On the logarithm of the minimizing integrand for certain variational problems in two dimensions

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Abstract

Let f be a smooth convex homogeneous function of degree $p, 1 < p < \infty$, on $\mathbb{C} \setminus \{0\}$. We show that if u is a minimizer for the functional whose integrand is $f(\nabla v), v$ in a certain subclass of the Sobolev space $W^{1,p}(\Omega)$, and $\nabla u \neq 0$ at $z \in \Omega$, then in a neighborhood of z, $\log f(\nabla u)$ is a sub, super, or solution (depending on whether $p > 2, p < 2$, or $p = 2$) to L where

$$L\zeta = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k \eta_j}(\nabla u(z)) \frac{\partial \zeta}{\partial x_j} \right),$$

We then indicate the importance of this fact in previous work of the authors when $f(\eta) = |\eta|^p$ and indicate possible future generalizations of this work in which this fact will play a fundamental role.

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1 Introduction

Let Ω denote a bounded region in the complex plane $\mathbb{C}$. Given $p, 1 < p < \infty$, let $z = x_1 + ix_2$ denote points in $\mathbb{C}$ and let $W^{1,p}(\Omega)$ denote equivalence classes of functions $h : \mathbb{C} \to \mathbb{R}$ with distributional gradient $\nabla h = h_{x_1} + ih_{x_2}$ and Sobolev norm

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|h|^p + |\nabla h|^p) dA \right)^{1/p} < \infty$$

where dA denotes two dimensional Lebesgue measure. Let $C_0^\infty(\Omega)$ denote infinitely differentiable functions with compact support in Ω and let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$. Let $f : \mathbb{C} \setminus \{0\} \to (0, \infty)$ be homogeneous of degree p on $\mathbb{C} \setminus \{0\}$. That is

$$f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0 \text{ when } \eta \in \mathbb{C} \setminus \{0\}. \tag{1.1}$$

Assume also that f is strictly convex in $\mathbb{C} \setminus \{0\}$. Given $h \in W^{1,p}(\Omega)$ let $E = \{h + \phi : \phi \in W_0^{1,p}(\Omega)\}$. It is well known (see [HKM, chapter 5]) that

$$\inf_{w \in E} \int_\Omega f(\nabla w) dA = \int_\Omega f(\nabla u) dA \text{ for some } u \in E.$$

Moreover u is a weak solution at $z \in \Omega$ to the Euler equation,

$$\nabla \cdot (\nabla f(\nabla u(z))) = \sum_{k=1,2} \frac{\partial}{\partial x_k} \left(\frac{\partial f}{\partial \eta_k} (\nabla u(z)) \right) = \sum_{k,j=1}^{2} f_{\eta_k \eta_j}(\nabla u(z))\, u_{x_k x_j}(z) = 0\,. \tag{1.2}$$

That is, $\int_\Omega \langle \nabla f(\nabla u(z)), \nabla \theta(z) \rangle dA = 0$ whenever $\theta \in W_0^{1,p}(\Omega)$. Here $\nabla \cdot$ denotes divergence in the $z = x_1 + ix_2$ variable and $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{C}$. Moreover if f is sufficiently ' smooth, ' it follows from either Schauder theory or the fact that ∇u is a quasiregular mapping of $\mathbb{C}$ that u has continous third derivatives in a neigborhood of z whenever $\nabla u(z) \neq 0$. In this case (1.2) holds pointwise and we can differentiate this equation with respect to $x_l, l = 1, 2$, to get

$$0 = \nabla \cdot \left(\frac{\partial}{\partial x_l} (\nabla f(\nabla u(z))) \right) = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(\frac{\partial^2 f}{\partial \eta_k \eta_j} (\nabla u(z))\, u_{x_j x_l} \right)$$

From this display we see that if $\nabla u(z) \neq 0$, and u, f are sufficiently smooth, then $\zeta = u_{x_l}$ satisfies

$$L\zeta = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(b_{kj}(z) \frac{\partial \zeta}{\partial x_j} \right) = 0 \tag{1.3}$$

where $b_{kj}(z) = f_{\eta_k \eta_j}(\nabla u(z))$ when $1 \leq k, j \leq 2$. We claim that also $\zeta = u$ is a solution to $L\zeta = 0$ in a neighborhood of z. To prove this claim and for later use note that from the homogeneity of f and Euler's formula it follows for $k = 1, 2$ that if $\eta \neq 0$, then

$$\sum_{j=1}^{2} \eta_j\, f_{\eta_k \eta_j}(\eta) = (p-1) f_{\eta_k}(\eta) \text{ and } \sum_{k=1}^{2} \eta_k\, f_{\eta_k}(\eta) = p f(\eta). \tag{1.4}$$

Putting u in for ζ in (1.3) and using (1.4), (1.2), it follows that

$$Lu = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k \eta_j}(\nabla u(z)) \frac{\partial u}{\partial x_j} \right) = (p-1) \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \left(f_{\eta_k}(\nabla u(z)) \right) = 0.$$

Using (1.3) for $\zeta = u_{x_l}, l = 1, 2,$ and $\zeta = u$ we prove

Theorem 1. *In a neighborhood of z and under the above assumptions, $\log f(\nabla u))$ is a sub solution, solution, or super solution to L in (1.3) respectively when $p > 2, p = 2, p < 2$.*

Before proving Theorem 1 we indicate its relevance and possible applications of this theorem. To this end we introduce the following notation. Let $B(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$ whenever $z \in \mathbb{C}$ and $r > 0$. Let $d(E, F)$ denote the distance between the sets $E, F \subset \mathbb{C}$. If $\lambda > 0$ is a positive function on $(0, r_0)$ with $\lim_{r \longrightarrow 0} \lambda(r) = 0$ define H^λ Hausdorff measure on $\mathbb{C}$ as follows: For fixed $0 < \delta < r_0$ and $E \subseteq \mathbb{C}$, let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \quad i = 1, 2, ...$ Set

$$\phi_\delta^\lambda(E) = \inf_{L(\delta)} \sum \lambda(r_i).$$

Then

$$H^\lambda(E) = \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

In case $\lambda(r) = r^\alpha$ we write H^α for H^λ.

Next suppose $D \subset \mathbb{C}$ is a bounded simply connected domain, $z_o \in D, \Omega = D \setminus B(z_0, \frac{1}{2} d(z_0, \partial D))$, and u is a minimizer for the above variational problem in Ω with boundary values $u = 1$ on $\partial B(z_0, \frac{1}{2} d(z_0, \partial D))$ and $u = 0$ on ∂D in the $W^{1,p}(\Omega)$ sense. Put $u \equiv 0$ outside of D. Then it follows from [HKM, ch 15] that there exists a unique finite positive Borel measure μ on ∂D satisfying

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \theta(z) \rangle dA = - \int \theta d\mu$$

whenever $\theta \in C_0^\infty(\mathbb{C} \setminus \bar{B}(z_0, \frac{1}{2} d(z_0, \partial D)))$. Define the Hausdorff dimension of μ denoted H-dim μ, by

$$\text{H-dim } \mu = \inf\{\alpha : \text{ there exists } E \text{ Borel} \subset \partial\Omega \text{ with } H^\alpha(E) = 0 \text{ and } \mu(E) = \mu(\partial\Omega)\}.$$

If $f(\nabla u) = |\nabla u|^2$, i.e, when u is harmonic, Makarov [M] essentially proved

Theorem A.

(*a*) *μ is concentrated on a set of σ finite H^1 measure .*
(*b*) *There exists $0 < A < \infty$, such that μ is absolutely continuous with respect to Hausdorff measure defined relative to $\tilde{\lambda}$ where*

$$\tilde{\lambda}(r) = r \, \exp[A\sqrt{\log 1/r \, \log\log\log 1/r}], 0 < r < 10^{-6}.$$

In [BL], [L], and [LNP] the second author and coauthors have attempted to generalize Theorem A to the case when $f(\eta) = |\eta|^p, p \neq 2,\ 1 < p < \infty$, i.e, when u is p harmonic in Ω. To briefly outline this work, in [BL] the first author, together with Bennewitz, proved the following theorem.

Theorem B. *If $\partial\Omega$ is a quasicircle, then H-dim $\mu \leq 1$ for $2 < p < \infty$, while H-dim $\mu \geq 1$ for $1 < p < 2$. Moreover, if $\partial\Omega$ is the von Koch snowflake then strict inequality holds for H-dim μ.*

In [L] we obtained the natural generalization of [M] to the p harmonic setting, at the expense of assuming more about $\partial\Omega$:

Theorem C. *Given $p, 1 < p < \infty, p \neq 2,$ there exists $k_0(p) > 0$ such that if $\partial\Omega$ is a k quasi-circle and $0 < k < k_0(p)$, then*

(a) *μ is concentrated on a set of σ finite H^1 measure when $p > 2$.*
(b) *There exists $A = A(p), 0 < A(p) < \infty$, such that if $1 < p < 2$, then μ is absolutely continuous with respect to Hausdorff measure defined relative to $\tilde{\lambda}$ (as in Theorem A).*

Finally in [LNP] we proved the following theorem.

Theorem D. *Let $D \subset \mathbb{C}$ be a bounded simply connected domain and $1 < p < \infty, p \neq 2$. Put*

$$\lambda(r) = r\ \exp[A\sqrt{\log 1/r\ \log\log 1/r}], 0 < r < 10^{-6}.$$

Then

(a) *If $p > 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of σ finite H^λ measure.*
(b) *If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^λ.*

The key ingredient used in the proof of Theorems $B - D$ was Theorem 1 when $f(\eta) = |\eta|^p$. Thus although we still need to check a few details, we hope to prove in future work that

Plausible Theorem. *Theorem A is valid when f is homogeneous of degree 2 and Theorem D holds for f homogenous of degree $p, p \neq 2$.*

We give two proofs of Theorem 1, in the order which they were obtained. The second proof illustrates the fact that hindsight is better than foresight.

2 Proof of Theorem 1

We first prove Theorem 1 when $p = 2$. Let $v(z) = \log f(\nabla u(z))$. Then for $k, j = 1, 2$ we have at z,

$$b_{kj} v_{x_j} = f^{-1}(\nabla u) \sum_{n=1}^{2} f_{\eta_n}(\nabla u) b_{kj} u_{x_n x_j}. \tag{2.1}$$

Summing (2.1) over $k, j = 1, 2$, and using (1.3) for $\zeta = u_{x_n}$ we get

$$Lv = \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k} \left(b_{kj} v_{x_j} \right) = f^{-1}(\nabla u) \sum_{n,j,k,l=1}^{2} b_{nl} b_{kj} u_{x_l x_k} u_{x_n x_j} - f^{-2}(\nabla u) \sum_{n,j,k,l=1}^{2} b_{kj} f_{\eta_n} f_{\eta_l} u_{x_l x_k} u_{x_n x_j}. \tag{2.2}$$

Multiplying (2.2) by $f^2(\nabla u(z))$ we rewrite this equation in the form ;

$$f^2(\nabla u)\, Lv = f(\nabla u)\, T_1 - T_2 \tag{2.3}$$

where at z,

$$T_1 = \sum_{n,j,k,l=1}^{2} b_{nl} b_{kj} u_{x_l x_k} u_{x_j x_n} \text{ and } T_2 = \sum_{n,j,k,l=1}^{2} b_{kj} f_{\eta_n} f_{\eta_l} u_{x_l x_k} u_{x_j x_n}. \tag{2.4}$$

We now use matrix notation. We write at z,

$$(b_{kj}(z)) = (f_{\eta_k \eta_j}(\nabla u(z))) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$(u_{x_k x_j}(z)) = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \tag{2.5}$$

$$\begin{pmatrix} u_{x_1} \\ u_{x_2} \end{pmatrix} = |\nabla u| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Let E^t, $\text{tr}\ E$, denote the transpose and trace of the matrix E. Observe that if

$$D = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ then } T_1 = \text{ tr } (D^2). \tag{2.6}$$

To simplify our calculations we choose an orthonormal matrix O such that

$$O^t \begin{pmatrix} A & B \\ B & C \end{pmatrix} O = \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix}$$
$$O^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} O = \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix}. \tag{2.7}$$

Then

$$T_1 = \text{ tr } D^2 = \text{ tr } [(O^t D O)^2] = (a'A')^2 + 2(b')^2 A'C' + (c'C')^2 \tag{2.8}$$

We also note that if

$$\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = O^t \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{2.9}$$

then from (1.4) with $p = 2$, (2.5), we find at z,

$$\begin{aligned} f(\nabla u) &= (1/2)|\nabla u|^2 (\cos\theta \, \sin\theta) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ &= (1/2)|\nabla u|^2 (\cos\phi \sin\phi) \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \\ &= (1/2)|\nabla u|^2 \, [a'(\cos\phi)^2 + 2b' \sin\phi \cos\phi + c'(\sin\phi)^2]. \end{aligned} \tag{2.10}$$

Putting (2.10) and (2.8) together we deduce that

$$f(\nabla u) T_1 = (1/2)|\nabla u|^2 \, [(a'A')^2 + 2(b')^2 A'C' + (c'C')^2] \, [a'(\cos\phi)^2 + 2b' \sin\phi \cos\phi + c'(\sin\phi)^2]. \tag{2.11}$$

We now consider T_2. Note that if $\lambda_m = \sum_{l=1,2} u_{x_m x_l} f_{\eta_l}(\nabla u)$, and $\lambda^t = (\lambda_1 \, \lambda_2)$, then from (2.4), (2.5), (2.7), (2.9), we get

$$T_2 = \lambda^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} \lambda = (\lambda')^t \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \lambda' \text{ where } \lambda' = O^t \lambda. \tag{2.12}$$

Also using the above displays and (1.4) with $p = 2$, we obtain at $\nabla u(z)$

$$\begin{aligned} &\begin{pmatrix} f_{\eta_1} \\ f_{\eta_2} \end{pmatrix} = |\nabla u| \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \\ &|\nabla u| \, O \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}. \end{aligned} \tag{2.13}$$

Next we have at z,

$$\begin{aligned} \lambda = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} f_{\eta_1}(\nabla u) \\ f_{\eta_2}(\nabla u) \end{pmatrix} &= |\nabla u| \, O \begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix} \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \\ &= |\nabla u| \, O \begin{pmatrix} a'A' \cos\phi + b'A' \sin\phi \\ b'C' \cos\phi + c'C' \sin\phi \end{pmatrix}. \end{aligned} \tag{2.14}$$

From (2.12), (2.14), we conclude that

$$T_2 = |\nabla u|^2 \left[\begin{pmatrix} a'A'\cos\phi + b'A'\sin\phi \\ b'C'\cos\phi + c'C'\sin\phi \end{pmatrix}^t \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \begin{pmatrix} a'A'\cos\phi + b'A'\sin\phi \\ b'C'\cos\phi + c'C'\sin\phi \end{pmatrix} \right]$$

$$= |\nabla u|^2 \left[a'(a'A'\cos\phi + b'A'\sin\phi)^2 + 2b'(a'A'\cos\phi + b'A'\sin\phi)(b'C'\cos\phi + c'C'\sin\phi) \right.$$

$$\left. + c'(b'C'\cos\phi + c'C'\sin\phi)^2 \right]. \tag{2.15}$$

To simplify (2.15) we observe from the Euler equation in (1.2) that

$$0 = \text{ tr } \left[\begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right] = \text{ tr } \left[\begin{pmatrix} A' & 0 \\ 0 & C' \end{pmatrix} \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} \right] = a'A' + c'C'. \tag{2.16}$$

To begin the estimation of T_2 we write $T_2 = |\nabla u|^2 \left[h_0 + h_1 b' + h_2 (b')^2 + h_3 (b')^3 \right]$ where $h_m, 0 \leq m \leq 3$, is independent of b'. From (2.15), (2.16), we conclude that

$$h_0 = a'(a'A')^2 \cos^2\phi + c'(c'C')^2 \sin^2\phi. \tag{2.17}$$

Also,

$$h_1 = 2(a'A')^2 \sin\phi\cos\phi + 2a'A'c'C' \sin\phi\cos\phi + 2(c'C')^2 \sin\phi\cos\phi$$

$$= \left[(a'A')^2 + (c'C')^2 \right] \sin\phi\cos\phi. \tag{2.18}$$

Next we have

$$h_2 = a'(A')^2 \sin^2\phi + 2(a'A'C')\cos^2\phi + 2(c'A'C')\sin^2\phi + c'(C')^2\cos^2\phi$$

$$= A'C'(a'\cos^2\phi + c'\sin^2\phi). \tag{2.19}$$

Finally we have

$$h_3 = 2A'C' \sin\phi\cos\phi \tag{2.20}$$

Adding (2.17) - (2.20), multiplying the resulting expression by $|\nabla u|^2$ and comparing with (2.11) we find in view of (2.16) that at z

$$f(\nabla u) T_1 = T_2. \tag{2.21}$$

From (2.21), (2.2), (2.3) we now have shown that $Lv = 0$ at z when $p = 2$.

The proof that $Lv \geq 0$ for $p > 2$ and $Lv \leq 0$ for $1 < p < 2$ is essentially the same only in these cases we use the fact that f is homogeneous of degree p and in particular (1.4) for p. More specifically the computation of T_1 is unchanged. However the right hand side in (2.10) should be multiplied by $\frac{2}{p(p-1)}$. The new (2.11) now becomes $\frac{2}{p(p-1)}$ times the old (2.11). Also, the right hand side in (2.13) should be multiplied by $1/(p-1)$. We then get a new expression for T_2 in (2.15) which is $1/(p-1)^2$ times the old expression. From this discussion and the $p = 2$ case we conclude that if $T = T_2$ when $p = 2$ then $T \geq 0$ and for fixed $p, 1 < p < \infty$, we have

$$Lv = \left(\frac{2}{p(p-1)} - \frac{1}{(p-1)^2} \right) T = \frac{p-2}{p\,(p-1)^2}\, T. \tag{2.22}$$

Thus $Lv \geq 0$ for $p > 2$ and $Lv \leq 0$ when $1 < p < 2$. The proof of Theorem 1 is now complete. □

3 Alternate Proof of Theorem 1

First some new notation, set

$$D^2f = D^2f(\nabla u(z)) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \text{ and } D^2u = D^2u(z) = \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

and let $\nabla u = (u_{x_1}, u_{x_2}) = (u_1, u_2)$ be a row vector, $Df = Df(\nabla u) = (f_{\eta_1}, f_{\eta_2}) = (f_1, f_2)$ also be a row vector. Then $b_{kj} = f_{kj}$ so that equation (1.2) is $\text{tr}(D^2f\ D^2u) = 0$ while the homogeneity conditions we will need are given by

$$D^2f\ \nabla u^t = (p-1)\ Df^t \text{ and } p(p-1)f = \nabla u\ D^2f\ \nabla u^t$$

where the exponent t indicates the transpose of ∇u and Df. In this notation we can rewrite equation (2.2), using these homogeniety conditions, as

$$\begin{aligned} f^2\ Lv &= \sum_{j,k,l,n=1}^{2} f f_{nl} f_{kj} u_{lk} u_{nj} - f_{kj} f_n f_l u_{lk} u_{nj} \\ &= \text{tr}\ \left(f\,(D^2f\ D^2u)^2 - Df^t\ Df\ D^2u\ D^2f\ D^2u \right) \\ &= \text{tr}\left(\frac{1}{p(p-1)} \nabla u D^2 f \nabla u^t (D^2f\ D^2u)^2 - \frac{1}{(p-1)^2} D^2f\nabla u^t (D^2f\nabla u^t)^t D^2u D^2f D^2u \right) \\ &= \text{tr}\ \left(\frac{1}{p(p-1)} \nabla u D^2 f \nabla u^t (D^2f\ D^2u)^2 - \frac{1}{(p-1)^2} D^2f\nabla u^t \nabla u (D^2f\ D^2u)^2 \right). \end{aligned}$$

Now $D^2fD^2u = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ since $\text{tr}(D^2fD^2u) = 0$, squaring gives

$$(D^2f\ D^2u)^2 = (\alpha^2 + \beta\gamma) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\text{det}\ (D^2fD^2u)\ I\,.$$

Finally note that $\text{tr}\ (D^2f\nabla u^t \nabla u) = \sum_{l,k=1}^{2} f_{lk} u_l u_k = \nabla u D^2 f \nabla u^t$. Substituting these into the display for $f^2\ Lv$ and noting that $\text{tr}\ I = 2$ we have

$$\begin{aligned} f^2\ Lv &= -\ \text{det}\ (D^2fD^2u)\ \nabla u D^2 f \nabla u^t \left(\frac{2}{p(p-1)} - \frac{1}{(p-1)^2} \right) \\ &= -\ \text{det}\ (D^2f)\ \text{det}\ (D^2u)\ \nabla u D^2 f \nabla u^t \left(\frac{p-2}{p(p-1)^2} \right). \end{aligned}$$

Since f is convex both the terms $\text{det}\ (D^2f)$ and $\nabla u D^2 f \nabla u^t$ are positive (for $\nabla u(z) \neq 0$). Since f_{11} is positive consider $f_{11}\text{det}\ (D^2u) = f_{11}u_{11}u_{22} - f_{11}u_{12}^2$, using the equation $\text{tr}(D^2f\ D^2u) = 0$ in the form $f_{11}u_{11} + 2f_{12}u_{12} + f_{22}u_{22} = 0$ we have $f_{11}\text{det}\ (D^2u) = -(2f_{12}u_{12} + f_{22}u_{22})u_{22} - f_{11}u_{12}^2 = -\nabla u_2 D^2 f \nabla u_2^t$ which is nonpositive. Altogether, see equation (2.22),

$$f_{11}\ f^2\ Lv = \text{det}\ (D^2f)\ \ \nabla u D^2 f \nabla u^t\ \ \nabla u_2 D^2 f \nabla u_2^t\ \ \left(\frac{p-2}{p(p-1)^2} \right).$$

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