Direct Kinematics in Analytical Form of the 6-4 Fully-Parallel Mechanism

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This paper presents the direct position analysis of the fully-parallel mechanism that features six connection points on the base and four on the platform. The shape of both base and platform is general and, in particular, not restricted to being planar. The analysis is performed first by writing a suitable set of five closure equations in five unknowns. Then, by a specifically-developed elimination scheme, the closure equation set is reduced to a 32nd-order algebraic equation with only one unknown, which proves free from extraneous roots. Hence 32 closure configurations of the 6-4 mechanism do exist in the complex field. A numerical example is reported that confirms the new theoretical results.

Introduction

In recent years, a surging interest has focused the attention of robotics researchers and designers on fully-parallel mechanisms, which generally consist (see Fig. 1) of two rigid bodies (base and platform) connected through spherical pairs by six variable-length legs. The legs provide the platform with up to six degrees of freedom with respect to the base. Wherever an operational device such as a robot end-effector, a radar antenna, etc., has to be steadily placed or accurately moved in cartesian space, fully-parallel kinematic chains offer a series of widely-recognized advantages over serial or hybrid kinematic chains (Hunt, 1983).

Despite their relatively simple arrangement, the kinematic analysis of fully-parallel mechanisms is challenging. In particular, the direct position analysis, also referred to as direct kinematics, is recognized as the most involved issue in kinematic characterization of fully-parallel chains. It is aimed at finding, for a given set of leg lengths, all possible configurations of the mechanism, i.e., all possible positions and orientations (locations) of the platform relative to the base. All closure configurations being of interest, an analytical-form solution is the most suitable one since it generally reduces the problem to solving one algebraic equation with only one unknown. Accordingly, the degree of the equation provides the number of platform locations in the complex field, and the locations themselves can be found by determining all roots of the equation.

The first fully-parallel mechanism whose direct kinematics has been solved in analytical form was the well-known Stewart platform, first described by Stewart (1965). This mechanism (see Fig. 1) features six legs that meet the platform pairwise, the base singly, and belongs to the class of 6-3 fully-parallel mechanisms (the two-digit code conveys the number of distinct connection points on base and platform, respectively). Griffis and Duffy (1989) analyzed the planar-base case, whereas Innocenti and Parenti-Castelli (1990a) solved the general-geometry scheme. In both cases, 16 closure configurations were stated possible in the complex field.

Since then, several fully-parallel arrangements have been considered, and their direct kinematics solved in analytical form. The majority of these arrangements feature less than six connection points on the base, and more than three on the platform. While only a few of them can be regarded as special cases of the 6-3 Stewart platform, no generalization of such an arrangement has so far been endowed with an analytical-form direct kinematics solution. Just to glance at some of the solved cases, Nanua and Waldron (1990) presented the direct position analysis of a 6-3 mechanism with a

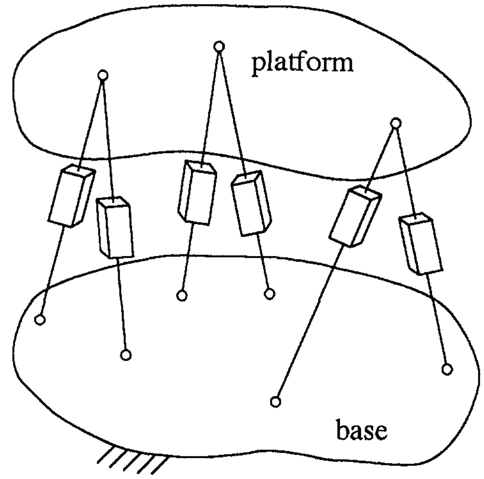

Fig. 1 The 6-3 Stewart platform

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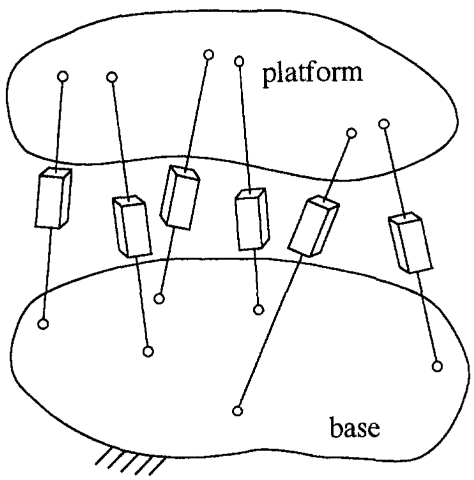

Fig. 2 The 6-6 generalized Stewart platform

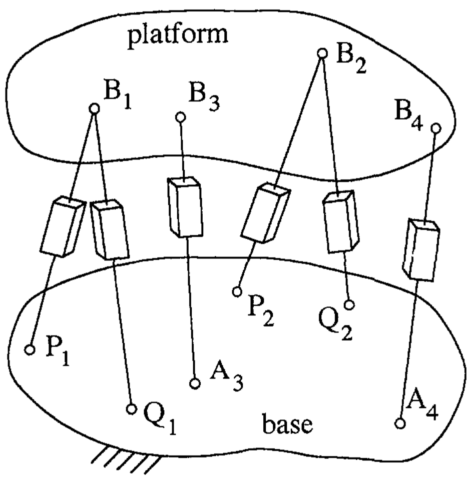

Fig. 3 The 6-4 fully-parallel mechanism

positional subchain, whereas in Lin et al. (1990) the direct kinematics of several 4-4 arrangements with planar base and platform was presented. The most involved direct kinematics of a general-geometry fully-parallel mechanism solved in analytical-form up to now seems to be that regarding the 5-5 arrangement (Innocenti and Parenti-Castelli, 1990b), for which 40 closure configurations have been found in the complex field.

The main goal in considering a variety of increasingly complex schemes lies in developing the skill to affront, in analytical form, the direct kinematics of the generalized Stewart platform. This mechanism, described by Fichter (1986), and represented in Fig. 2, is a 6-6 arrangement; it represents the utmost generalization of the 6-3 Stewart platform. Although some valuable efforts have been devoted to finding an upper bound to the number of closure configurations of the generalized Stewart platform (Merlet, 1991; Raghavan, 1991), the author is still unaware of any definitive method able to solve the direct kinematics of that mechanism in analytical form.

This paper presents the analytical-form direct kinematics solution of the 6-4 fully-parallel mechanism represented in Fig. 3. The mechanism features six variable-length legs, four of which meet the platform pairwise, while the remaining two meet both base and platform singly. The shape of base and platform is general and, in particular, not restricted to being planar. The 6-4 mechanism represents a first step in generalization of the 6-3 Stewart platform: indeed it can be thought of as derived from the 6-3 Stewart platform by disjoining a pair of touching legs (compare Figs. 1 and 3). To further connote the mechanism it is noticed that, if all legs were considered to have constant length, and the point where one of the two single legs meets the base were allowed to move in a circular arc, the 6-4 arrangement would reproduce the double-wishbone suspension linkage of present-day racing cars (Bastow, 1987).

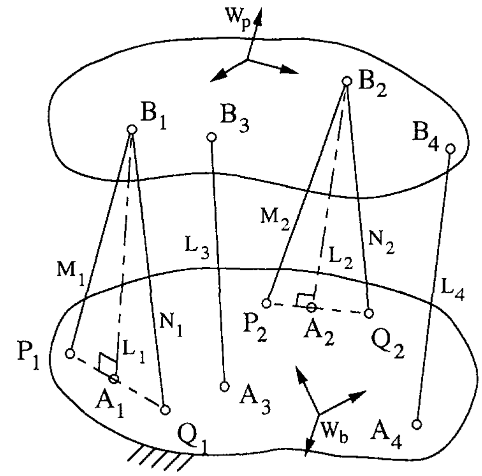

Fig. 4 The 6-4 structure

Only one other 6-4 fully-parallel arrangement could be devised, namely, the one studied by Innocenti and Parenti-Castelli (1991). It features three legs out of six that meet the platform at the same point, the remaining legs being singly connected to base and platform. Two sets of eight closure configurations each were found possible in the complex field for that arrangement. However, between the two possible 6-4 mechanisms, only the one considered in the present paper is a generalization on the 6-3 Stewart platform; furthermore, it is believed to represent the first generalization of the Stewart platform to be endowed with an analytical-form direct kinematics solution.

Unlike all fully-parallel arrangements studied in the literature, which required writing a maximum of three closure equations, the direct kinematics solution of the 6-4 mechanism here presented calls for a set of as many as five closure equations with five unknowns to be laid down. This imposes the adoption of a specifically-developed solution procedure that is able to drop four unwanted unknowns by means of a two-step elimination. As a result, a final algebraic equation of 32nd order, free from extraneous roots, is obtained. For every root, via back substitution, one location of the platform can be determined: hence the direct kinematics solution of the 6-4 mechanism admits of 32 closure configurations in the complex field.

A numerical example is reported that confirms the new theoretical results.

Kinematic Model

When performing the direct position analysis of a mechanism, the displacement values of all actuated kinematic pairs are known. Accordingly, all actuators can be thought of as frozen, and the mechanism itself can be regarded as a structure. The direct kinematics of the mechanism is then equivalent to finding all closure configurations of the structure.

Figure 4 shows the 6-4 structure, derived from the 6-4 fully-parallel mechanism in Fig. 3 by freezing all actuators. The leg lengths M_1, N_1, M_2, N_2, L_3, and L_4 are given; moreover, the position of points P_1, Q_1, P_2, Q_2, A_3, A_4 is known in an arbitrary reference frame W_b fixed to the base, whereas points B_j, $j = 1, \ldots, 4$, are given in an arbitrary

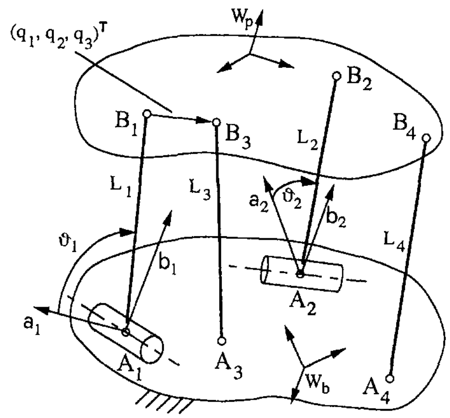

Fig. 5 The auxiliary structure

reference frame W_p fixed to the platform. It is worth noting that, although each leg of the 6-4 structure can still rotate about the line through the centers of the extremity spherical pairs, this freedom does not affect the location of the platform with respect to the base.

The Auxiliary Structure. Due to the couple of legs P_1B_1 and Q_1B_1—whose length is M_1 and N_1, respectively–point B_1 is confined to a circle which lies on a plane orthogonal to line P_1Q_1; the circle center A_1 and the circle radius L_1 are given by:

$$A_1 = P_1 + \frac{M_1^2 - N_1^2 + (Q_1 - P_1)^2}{2(Q_1 - P_1)^2}(Q_1 - P_1) \quad (1)$$

$$L_1 = \left\{ M_1^2 - \frac{\left[M_1^2 - N_1^2 + (Q_1 - P_1)^2\right]^2}{4(Q_1 - P_1)^2} \right\}^{1/2} \quad (2)$$

Similarly, point B_2 is confined to a circle which lies on a plane orthogonal to line P_2Q_2; the center, A_2, and the radius, L_2, of such a circle are still given by expressions (1) and (2), provided index 2 is throughout substituted for index 1.

From a kinematic standpoint, the 6-4 structure is thus equivalent to the auxiliary structure represented in Fig. 5, where links A_1B_1 and A_2B_2 are hinged to the base and joined through spherical pairs to the platform. In Figs. 4 and 5, points identified by the same label correspond.

In order to determine the set of five closure equations, the auxiliary structure is first disassembled, then gradually reassembled. At the beginning, all six spherical pairs are released, while links A_1B_1 and A_2B_2 still remain hinged at the base.

The First Closure Equation. In the disassembled auxiliary structure (see Fig. 5), links A_1B_1 and A_2B_2 are free to rotate with respect to the base. With reference to the jth revolute pair ($j = 1, 2$), mutually orthogonal unit vectors $\mathbf{a}_j$ and $\mathbf{b}_j$ are both chosen fixed to the base and orthogonal to the revolute pair axis. Hence the position of point B_j in reference frame W_b is given by:

$$(B_j - A_j) = L_j(\cos\theta_j \mathbf{a}_j + \sin\theta_j \mathbf{b}_j) \quad (j = 1, 2) \quad (3)$$

where θ_j is the angle between unit vector $\mathbf{a}_j$ and vector $(B_j - A_j)$.

The first step in the reassembling course consists in joining the platform to links A_1B_1 and A_2B_2 at points B_1 and B_2. This is feasible if and only if the following condition is satisfied:

$$(B_2 - B_1)_b^2 = (B_2 - B_1)_p^2 \quad (4)$$

where index b or p affixed to a vector specifies that the vector is to be resolved into components in reference frame W_b or W_p, respectively.

If the following relation is introduced:

$$(B_2 - B_1)_b = (B_2 - A_2)_b + (A_2 - A_1)_b - (B_1 - A_1)_b \quad (5)$$

and relations (3) are taken into account, Eq. (4) can be rearranged as:

$$(B_1 - A_1)_b \cdot (B_2 - A_2)_b + (A_2 - A_1)_b \cdot (B_1 - A_1)_b - (A_2 - A_1)_b \cdot (B_2 - A_2)_b = \left[L_1^2 + L_2^2 + (A_2 - A_1)_b^2 - (B_2 - B_1)_p^2\right]/2 \quad (6)$$

where the right-hand side is a constant quantity and the left-hand side contains unknowns θ_1 and θ_2 [see Eq. (3)]. Equation (6) represents the first closure equation of the auxiliary structure.

At this stage of reassembling, the linkage formed by base, platform, and links A_1B_1, A_2B_2 can be thought of as an *RSSR* closed loop mechanism (here *R* and *S* stand for revolute and spherical pair, respectively). Even if the position of the two links hinged at the base were known, the orientation of the platform about line B_1B_2 would remain undefined.

The Second Closure Equation. Three more unknowns —besides θ_1 and θ_2—are now introduced, namely the components q_1, q_2, and q_3 in reference frame W_b of vector $(B_3 - B_1)$:

$$(B_3 - B_1)_b = (q_1, q_2, q_3)^T \quad (7)$$

A first condition that vector $(B_3 - B_1)_b$ must satisfy regards its magnitude:

$$(B_3 - B_1)_b^2 = (B_3 - B_1)_p^2 \quad (8)$$

where the right-hand side is obviously a known quantity. Equation (8) represents the second closure equation; it contains unknowns q_1, q_2, and q_3 only.

The Third Closure Equation. One further condition expresses the invariance of angle between vectors $(B_2 - B_1)$ and $(B_3 - B_1)$ when seen from reference frames W_b and W_p:

$$(B_2 - B_1)_b \cdot (B_3 - B_1)_b = (B_2 - B_1)_p \cdot (B_3 - B_1)_p \quad (9)$$

By taking into account relation (5), the third closure equation is determined as:

$$(B_2 - A_2)_b \cdot (B_3 - B_1)_b + (A_2 - A_1)_b \cdot (B_3 - B_1)_b - (B_1 - A_1)_b \cdot (B_3 - B_1)_b = (B_2 - B_1)_p \cdot (B_3 - B_1)_p \quad (10)$$

where the right-hand side is a constant quantity and the left-hand side [see Eqs. (3) and (7)] contains all five unknowns.

The Fourth Closure Equation. Now leg A_3B_3, whose length is L_3, is added to the previously obtained linkage. This is feasible (see Fig. 5) if, and only if, the following condition holds:

$$(B_3 - A_3)_b^2 = L_3^2 \quad (11)$$

Furthermore, if relation

$$(B_3 - A_3)_b = (B_3 - B_1)_b + (B_1 - A_1)_b - (A_3 - A_1)_b \quad (12)$$

is introduced, and conditions (3) and (8) are taken into account, the following equation can be obtained:

$$(A_3 - A_1)_b \cdot (B_3 - B_1)_b + (A_3 - A_1)_b \cdot (B_1 - A_1)_b - (B_1 - A_1)_b \cdot (B_3 - B_1)_b = \left[L_1^2 - L_3^2 + (B_3 - B_1)_p^2 + (A_3 - A_1)_b^2\right]/2 \quad (13)$$

whose right-hand side is a constant term. Equation (13) represents the fourth closure equation, and contains unknowns θ_1, q_1, q_2, and q_3.

The Fifth Closure Equation. The last step in reassembling the auxiliary structure is the insertion of leg A_4B_4. This is feasible (see Fig. 5) if, and only if, the following condition is satisfied:

$$(B_4 - A_4)_b^2 = L_4^2 \quad (14)$$

In order to rearrange Eq. (14), the following relations are introduced:

$$(B_4 - A_4)_b = (B_4 - B_1)_b + (B_1 - A_1)_b - (A_4 - A_1)_b \quad (15)$$

$$(B_4 - B_1) = \lambda(B_2 - B_1) + \mu(B_3 - B_1) + \sigma(B_2 - B_1) \times (B_3 - B_1) \quad (16)$$

In relation (16), coefficients λ, μ, and σ are constant quantities; they can be determined as explained in the following.

Relation (16) holds in both reference frames W_b and W_p (that is why, in (16), the reference frame index has been dropped). If reference frame W_p is considered, all vectors appearing in Eq. (16) are known quantities. Hence, provided $(B_2 - B_1)_p$ and $(B_3 - B_1)_p$ are not collinear, constants λ, μ, and σ are given by:

$$\lambda = \left\{\left[(B_2 - B_1)_p \cdot (B_4 - B_1)_p\right](B_3 - B_1)_p^2 - \left[(B_3 - B_1)_p \cdot (B_4 - B_1)_p\right]\left[(B_2 - B_1)_p \cdot (B_3 - B_1)_p\right]\right\}/\zeta \quad (17)$$

$$\mu = \left\{\left[(B_3 - B_1)_p \cdot (B_4 - B_1)_p\right](B_2 - B_1)_p^2 - \left[(B_2 - B_1)_p \cdot (B_4 - B_1)_p\right]\left[(B_2 - B_1)_p \cdot (B_3 - B_1)_p\right]\right\}/\zeta \quad (18)$$

$$\sigma = \left\{\left[(B_2 - B_1)_p \times (B_3 - B_1)_p\right] \cdot (B_4 - B_1)_p\right\}/\zeta \quad (19)$$

where

$$\zeta = (B_2 - B_1)_p^2 (B_3 - B_1)_p^2 - \left[(B_2 - B_1)_p \cdot (B_3 - B_1)_p\right]^2 \quad (20)$$

Now relation (15), together with relation (16) referred to W_b, are inserted in (14). If conditions (3), (4), (5), and (8) are taken into account, the following equation can be obtained:

$$\begin{aligned}
&\lambda(B_1 - A_1)_b \cdot (B_2 - A_2)_b + \lambda(A_2 - A_1)_b \cdot (B_1 - A_1)_b \\
&- \lambda(A_4 - A_1)_b \cdot (B_2 - A_2)_b + \lambda(A_4 - A_1)_b \cdot (B_1 - A_1)_b \\
&+ \mu(B_1 - A_1)_b \cdot (B_3 - B_1)_b - \mu(A_4 - A_1)_b \cdot (B_3 - B_1)_b \\
&+ \sigma\left[(B_2 - A_2)_b \times (B_3 - B_1)_b\right] \cdot (B_1 - A_1)_b \\
&+ \sigma\left[(A_2 - A_1)_b \times (B_3 - B_1)_b\right] \cdot (B_1 - A_1)_b \\
&- \sigma\left[(B_2 - A_2)_b \times (B_3 - B_1)_b\right] \cdot (A_4 - A_1)_b \\
&+ \sigma\left[(B_1 - A_1)_b \times (B_3 - B_1)_b\right] \cdot (A_4 - A_1)_b \\
&- \sigma\left[(A_2 - A_1)_b \times (B_3 - B_1)_b\right] \cdot (A_4 - A_1)_b \\
&- (A_4 - A_1)_b \cdot (B_1 - A_1)_b \\
&= \left[L_4^2 - L_1^2 - (A_4 - A_1)_b^2 - (B_4 - B_1)_p^2\right]/2 \\
&\quad + \lambda L_1^2 + \lambda(A_2 - A_1)_b \cdot (A_4 - A_1)_b \quad (21)
\end{aligned}$$

which represents the fifth closure equation. The terms on the right-hand side of Eq. (21) are constant, while the left-hand side contains [see Eqs. (3) and (7)] all five unknowns θ_1, θ_2, q_1, q_2, and q_3.

Solution of the Closure Equation Set

The five closure Eqs. (6), (8), (10), (13), and (21) represent a set of necessary and sufficient conditions for assembly of the auxiliary structure. Prior to affronting solution of the closure equation set, each equation is rewritten in order to show the dependence on each of the unknowns.

Direct inspection of Eq. (6) shows that it has the form:

$$\sum_{i,j=0,1,2} a_{ij} C_1^{u(i)} S_1^{v(i)} C_2^{u(j)} S_2^{v(j)} = 0 \quad (22)$$

where coefficients a_{ij} (i,j = 0, 1, 2) are constant quantities that depend only on the geometry of the auxiliary structure; they can be determined by comparison with Eq. (6). In Eq. (22), the following notation has been introduced:

$$C_j = \cos\theta_j \quad S_j = \sin\theta_j \quad (j = 1, 2) \quad (23)$$

Moreover, integer exponents $u(k)$ and $v(k)$ are defined as:

$$u(k) = [k - \mathrm{mod}(k, 2)]/2; \quad v(k) = \mathrm{mod}(k, 2) \quad (24)$$

and mod(k, 2) is the remainder of the division of integer k by 2.

Obviously closure Eq. (7) can be rewritten as:

$$q_1^2 + q_2^2 + q_3^2 + b = 0 \quad (25)$$

where b is a constant term.

Inspection of closure Eqs. (10), (13), and (21) proves that they have the form:

$$\sum_{\substack{i=0,1,2\\k=0,\ldots,3}} \left(c_{i0k} C_1^{u(i)} S_1^{v(i)} + c_{0ik} C_2^{u(i)} S_2^{v(i)}\right) q_k = 0 \quad (26)$$

$$\sum_{\substack{i=0,1,2\\k=0,\ldots,3}} d_{i0k} C_1^{u(i)} S_1^{v(i)} q_k = 0 \quad (27)$$

$$\sum_{\substack{i,j=0,1,2\\k=0,\ldots,3}} e_{ijk} C_1^{u(i)} S_1^{v(i)} C_2^{u(j)} S_2^{v(j)} q_k = 0 \quad (28)$$

where

$$q_0 = 1 \quad (29)$$

must be considered. In Eqs. (26), (27), and (28), coefficients c_{i0k}, c_{0ik}, d_{i0k}, and e_{ijk} depend only on the geometry of the auxiliary structure.

To sum up, the closure equation set can henceforth be considered as represented by Eqs. (22), (25), (26), (27), and (28).

Elimination of q_1, q_2, and q_3. The left-hand sides of Eqs. (26), (27), and (28) depend linearly on q_1, q_2, and q_3. Hence those equations can be linearly solved for q_1, q_2, and q_3, which turn out to be functions of θ_1 and θ_2. By Cramer's rule, the following expression for q_1, q_2, and q_3 can be expected:

$$q_1 = D_1/D_0; \quad q_2 = D_2/D_0; \quad q_3 = D_3/D_0 \quad (30)$$

where:

$$D_k = \sum_{(i,j=0,\ldots,4)\cup(i=5,6;\, j=0,1,2)} f_{ijk} C_1^{u(i)} S_1^{v(i)} C_2^{u(j)} S_2^{v(j)} \quad (k = 0, \ldots, 3) \quad (31)$$

and coefficients f_{ijk} depend on coefficients c_{i0k}, c_{i0k}, d_{i0k}, and e_{ijk} of Eqs. (26), (27), and (28) only.

Actually, numerical computation shows that quantity D_0 admits the following expression:

$$D_0 = \sum_{i,j=0,\ldots,4} f_{ij0} C_1^{u(i)} S_1^{v(i)} C_2^{u(j)} S_2^{v(j)} \quad (32)$$

which is slightly simpler than the first of relations (31).

The elimination of unknowns q_1, q_2, and q_3 from the set of five closure equations is accomplished by substituting expressions (30) in Eq. (25). The following condition is thus obtained:

$$D_1^2 + D_2^2 + D_3^2 + bD_0^2 = 0 \quad (33)$$

and numerical computation shows it can be put in the form:

$$\sum_{\substack{i=0,\ldots,10;\\ j=0,\ldots,8}} g_{ijk} C_1^{u(i)} S_1^{v(i)} C_2^{(j)} S_2^{v(j)} = 0 \quad (34)$$

where coefficients g_{ijk} are constant quantities. Equations (22) and (35) together represent a reduced set of closure equations whose only unknowns are θ_1 and θ_2.

Elimination of θ_2. By substituting for C_j and S_j ($j = 1$, 2) the well-known identities:

$$C_j = (1 - t_j^2)/(1 + t_j^2); \quad S_j = 2t_j/(1 + t_j)^2 \quad (35)$$

Eqs. (22) and (34) can be respectively written in the following algebraic form:

$$\sum_{i,j=0,1,2} m_{ij} t_1^i t_2^j = 0 \quad (36)$$

$$\sum_{\substack{i=0,\ldots,10;\\ j=0,\ldots,8}} n_{ij} t_1^i t_2^j = 0 \quad (37)$$

where coefficients m_{ij} and n_{ij} depend only on the geometry of the auxiliary structure.

In order to simultaneously solve Eqs. (36) and (37) for t_1 and t_2, one unknown, say t_2, must be eliminated. By adopting the notation:

$$E_j = \sum_{i=0,1,2} m_{ij} t_1^i \quad (j = 0, 1, 2) \quad (38)$$

$$F_j = \sum_{i=0,\ldots,10} n_{ij} t_1^i \quad (j = 0, \ldots, 8) \quad (39)$$

the eliminant of Eqs. (36) and (37) can be written as:

$$\begin{vmatrix} E_0 & E_1 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_0 & E_1 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_0 & E_1 & E_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_0 & E_1 & E_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_0 & E_1 & E_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E_0 & E_1 & E_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & E_0 & E_1 & E_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_0 & E_1 & E_2 \\ F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & 0 \\ 0 & F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 \end{vmatrix} = 0 \quad (40)$$

Equation (40) represents the condition for Eqs. (36) and (37) to have a common root for t_2. Based on the order of terms E_j and F_j (see relations (38) and (39)), the degree of Eq. (40) cannot exceed 36. Actually, numerical computation shows that the real degree of Eqs. (40) is 32; hence this can be put in the form:

$$\sum_{j=0,\ldots,32} G_j t_1^j = 0 \quad (41)$$

where coefficients G_j, $j = 0, \ldots, 32$, depend only on the geometry of the auxiliary structure.

Equation (41) provides 32 solutions for t_1 in the complex field. It represents the final result of elimination of unknowns θ_2, q_1, q_2, and q_3 from the original set of five closure equations.

Back Substitution

For every root of Eq. (41), one closure configuration of the auxiliary structure can be determined by the following back-substitution procedure.

Let t_{1h} $(1 \le h \le 32)$ be the generic root of Eq. (41). For $t_1 = t_{1h}$, the left-hand sides of Eqs. (36) and (37) can be considered as polynomials in t_2. Such polynomials generally admit a G.C.D. of the first order, whose root directly provides the solution sought, t_{2h}, for t_2. Since t_{1h} and t_{2h} are known, so are (see relations (35)) the values θ_{1h} and θ_{2h} for θ_1 and θ_2. Hence, by Eqs. (32), (31), and (30), values q_{1h}, q_{2h}, and q_{3h} for q_1, q_2, and q_3 can be univocally determined.

By referring to Fig. 5 or, equivalently, to Eqs. (3), it is easy to recognize that θ_{1h} and θ_{2h} are responsible for the position of points B_1 and B_2 with respect to the base reference frame W_b. The position of B_1 being known, vector (q_{1h}, q_{2h}, q_{3h}) (see relation (7)) provides the position of point B_3 with respect to W_b. Since the positions of three distinct points of the platform are known, so is the position of every other point of the platform. In particular, the position of B_4 can be directly obtained from Eq. (16), if this is considered as referred to W_b.

Concluding, the auxiliary structure, as well as the 6-4 fully-parallel mechanism, exhibit 32 closure configurations in the complex field.

Numerical Example

With reference to Fig. 3, a 6-4 fully-parallel mechanism is considered as having the dimensions listed below (all lengths are given in arbitrary length unit). The coordinates of points on the base, referred to W_b, are: $P_1 = (0, 0, 0)$, $Q_1 = (5, 0, 1)$, $P_2 = (-2, 4, -1)$, $Q_2 = (-3, -1, 1)$, $A_3 = (6, -2, 2)$, $A_4 = (-3, 5, -1)$. The coordinates of points B_j, ($j = 1, \ldots, 4$) on the platform, referred to W_p, are: $B_1 = (4, 1, 4)$, $B_2 = (0, 3, 3)$, $B_3 = (4, 1, 5)$, $B_4 = (-4, 3, 3)$. As for the leg lengths, the following values are assumed: $P_1B_1 = 5.74$, $Q_1B_1 = 3.32$, $P_2B_2 = 4.58$, $Q_2B_2 = 5.39$, $A_3B_3 = 4.69$, $A_4B_4 = 4.58$.

By adopting the proposed solution procedure, 32 closure configurations have been determined, ten of which are real, and the remainder complex. They are reported in Table 1 in terms of (x, y, z) coordinates of points B_j ($j = 1, \ldots, 4$) in reference frame W_b. Only one solution is reported in Table 1 out of every couple of complex conjugate solutions.

It has been numerically proved that each of the 32 solutions satisfies the original system of five closure equations, thus confirming that no extraneous solution has been introduced by the adopted elimination procedure.

Moreover, for each closure configuration it has been verified that the distance between any couple of points connected by a leg equals the corresponding leg length. Such an overall test proves that even the adopted five closure equations are free from extraneous roots, and represents an ultimate check on the correctness of the results presented.

Conclusions

The analytical-form direct kinematics of the 6-4 fully-parallel mechanism with general geometry has been presented.

When all actuator displacements are given, the mechanism reduces to a structure. Inspection of the kinematic peculiarities of this structure has allowed a system of five closure

Table 1 Closure configurations in terms of (x, y, z) coordinates (real and imaginary parts) in reference frame W_b of points B_j, $j = 1, \ldots, 4$

x	y	z
——— configuration 1		
(4.60799399e+00, 0.00000000e+00)	(3.29586789e+00, 0.00000000e+00)	(9.22630038e−01, 0.00000000e+00)
(4.14439916e−01, 0.00000000e+00)	(2.80642465e+00, 0.00000000e+00)	(2.70435660e+00, 0.00000000e+00)
(4.85721886e+00, 0.00000000e+00)	(2.35584239e+00, 0.00000000e+00)	(6.89737312e−01, 0.00000000e+00)
(−3.45656592e+00, 0.00000000e+00)	(1.87588873e+00, 0.00000000e+00)	(2.31781198e+00, 0.00000000e+00)
——— configuration 2		
(4.06460986e+00, 0.00000000e+00)	(1.78331636e+00, 0.00000000e+00)	(3.63955072e+00, 0.00000000e+00)
(−2.89481075e−01, 0.00000000e+00)	(3.12878898e+00, 0.00000000e+00)	(3.15830691e+00, 0.00000000e+00)
(4.54375754e+00, 0.00000000e+00)	(2.35130403e+00, 0.00000000e+00)	(2.97036652e+00, 0.00000000e+00)
(−3.13862428e+00, 0.00000000e+00)	(5.91726751e+00, 0.00000000e+00)	(3.48506451e+00, 0.00000000e+00)
——— configuration 3		
(3.98438666e+00, 0.00000000e+00)	(8.63293514e−01, 0.00000000e+00)	(4.04066668e+00, 0.00000000e+00)
(6.55575203e−02, 0.00000000e+00)	(2.97759797e+00, 0.00000000e+00)	(2.95784869e+00, 0.00000000e+00)
(4.03471251e+00, 0.00000000e+00)	(9.90900822e−01, 0.00000000e+00)	(5.03121384e+00, 0.00000000e+00)
(−3.92224262e+00, 0.00000000e+00)	(3.24021084e+00, 0.00000000e+00)	(3.12662218e+00, 0.00000000e+00)
——— configuration 4		
(3.99904406e+00, 0.00000000e+00)	(1.10233617e+00, 0.00000000e+00)	(3.96737968e+00, 0.00000000e+00)
(−5.13755195e−02, 0.00000000e+00)	(3.02968925e+00, 0.00000000e+00)	(3.02961036e+00, 0.00000000e+00)
(3.97567551e+00, 0.00000000e+00)	(1.01911686e+00, 0.00000000e+00)	(4.96363691e+00, 0.00000000e+00)
(−4.04724560e+00, 0.00000000e+00)	(2.88212664e+00, 0.00000000e+00)	(2.92355567e+00, 0.00000000e+00)
——— configuration 5		
(3.96717364e+00, 0.00000000e+00)	(4.23341311e−01, 0.00000000e+00)	(4.12673181e+00, 0.00000000e+00)
(1.66935152e+00, 0.00000000e+00)	(1.61131733e+00, 0.00000000e+00)	(3.44044076e−01, 0.00000000e+00)
(3.28534369e+00, 0.00000000e+00)	(5.54339157e−01, 0.00000000e+00)	(4.84641750e+00, 0.00000000e+00)
(−1.25456744e+00, 0.00000000e+00)	(9.73352629e−01, 0.00000000e+00)	(−2.30995271e+00, 0.00000000e+00)
——— configuration 6		
(5.21727253e+00, 0.00000000e+00)	(−1.10331294e+00, 0.00000000e+00)	(−2.12376267e+00, 0.00000000e+00)
(1.64772745e+00, 0.00000000e+00)	(1.27421044e+00, 0.00000000e+00)	(−5.09535164e−01, 0.00000000e+00)
(5.85286920e+00, 0.00000000e+00)	(−3.82294290e−01, 0.00000000e+00)	(−2.39971371e+00, 0.00000000e+00)
(2.85559978e−02, 0.00000000e+00)	(3.73665292e+00, 0.00000000e+00)	(2.19502781e+00, 0.00000000e+00)
——— configuration 7		
(5.20935800e+00, 0.00000000e+00)	(1.21092582e+00, 0.00000000e+00)	(−2.08418999e+00, 0.00000000e+00)
(1.61648429e+00, 0.00000000e+00)	(1.76649011e+00, 0.00000000e+00)	(7.05542409e−01, 0.00000000e+00)
(5.33885116e+00, 0.00000000e+00)	(2.19362359e−01, 0.00000000e+00)	(−2.07840814e+00, 0.00000000e+00)
(−2.26198370e+00, 0.00000000e+00)	(1.26488087e+00, 0.00000000e+00)	(1.54570561e+00, 0.00000000e+00)
——— configuration 8		
(5.25353810e+00, 0.00000000e+00)	(1.85729160e−01, 0.00000000e+00)	(−2.30509048e+00, 0.00000000e+00)
(7.00951905e−01, 0.00000000e+00)	(6.44706293e−01, 0.00000000e+00)	(−2.55668332e+00, 0.00000000e+00)
(5.50915871e+00, 0.00000000e+00)	(8.99752342e−01, 0.00000000e+00)	(−1.65330141e+00, 0.00000000e+00)
(−2.92814032e+00, 0.00000000e+00)	(4.21901089e−01, 0.00000000e+00)	(−8.89335883e−01, 0.00000000e+00)
——— configuration 9		
(5.23408780e+00, 0.00000000e+00)	(−8.23025949e−01, 0.00000000e+00)	(−2.20783902e+00, 0.00000000e+00)
(9.16391747e−01, 0.00000000e+00)	(7.10472134e−01, 0.00000000e+00)	(−2.28454879e+00, 0.00000000e+00)
(5.14813914e+00, 0.00000000e+00)	(−1.73696679e+00, 0.00000000e+00)	(−2.60448114e+00, 0.00000000e+00)
(−2.33518078e+00, 0.00000000e+00)	(1.88851274e+00, 0.00000000e+00)	(−4.29439865e+00, 0.00000000e+00)
——— configuration 10		
(4.11895807e+00, 0.00000000e+00)	(2.15398296e+00, 0.00000000e+00)	(3.36780965e+00, 0.00000000e+00)
(1.66093285e+00, 0.00000000e+00)	(1.31253620e+00, 0.00000000e+00)	(−4.07118080e−01, 0.00000000e+00)
(4.99080592e+00, 0.00000000e+00)	(2.47088835e+00, 0.00000000e+00)	(2.99437654e+00, 0.00000000e+00)
(−2.12216657e−01, 0.00000000e+00)	(2.57653137e+00, 0.00000000e+00)	(−3.70766760e+00, 0.00000000e+00)
——— configurations 11 and 12		
(4.07647614e+00, −2.42063994e−03)	(−1.87415135e+00, 1.78557768e−02)	(3.58021929e+00, 1.21031997e−02)
(1.92319852e+00, 5.02149699e−01)	(1.88908069e+00, −7.61147214e−01)	(1.16537599e+00, −1.65179318e+00)
(2.37845932e+00, 1.07330870e+00)	(−2.26391745e+00, 2.64774635e+00)	(6.25830719e+00, 1.07691093e+00)
(−3.39410309e+00, −1.28285989e+00)	(−2.89860091e−01, 2.51903161e−01)	(−6.49766249e−01, 2.36114718e+00)
——— configurations 13 and 14		
(5.05222750e+00, 4.20853115e−02)	(−2.41308709e+00, 2.01348451e−01)	(−1.29853751e+00, −2.10426558e−01)
(1.62853572e+00, −4.97125208e−01)	(9.03273023e−01, −4.01826364e−01)	(−1.44647458e+00, −1.25312851e+00)
(5.14129988e+00, 1.01648689e+00)	(−2.89846478e+00, 6.89058354e−01)	(−2.69685460e+00, −3.17649194e−01)
(1.07377556e+00, −1.70527112e−01)	(4.65876209e+00, 4.90459460e−02)	(−3.10013046e+00, −3.38753012e−01)
——— configurations 15 and 16		
(3.77759996e+00, 1.02573191e+00)	(−5.87387440e+00, −3.77112113e+00)	(5.07460020e+00, −5.12865953e+00)
(1.65879427e+01, 1.40823497e+01)	(4.30378568e+00, −9.48243555e+00)	(1.45345105e+01, −1.66649140e+01)
(2.70929972e+00, 5.48098225e−01)	(−7.10973801e+00, −3.08071019e+00)	(5.40565933e+00, −4.09259813e+00)
(2.07763437e+01, 2.89993630e+01)	(1.39890834e+01, −9.58355733e+00)	(2.68655083e+01, −2.16522671e+01)
——— configurations 17 and 18		
(4.89985332e+00, 9.27090913e−01)	(5.70244685e+00, −1.23285726e+00)	(−5.36666576e−01, −4.63545457e+00)
(4.80724004e+00, 4.02962193e+00)	(2.66699669e+00, −3.07769058e+00)	(4.55218676e+00, −5.67941549e+00)
(4.25483181e+00, 1.78853128e+00)	(4.46297950e+00, −1.62597981e+00)	(−7.51110724e−01, −4.95434877e+00)
(1.50493222e+00, 6.37272357e+00)	(6.04002397e−01, −7.14521795e+00)	(9.33678550e+00, −5.81602943e+00)
——— configurations 19 and 20		
(4.13840370e+00, −1.28202000e+00)	(−7.24747324e+00, 2.16064793e+00)	(3.27058148e+00, 6.41010000e+00)
(1.44685105e+01, 1.30553888e+01)	(−5.75129903e+00, 3.99352274e+00)	(−1.16629173e+01, 1.65115013e+01)
(1.55486878e+00, −1.20311564e+00)	(−6.75065633e+00, 2.11064794e+00)	(3.17660182e+00, 3.97666908e+00)
(1.61722843e+01, 2.84197639e+01)	(−5.70326498e+00, 5.73606867e+00)	(−2.76285330e+01, 1.81563562e+01)
——— configurations 21 and 22		
(5.30226550e+00, 1.72463956e+00)	(9.31811256e+00, −3.34001957e+00)	(−2.54872750e+00, −8.62319782e+00)
(−8.20463268e+00, −3.10585423e+00)	(1.73867445e+00, 2.33121151e+00)	(−4.27455523e+00, 4.27510167e+00)
(1.22262093e+01, 5.25858396e−01)	(1.25337988e+01, −9.47273945e+00)	(−7.05981895e+00, −1.48348202e+01)
(−1.18178551e+01, 4.29622070e+00)	(9.21285104e+00, 7.68784437e+00)	(1.68669902e+00, 2.04552438e+00)
——— configurations 23 and 24		
(4.88758819e+00, 1.56291395e+00)	(8.60716460e+00, −1.31907139e+00)	(−4.75340929e−01, −7.81456975e+00)
(−1.22998994e+01, 1.38606943e+00)	(4.45791784e+00, 3.19537472e+00)	(4.75919898e−01, 8.68147151e+00)
(8.75075636e+00, 3.49943139e+00)	(1.13907444e+01, −1.63634426e+00)	(1.00006071e+00, −1.22865305e+01)
(−2.54333403e+01, 1.09539468e+01)	(4.21366499e+00, 1.09658085e+01)	(1.16681735e+01, 2.00783812e+01)
——— configurations 25 and 26		
(2.69064597e+00, −4.26318176e+00)	(2.19455294e+01, −9.68516452e+00)	(1.05093702e+01, 2.13159088e+01)
(−6.46261407e+00, 1.01576967e−02)	(2.90468026e+00, 2.33061321e−01)	(−4.88531390e−01, 5.87732150e−01)
(3.36486853e+00, −1.00539044e+01)	(2.11991552e+01, −1.83074281e+01)	(2.08979987e+01, 2.10722583e+01)
(−3.29525335e+00, −1.55066852e−01)	(4.33916615e−01, 3.54068362e−02)	(−6.08626134e−01, 2.96293536e−01)
——— configurations 27 and 28		
(4.77139878e+00, −1.52085342e+00)	(8.41994850e+00, 7.66458508e−01)	(1.05606098e−01, 7.60426709e+00)
(−1.09043534e+01, −3.71819511e+00)	(5.19872113e+00, −2.27506093e+00)	(3.02570113e+00, −7.54674989e+00)
(5.72219330e+00, −2.37150558e+00)	(7.84181179e+00, 6.73909741e−01)	(1.12619448e+00, 8.34432016e+00)
(−2.35058237e+01, −5.65739583e−01)	(4.55027403e−01, −3.56390787e+00)	(3.74501110e−01, −2.02246616e+01)
——— configurations 29 and 30		
(4.57018069e+00, 1.38043608e+00)	(−7.77185259e+00, −1.75542186e−01)	(1.11169657e+00, −6.90218042e+00)
(3.14884680e+00, −9.38730931e+00)	(−2.56877756e+00, −3.29803439e−01)	(−9.36644549e+00, −5.51816325e+00)
(2.16050348e+00, 2.21257695e+00)	(−8.21278734e+00, −1.28717100e+00)	(1.89995078e+00, −4.98016706e+00)
(7.88378854e−01, −1.68722887e+01)	(−5.67811237e−01, −1.00860287e+00)	(−1.75448933e+01, −3.52392187e+00)
——— configurations 31 and 32		
(4.55317516e+00, −2.43281914e+00)	(1.28351623e+01, −2.71131418e−01)	(1.19672418e+00, 1.21640957e+01)
(1.95869356e+00, 1.36620144e−01)	(1.28394476e+00, 5.63874301e−01)	(−3.29716328e−01, 1.47799583e+00)
(7.72338603e+00, −1.21927649e+01)	(2.08951971e+01, −1.13757487e+01)	(1.57824770e+01, 2.04217900e+01)
(2.01833621e+01, −4.80983761e−03)	(1.85209112e+01, 6.36097849e+00)	(−4.31875175e+00, 2.58816336e+01)

equations with five unknowns to be found. The way the equations have been determined guarantees they are free from extraneous roots.

By a specifically-developed solution procedure, four unwanted unknowns have been eliminated, still avoiding the inclusion of extraneous roots. As a result, an algebraic equation of 32nd order has been obtained which exhibits 32 roots in the complex field. For every root, a closure configuration of the 6-4 fully-parallel mechanism can be determined. Hence the direct position analysis of the general-geometry 6-4 fully-parallel mechanism admits of 32 solutions in the complex field.

Finally, a numerical example has been reported which confirms the new theoretical results.

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References

Bastow, D., 1987, *Car Suspension and Handling*, Pentech Press, London.

Fichter, E. F., 1986, "A Stewart Platform-Based Manipulator: General Theory and Practical Construction," *Int. Journal of Robotics Research*, Vol. 5, No. 2, pp. 157–182.

Griffis, M., and Duffy, J., 1989, "A Forward Displacement Analysis of a Class of Stewart Platforms," *Journal of Robotic Systems*, John Wiley, Vol. 6, No. 6, pp. 703–720.

Hunt, K. H., 1983, "Structural Kinematics of In-Parallel-Actuated Robot-Arms," ASME JOURNAL OF MECHANISM, TRANSMISSIONS, AND AUTOMATION IN DESIGN, Vol. 105, pp. 705–712.

Innocenti, C., and Parenti-Castelli, V., 1990a, "Direct Position Analysis of the Stewart Platform Mechanism," *Mechanism and Machine Theory*, Vol. 25, No. 6, pp. 611–621.

Innocenti, C., and Parenti-Castelli, V., 1990b, "Closed-Form Direct Position Analysis of a 5-5 Parallel Mechanism," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 115, No. 3, 1993, pp. 515–521.

Innocenti, C., and Parenti-Castelli, V., 1991, "Direct Kinematics of the 6-4 Fully Parallel Manipulator with Position and Orientation Uncoupled," Proc. of the European Robotics and Intelligent Systems Conference, Corfù, Greece, June 23–28, pp. 3–10.

Lin, W., Griffis, M., and Duffy, J., 1990, "Forward Displacement Analyses of the 4-4 Stewart Platforms," *Proc. of the 21st ASME Mechanisms Conference*, Chicago, IL, Sept. 16–19, pp. 263–269.

Merlet, J. P., 1991, "An Algorithm for the Forward Kinematics of General Parallel Manipulators," *Proc. of the Fifth Int. Conf. on Advanced Robotics*, Pisa, Italy, June 19–22, pp. 1136–1140.

Nanua, P., and Waldron, K. J., 1990, "Direct Kinematic Solution of a Special Parallel Robot Structure," *Proc. 8th CISM-IFToMM Symposium on Theory and Practice of Robots and Manipulators*, Cracow, Poland, July 2–6, pp. 134–142.

Raghavan, 1991, "The Stewart Platform of General Geometry has 40 Configurations," *Proc. of the ASME Design Engineering Conference*, Miami, FL, Sept. 22–25.

Stewart, D., 1965, "A Platform with Six Degrees of Freedom," *Proc. Instn. Mech. Engrs.*, London, Vol. 180, Pt. 1, No. 15, pp. 371–386.