is suggested in the paper which performed very well in terms of CPU time. The managerial insights are derived based on the analysis of an illustrative case study of network of hospitals in Shiraz, Iran.

Our model can be generalized or extended in a number of ways. First, our model can be extended to the case where the demand originates over a region or plane. Another extension is to consider a fixed cost for each potential location to establish a facility. Third, minimum capacity requirements at each open facility can be enforced and more carefulness is needed in applying Rule 1. Fourth, multi-server queue can be considered at each facility instead of a single-server one.

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Appendix 1

Proof. Suppose that for j_1, j_2: $y_{ij^*1} > 0$, $y_{ij^*2} > 0$. We will show that we can substitute the addition of y_{ij^*1}, y_{ij^*2} by two new variables y'_{ij1} , y'_{ij2} such that: $y'_{ij1} = y_{ij*1} + y_{ij*2}$ or $y'_{ij2} = y_{ij*1} + y_{ij*2}$ $\mu'_{j1} = \mu*_{j1} + \lambda^{max}_i y_{ij*2}$
$\mu'_{j2} = \mu*_{j2} - \lambda^{max}_i y_{ij*2}$.

Now, we show that after this substitution in the objective function and constraints, the optimal solution remains optimal. Objective function:

$$Z^*(X^*, \mu^*, Y^*) = \sum_{i \in N} \lambda_i^{max}(y_{i1}^* + y_{i2}^* + \ldots + \underbrace{y_{ij1}^* + y_{ij2}^*}_{y'_{ij}1 \text{ or } y'_{ij}2} + \ldots + y_{im}^*)$$

$$\Longrightarrow Z^*(X^*, \mu^*, Y^*) = Z0^*(X'^*, \mu'^*, Y'^*)$$

Constraint 8:

$$\sum_{j \in M} y_{ij} = (y_{i1} + y_{i2} + \ldots + y_{im}) \leq 1, \qquad i \in N$$

$$\Longrightarrow (0 + 0 + \ldots + y_{ij^*1} + y_{ij^*2} + 0 + \ldots + 0) \leq 1$$

Constraint 9:

$$y_{ij}^{*1} \leq x^{*}_{j1} \implies x^{*}_{j1} = 1$$

$$y_{ij}^{*2} \leq x^{*}_{j2} \implies x^{*}_{j2} = 1$$

Constraint 10:

$$\mu_{j1}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{*} - \frac{x_{j1}^{*}}{W^{max}} \geq 0$$

$$\mu\prime_{j1} - \lambda_k^{max} y_{kj2}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{*} - \frac{x_{j1}^{*}}{W^{max}} \geq 0$$

$$\mu\prime_{j1} - \lambda_k^{max} \underbrace{(y_{kj1}^{*} + y_{kj2}^{*})}_{y\prime_{kj1}} - \sum_{i \in N, i \neq k} \lambda_i^{max} y_{ij1}^{*} - \frac{x_{j1}^{*}}{W^{max}} \geq 0$$

$$\mu_{j2}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij2}^{*} - \frac{x_{j2}^{*}}{W^{max}} \geq 0$$

$$\mu\prime_{j2} + \lambda_k^{max} y_{kj2}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij2}^{*} - \frac{x_{j2}^{*}}{W^{max}} \geq 0$$

$$\mu\prime_{j2} - \lambda_k^{max} \underbrace{(y_{kj2}^{*} - y_{kj2}^{*})}_{0} - \sum_{i \in N, i \neq k} \lambda_i^{max} y_{ij2}^{*} - \frac{x_{j2}^{*}}{W^{max}} \geq 0$$

Constraint 11:

$$\sum_{j \in M} \mu_j = \sum_{j \in M, j \neq j1, j2} \mu_j^{*} + \underbrace{\mu_{j1}^{*} + \mu_{j2}^{*}}_{\mu\prime_{j1} \text{ or } \mu\prime_{j2}} = C^{max}$$

$$\implies \sum_{j \in M} \mu'^{*}_{j} = C^{max}$$

Constraint 12:

$$\mu_{j1}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{*} - \geq \frac{1}{v - t_{ij1}}$$

$$\mu\prime_{j1} - \lambda_k^{max} y_{kj2}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{*} \geq -\frac{1}{v - t_{ij1}}$$

$$\mu\prime_{j1} - \lambda_k^{max} \underbrace{(y_{kj1}^{*} + y_{kj2}^{*})}_{y\prime_{kj1}} - \sum_{i \in N, i \neq k} \lambda_i^{max} y_{ij1}^{*} \geq \frac{1}{v - t_{ij1}}$$

$$\mu_{j2}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij2}^{*} \geq \frac{1}{v - t_{ij2}}$$

$$\mu\prime_{j2} + \lambda_k^{max} y_{kj2}^{*} - \sum_{i \in N} \lambda_i^{max} y_{ij2}^{*} \geq \frac{1}{v - t_{ij2}}$$

$$\mu\prime_{j2} - \lambda_k^{max} \underbrace{(y^*_{kj2} - y^*_{kj2})}_{0} - \sum_{i \in N, i \neq k} \lambda_i^{max} y^*_{ij2} \geq \frac{1}{v - t_{ij2}}$$

Constraint 13:

$$y^*_{ij1}(v - t_{ij1} - \frac{1}{\mu^*_{j1} - \sum_{i \in N} \lambda_i^{max} y^*_{ij1}}) \geq 0$$

$yij{*}\ 1 > 0 \quad -\!\!\longrightarrow$

$$v - t_{ij1} - \frac{1}{\mu^*_{j1} - \sum_{i \in N} \lambda_i^{max} y^*_{ij1}} \geq 0$$

$$\mu^*_{j1} - \sum_{i \in N} \lambda_i^{max} y^*_{ij1} - \geq \frac{1}{v - t_{ij1}}$$

It's done due to previous constraint.

$$y^*_{ij2}(v - t_{ij2} - \frac{1}{\mu^*_{j2} - \sum_{i \in N} \lambda_i^{max} y^*_{ij2}}) \geq 0$$

$yij{*}\ 2 > 0 \quad -\!\!\longrightarrow$

$$v - t_{ij2} - \frac{1}{\mu^*_{j2} - \sum_{i \in N} \lambda_i^{max} y^*_{ij2}} \geq 0$$

$$\mu^*_{j2} - \sum_{i \in N} \lambda_i^{max} y^*_{ij2} - \geq \frac{1}{v - t_{ij2}}$$

It's done due to previous constraint. □

Appendix 2

Proof.

$$y^n_{ij*} + y^n_{ij1} = y^p_{ij1} = y^p_{ij*} + y^p_{ij1} \tag{35}$$

,

$$\mu^n_{j*} + \mu^n_{j1} = \mu^p_{j*} + \mu^p_{j1} = \lambda_k^{max} y^n_{kj*} = \lambda_k^{max} y^p_{kj1} \tag{36}$$

Objective function: We have:

$$Z^p(x,y,\mu) = \sum_{i \in N} \sum_{j \in M} \lambda^{max}{}_i\, yij^p$$

$$= \sum_{i \in N} \lambda_i^{max} (y^p_{i1} U_{i1} + \cdots + y^p_{ij1} U_{ij1} + \underbrace{y^p_{ij*} U_{ij*}}_{0} + \cdots + y^p_{im} U_{im})$$

After substitution Constraint 8:

$$= \sum_{i \in N} \lambda_i^{max} (y^n_{i1} U_{i1} + \cdots + \underbrace{y^n_{ij1} U_{ij1}}_{0} + y^n_{ij*} U_{ij*} + \cdots + y^n_{im} U_{im})$$

Since

$$\forall j \neq j1, j*, \qquad yij^p = yij^n$$

$$\sum_{j \in M} y^p_{ij} = y^n_{i1} + \cdots + \underbrace{y^n_{ij1} + y^n_{ij*}}_{y^p_{ij1}} + \cdots + y^n_{im} = \sum_{j \in M} y^n_{ij} \leq 1$$

Constraint 9:

$$j1, j* \in JOpen \qquad -\!\!\longrightarrow xj1 = xj* = 1,$$

$$y^{n}_{ij1} \leq x_{j1}$$
$$y^{n}_{ij*} \leq x_{j*}$$

Constraint 10:

$$\mu^{p}_{j1} - \sum_{i \in N} \lambda^{max}_{i} y^{p}_{ij1} - \frac{x_{j1}}{w^{max}} \geq 0$$

Due to (34)

$$\mu^{n}_{j*} + \mu^{n}_{j1} - \mu^{p}_{j*} - \sum_{i \in N} \lambda^{max}_{i} y^{p}_{ij1} - \frac{x_{j1}}{w^{max}} \geq 0$$

→

$$\mu^{n}_{j1} + \underbrace{\lambda^{max}_{k} y^{p}_{kj1} - \lambda^{max}_{k} y^{p}_{kj1}}_{0} - \sum_{i \in N, i \neq k} \lambda^{max}_{i} y^{p}_{ij1} - \frac{x_{j1}}{w^{max}} \geq 0$$

$y^{n}_{ij1} = y^{p}_{ij1}$ $\quad$ i 6= k, $\quad$ $y_{ij1}^{n} = 0$ $\quad$ $i = k$

$$\mu^{n}_{j1} - \underbrace{\lambda^{max}_{k} y^{n}_{kj1}}_{0} - \sum_{i \in N, i \neq k} \lambda^{max}_{i} y^{n}_{ij1} - \frac{x_{j1}}{w^{max}} \geq 0$$

→

$$\mu^{n}_{j1} - \sum_{i \in N} \lambda^{max}_{i} y^{n}_{ij1} - \frac{x_{j1}}{w^{max}} \geq 0$$

$$\mu^{p}_{j*} - \sum_{i \in N} \lambda^{max}_{i} y^{p}_{ij*} - \frac{x_{j*}}{w^{max}} \geq 0$$

Due to (34)

$$\mu^{n}_{j*} + \mu^{n}_{j1} - \mu^{p}_{j1} - \sum_{i \in N} \lambda^{max}_{i} y^{p}_{ij*} - \frac{x_{j*}}{w^{max}} \geq 0$$

→

$$\mu^{n}_{j*} - \lambda^{max}_{k} y^{n}_{kj*} - \sum_{i \in N} \lambda^{max}_{i} y^{p}_{ij*} - \frac{x_{j*}}{w^{max}} \geq 0$$

$y^{n}_{ij*} = y^{p}_{ij*}$ $\quad$ i 6= k, $\quad$ $y_{ij*}^{p} = 0$ $\quad$ $i = k$

$$\mu^{n}_{j*} - \lambda^{max}_{k} y^{n}_{kj*} - \sum_{i \in N, i \neq k} \lambda^{max}_{i} y^{n}_{ij*} - \frac{x_{j*}}{w^{max}} \geq 0$$

→

$$\mu^{n}_{j*} - \sum_{i \in N} \lambda^{max}_{i} y^{n}_{ij*} - \frac{x_{j*}}{w^{max}} \geq 0$$

Constraint 11:

$$\sum_{j \in M} \mu j1^{p} \leq C^{max}$$

$$\mu^{p}_{1} + \cdots + \mu^{p}_{j1} + \mu^{p}_{j*} + \cdots + \mu^{p}_{m} \leq C^{max}$$

From (36):

$$\mu^{p}_{j} = \mu^{n}_{j} \qquad j \neq j1, j*, \qquad \mu^{p}_{j1} + \mu^{p}_{j*} = \mu^{n}_{j1} + \mu^{n}_{j*}$$

→

$$\mu^{n}_{1} + \cdots + \mu^{n}_{j1} + \mu^{n}_{j*} + \cdots + \mu^{n}_{m} \leq C^{max}$$

Constraint 12:

$$\mu_{j1}^{p} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{p} - \frac{1}{v - t_{ij1}} \geq 0 \quad (yijp\ 1 > 0 \longrightarrow zijp\ 1 = 1)$$

$$\mu_{j*}^{n} + \mu_{j1}^{n} - \mu_{j*}^{p} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{p} - \frac{1}{v - t_{ij1}} \geq 0$$

From (34)

$$\mu_{j1}^{n} + \lambda_k^{max} y_{kj*}^{n} - \lambda_k^{max} y_{kj1}^{p} - \sum_{i \in N, i \neq k} \lambda_i^{max} y_{ij1}^{p} - \frac{1}{v - t_{ij1}} \geq 0$$

$$\mu_{j1}^{n} - \sum_{i \in N} \lambda_i^{max} y_{ij1}^{n} - \frac{1}{v - t_{ij1}} \geq 0$$

$$\mu_{j*}^{p} - \sum_{i \in N} \lambda_i^{max} y_{ij*}^{p} - \frac{1}{v - t_{ij*}} \geq 0$$

$$\mu_{j*}^{n} + \mu_{j1}^{n} - \mu_{j1}^{p} - \sum_{i \in N} \lambda_i^{max} y_{ij*}^{p} - \frac{1}{v - t_{ij*}} \geq 0$$

From (34)

$$\mu_{j*}^{n} - \sum_{i \in N} \lambda_i^{max} y_{ij*}^{n} - \frac{1}{v - t_{ij*}} \geq 0$$

□