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Abstract

In this note we provide a simple alternative derivation of an explicit formula of Kwan and Yang [14] for the probability of ruin in a risk model with a certain dependence between general claim inter-occurrence times and subsequent claim sizes of conditionally exponential type. The approach puts the type of formula in a general context, illustrating the potential for similar simple ruin probability expressions in more general risk models with dependence.

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1 Introduction

Consider a renewal process N_t with inter-epoch times A_k, and let

$$Y_t = x + ct - \sum_{k=1}^{N_t} B_k$$

be the surplus process of an insurance portfolio with initial capital $x \geq 0$, premium intensity $c \geq 0$ and claims B_k. The study of the ruin probability $\psi(x) = \mathbb{P}_x(T < \infty)$, with $T = \inf\{t \geq 0 : Y_t < 0\}$, is a classical topic in risk theory. Whereas the usual assumption is that the i.i.d. sequences of random variables $(A_k)_{k\geq 1}$ and $(B_k)_{k\geq 1}$ are independent (in which case the model is referred to as the Sparre-Andersen model), in recent years there has been increased interest in models with certain types of dependence. In that context, [3] considered a model in which A_{k+1} depends on the previous claim size B_k and derived some explicit expressions for the probability of ruin in this setting. In [4] this analysis was extended to a semi-Markovian risk model (see also [2]). Due to a sample path duality (see e.g. [7, Ch.III.2]), risk processes of that type have a counterpart in workload models of queueing theory, and a similar semi-Markovian structure was considered in [1] in a queueing context.
In [5] it was proposed to assume that (A_k, B_k) are i.i.d. pairs of positive random variables, but for each k, A_k and B_k may be dependent. Under this assumption the random walk structure of the process (when observed immediately after the claim occurrences only) is still preserved and allows for some explicit analysis (this model is also referred to as the dependent Sparre Andersen model, cf. [10]). In [8] an explicit expression for $\psi(x)$ could be obtained for a particular type of dependence between A_k and B_k. In [14], another explicit dependence structure in the framework of [5] was considered: if $A_k < a$, then B_k is distributed according to $B^{(1)}$, otherwise according to $B^{(2)}$, where a is a fixed threshold. It could be shown in [14] that the ruin probability for this model has a remarkably simple form

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if the inter-occurrence times are exponentially distributed and also $B^{(1)}$ and $B^{(2)}$ are exponentially distributed (with rates μ_1 and μ_2 respectively). In particular, in that case

$$\psi(x) = c_1 e^{-\theta_1 x} + c_2 e^{-\theta_2 x}, \tag{1}$$

where $c_1, c_2 \in \mathbb{R}$ and θ_1, θ_2 are positive roots of a certain characteristic equation. This result was proven in [14] with quite some effort by deriving an integro-differential equation for the ruin probability and reformulating it into a delay differential equation of a particular, tractable, type. In this note we provide a quick alternative derivation of formula (1) for general inter-occurrence times (not just exponentially distributed), which can be easily extended to the case of multiple thresholds and phase-type distributed claims.

2 A short proof of (1) for general inter-occurrence times

The idea is to view this problem as the problem of a continuous first passage downwards for a related spectrally-positive Markov Additive Process (MAP) [6, Ch. XI] on two states. In fact, only some very basic observations are required to establish that the ruin probability is given by a sum of exponential terms. Some deeper theory is needed to identify the constants.

Let us interchange jumps and inter-occurrence times, so that cA_k becomes a jump and B_k becomes the subsequent inter-occurrence time during which the process decreases linearly with slope -1; the initial position of this auxiliary process is given by $x + cA_1$. Note that the events of ruin for both processes coincide. This auxiliary process $X(t)$ is a MAP with only positive jumps, for which the ruin probability is known. In fact, it is a Markov-modulated linear drift model on two states with jumps at switching times.

Let $J(t) \in \{1, 2\}$ denote the state. According to our construction, J stays in state 1 for an $\mathrm{Exp}(\mu_1)$ time, and moves to the states 1 or 2 with probabilities p and $1-p$ respectively, where $p = \mathbb{P}(A_1 < a)$. Similarly, it stays in state 2 for an $\mathrm{Exp}(\mu_2)$ time, and moves to the states 1 or 2 with probabilities p and $1-p$. The moves of J into state 1 (irrespective of the previous state) cause a jump of $X(t)$ distributed as cA_1 given $A_1 < a$, and the moves into state 2 cause a jump of $X(t)$ distributed as cA_1 given $A_1 \geq a$ (this is the analogous interpretation to the one in [4] for a risk model with dependence between claims and subsequent inter-occurrence times). Let $A^{(1)}$ and $A^{(2)}$ denote random variables distributed as cA_1 given $A_1 < a$ and $A_1 \geq a$, respectively. Finally, in between the moves of the J process, X decreases linearly with slope -1. For a certain initial distribution of (X, J), to be specified later, we need to determine $\mathbb{P}(\tau_0^- < \infty)$, where for $y \geq 0$ we define $\tau_y^- = \inf\{t \geq 0 : X(t) < -y\}$.

Suppose for a moment that $X(0) = 0$. Using the memoryless property of the exponential distribution it is not difficult to see that $J(\tau_y^-), y \geq 0$ is also a Markov chain with some transition rate matrix Λ. It is transient if $X(t)$ (equivalently, the original process) drifts to ∞ a.s. (one can also think of adding an additional absorbing state to the state space of this Markov chain). Observe that

$$\mathbb{P}(\tau_y^- < \infty | J(0) = 1) = (1, 0) e^{\Lambda y} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and a similar identity is true for $J(0) = 2$, see e.g. [6, Cor. II.3.5]. The initial distribution of J is given by $(p, 1-p)$, and $X(0)$ is distributed as $x + A^{(i)}$ on $J(0) = i$ for $i = 1, 2$, and hence we obtain

$$\mathbb{P}_x(T < \infty) = p(1, 0)\mathbb{E}e^{\Lambda(x + A^{(1)})} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1-p)(0, 1)\mathbb{E}e^{\Lambda(x + A^{(2)})} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{2}$$

Remark 2.1. *Replacing the vectors $(1, 1)^T$ by $(1, 0)^T$ on the right hand side of (2) would lead to the probability of ruin caused by a claim of type 1.*

It is well-known that all the eigenvalues of a transient irreducible transition rate matrix belong to the left half of the complex plane. To see this, one can use Gershgorin's theorem and the fact that irreducibly diagonally dominant matrices are invertible, see e.g. [12]. Furthermore, these eigenvalues are distinct and real for a 2×2 matrix, which is easily shown by the examination of the corresponding characteristic equation. Let us denote the eigenvalues of Λ by $-\theta_1$ and $-\theta_2$, where $\theta_1 \neq \theta_2$ and $\theta_1, \theta_2 > 0$. Then Λ can be written as

$$\Lambda = V \begin{pmatrix} -\theta_1 & 0 \\ 0 & -\theta_2 \end{pmatrix} V^{-1},$$

where the 2×2 matrix V is formed from the corresponding eigenvectors. Substituting this into (2) we get the following expression for the ruin probability

$$\mathbb{P}_x(T<\infty) = p(1,0)V \begin{pmatrix} e^{-\theta_1 x}G_1(\theta_1) & 0 \\ 0 & e^{-\theta_2 x}G_1(\theta_2) \end{pmatrix} V^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1-p)(0,1)V \begin{pmatrix} e^{-\theta_1 x}G_2(\theta_1) & 0 \\ 0 & e^{-\theta_2 x}G_2(\theta_2) \end{pmatrix} V^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $G_i(\theta)$ is the transform of $A^{(i)}$; e.g., $G_1(\theta) = \mathbb{E}e^{-\theta A^{(1)}} = \mathbb{E}[e^{-\theta c A_1} | A_1 < a]$. This shows that the ruin probability is of the form (1). It is only left to identify the spectrum of Λ, i.e. the numbers θ_1, θ_2 and the matrix V.

3 Concluding comments

Identification of the matrix Λ is a well-studied problem, see [9] for an iterative procedure and [11] for a spectral method, where the latter also contains a list of earlier works. Firstly, note that there exists a matrix $F(\theta)$ for $\theta \geq 0$, which characterizes our MAP: $\mathbb{E}[e^{-\theta X(t)}; J(t) = j | X(0) = 0, J(0) = i] = [e^{F(\theta)t}]_{ij}$ with $i, j \in 1, 2$. For the present model this matrix is given by

$$F(\theta) = \begin{pmatrix} -\mu_1 + \theta + p\mu_1 G_1(\theta) & (1-p)\mu_1 G_2(\theta) \\ p\mu_2 G_1(\theta) & -\mu_2 + \theta + (1-p)\mu_2 G_2(\theta) \end{pmatrix},$$

see also [6, Prop.XI.2.2]. Secondly, the spectral method of [11] states that θ_1 and θ_2 are the two zeros of $\det(F(\theta))$ in the right half complex plane, which are positive in our case. Furthermore, the corresponding eigenvectors $\boldsymbol{v}_i$, which define $V = [\boldsymbol{v}_1, \boldsymbol{v}_2]$, are found from $F(\theta_i)\boldsymbol{v}_i = \mathbf{0}$.

Example 3.1. Let us consider the particular case when A_k have an exponential distribution of rate λ. Then

$$pG_1(\theta) = \mathbb{E}[e^{-\theta c A_1}; A_1 < a] = \int_0^a \lambda e^{-\lambda t} e^{-\theta c t} \mathrm{d}t = \frac{\lambda}{\lambda + \theta c}(1 - e^{-(\lambda+\theta c)a}),$$
$$(1-p)G_2(\theta) = \mathbb{E}[e^{-\theta c A_1}; A_1 \geq a] = \frac{\lambda}{\lambda + \theta c} e^{-(\lambda + \theta c)a},$$

which reduces the equation $\det(F(\theta)) = 0$ to

$$(\theta - \mu_2)(\lambda + \theta c - \mu_1 c) + \lambda(\mu_2 - \mu_1)e^{-(\lambda+\theta c)a} = 0,$$

and θ_1, θ_2 in (1) are the positive solutions of this equation. This result indeed coincides with the result of [14, Eqns. 8 and 15].

Remark 3.1. The method easily carries over to more general models. Firstly, one can have multiple thresholds for A_k. Then for m intervals the dimension of the matrices Λ and $F(\theta)$ will be $m \times m$. Moreover, one can extend the model to claim sizes of phase-type with $(n_i)_{i=1,\ldots,m}$ phases. Then the dimension of the matrices becomes $(\sum_{i=1}^{m} n_i) \times (\sum_{i=1}^{m} n_i)$. In that case the eigenvalues of Λ are not necessarily real and distinct. Nevertheless, the spectrum of Λ can still be identified, see [11]. The corresponding expression (1) may then include terms of the form $x^k e^{-\theta_i x}$ for certain k.

Finally, we note that the concept of killing (cf. [13]) can be used in the present context to extend this result on the ruin probability to other ruin-related quantities, e.g. the joint transform of the time to ruin and the deficit at ruin.

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