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On the Use of Adaptive Meshes to Counter Overshoot in Solutions of Discretised Nonlinear Stochastic Differential Equations

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Abstract

We consider two classes of nonlinear stochastic differential equation with a.s. positive solutions. In the first case the drift coefficient is strongly zero-reverting, and dominates the diffusion, whereas in the second the diffusion is highly variable and dominates the drift. In each case, the tendency to overshoot zero prevents a uniform Euler discretisation from preserving positivity in solutions. To address this, we construct adaptive meshes allowing the generation of positive trajectories with arbitrarily high probability. For completeness, we generalise the analysis to finite-dimensional systems of stochastic differential equations, investigating the effect of a uniform Euler discretisation on the positivity of systems with coefficients satisfying linear bounds, and introducing an adaptive mesh to counter overshoot when those bounds are violated.

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1 Introduction

In Appleby et al [1], the preservation of positivity in solutions of discretised stochastic differential equations was addressed within the framework of numerical analysis. Using a linear scalar test equation, it was shown that, for the Euler–Maruyama class of numerical methods with uniform stepsize, the probability of positivity over a fixed interval of simulation can be made arbitrarily close to one by choosing a sufficiently large number of meshpoints. This analysis was generalised to a broad class of nonlinear equations with coefficients satisfying linear bounds, and including weak discretisations. For the linear case, the authors developed an a priori estimate of the number of meshpoints required to generate positive trajectories with a given probability of success. We refer the reader to the introduction of that paper for an overview of the literature surrounding positivity in solutions of deterministic and stochastic difference equations.

This article is a sequel to [1], and is intended to progress the analysis in two directions. First, we show that uniform discretisations fail to preserve positivity when coefficients violate linear bounds, and investigate the usefulness of adaptive meshes in this context. Second, we generalise this analysis to include systems of stochastic differential equations, both for uniform and adaptive meshes. We restrict our investigation to the Euler class of numerical methods in this paper; for an overview of numerical methods for stochastic differential equation we refer the reader to, among others, [3, 4, 6, 9, 10].

Consider the scalar Itô stochastic differential equation

$$\begin{aligned} &dX(t) = f(X(t))\,dt + g(X(t))\,dB(t), \quad t \geq 0, \quad \text{a.s.},\\ &X(0) = \zeta > 0. \end{aligned} \tag{1.1}$$

Throughout the paper, an almost sure (a.s.) event is an event of probability one. In [1] the following conditions were imposed on the drift coefficient:

$$f \text{ is locally Lipschitz continuous on } (0,\infty); \tag{1.2}$$

$$f(0) \geq 0; \tag{1.3}$$

$$\text{there is } K_1 > 0 \text{ such that} \quad f(x) \geq -K_1 x \quad \text{for all } x > 0; \tag{1.4}$$

and on the diffusion coefficient:

$$g \text{ is locally Lipschitz continuous on } (0,\infty); \tag{1.5}$$

$$g(0) = 0, \quad g(x) > 0 \quad \text{for all } x > 0; \tag{1.6}$$

$$\text{there exists } K_2 > 0 \text{ such that} \quad g(x) \leq K_2 x \quad \text{for all } x > 0. \tag{1.7}$$

Solutions of (1.1) are defined with respect to the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, \mathbb{P})$, and $(\mathcal{F}(t))_{t\geq 0}$ is the natural filtration of a scalar standard Brownian motion $B = \{B(t); 0 \leq t < \infty; \mathcal{F}^B(t)\}$, i.e., $\mathcal{F}(t) = \mathcal{F}^B(t) := \sigma\{B(s) : 0 \leq s \leq t\}$.

Under conditions (1.2) and (1.5) there is a unique continuous adapted process X which obeys (1.1) on the interval $[0, \tau_e^\zeta)$. Here τ_e^ζ is the explosion time, defined by

$\tau_e^\zeta = \inf\{t > 0 : |X(t,\zeta)| \notin [0,\infty)\}$. It was proved in Appleby, Mao and Rodkina [2], that under conditions (1.2), (1.5), (1.6) and $f(0) = 0$ the solution X cannot reach zero in finite time if it does not start from the origin: they prove this by showing that $\tau_e^\zeta \le \vartheta_0^\zeta$ a.s., where $\vartheta_0^\zeta = \inf\{t > 0 : |X(t,\zeta)| = 0\}$ (see also Karatzas and Shreve [5] and Mao [7]). In other words, $X(t) > 0$ for all $t \in [0, \tau_e^\zeta)$. When the condition $f(0) = 0$ is replaced by the nonnegativity condition $f(0) \ge 0$, i.e., condition (1.3), by applying a stochastic comparison principle (see [5, Proposition 5.2.18]), we can also conclude that $X(t) > 0$ for all $t \in [0, \tau_e^\zeta)$.

Applying a uniform Euler discretisation with N meshpoints to solutions of (1.1) over the interval $[0,T]$ requires that for each N the stepsize satisfy $h_N = T/N$, and the resulting discretisation is

$$\begin{aligned} x_N(n+1) &= x_N(n) + h_N f(x_N(n)) + \sqrt{h_N} g(x_N(n))\xi_N(n+1), \\ x_N(0) &= \zeta > 0, \end{aligned} \tag{1.8}$$

for $n = 0, \dots, N-1$, where the stochastic component of (1.8) satisfies

$\xi_N = \{\xi_N(n) : n = 1, \dots, N\}$ is a sequence of independent and identically distributed $\mathcal{F}_N$–measurable r.vs.; (1.9a)

each $\xi_N(n)$ has density symmetric about 0, and corresponding distribution function F; (1.9b)

$$\mathbb{E}_N[\xi_N(n)^2] =: \varsigma^2 < +\infty, \quad n = 1, \dots, N. \tag{1.9c}$$

We may interpret each $x_N(n)$ as the simulated approximation of $X(h_N n)$, and for each N there is a probability triple $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ on which solutions of (1.8) are defined. Moreover the sequence ξ_N generates a natural discrete filtration $\{\mathcal{F}_N(n)\}_{n=1}^N$, where $\mathcal{F}_N(n) = \sigma\{\xi_N(j) : j \le n\}$ for $n = 1, \dots, N$. Clearly, if x_N is a solution of (1.8) then it is a $\mathcal{F}_N(n)$–adapted process. In particular, if each $\xi_N(n)$ is Gaussian, then (1.8) is the uniform Euler–Maruyama discretisation of (1.1).

In Appleby et al [1] it was shown that, if we define the event

$$P_N = \{\omega \in \Omega_N : x_N(n,\omega) > 0 \text{ for } n = 0, \dots, N\},$$

then

$$\lim_{N\to\infty} \mathbb{P}_N[P_N] = 1.$$

Additionally, in the special case where (1.1) is a geometric Brownian motion with drift coefficient λ and diffusion coefficient σ, and each $\xi_N(n)$ in (1.8) is Gaussian, it was shown that positive trajectories of (1.8) could be generated on $[0,T]$ with probability approximately $1 - \eta$, $\eta \ll 1$, by choosing

$$N > 2\sigma^2 T \log\left(\frac{\sigma^2 T e^{\lambda/\sigma^2}}{\sqrt{\pi}} \frac{\sqrt{\log(1/\eta^2)}}{\eta} \right). \tag{1.10}$$

In Section 2 we consider two particular violations of the linear bounds (1.4) and (1.7). First, we relax condition (1.4) to introduce a dominant and strongly zero-reverting drift coefficient. It turns out that the probability of positivity over the interval $[0,T]$ following a uniform Euler discretisation tends to zero as the initial value grows large; trajectories overshoot under pressure from the drift. A carefully chosen adaptive mesh is then introduced allowing positive trajectories to be generated with an arbitrarily high probability of success by adjusting a density parameter. Second, we demonstrate the versatility of such meshes by relaxing condition (1.7) to allow a highly variable and dominant diffusion coefficient. Even under such circumstances, an adaptive mesh may be constructed to ensure positivity with arbitrarily high probability.

The use of adaptive, and therefore random, meshes means that our analysis no longer aims to preserve positive trajectories over a fixed simulation interval $[0,T]$, but rather over a fixed number of timesteps N. This excludes the possibility of developing a priori estimates of the form (1.10).

In Section 3 we develop an analogue of the analysis in [1] for finite-dimensional systems with diagonal noise, placing similar linear constraints on coefficients. Once again we demonstrate that the probability of generating positive trajectories over an interval of simulation $[0,T]$ can be made arbitrarily close to one by increasing the density of meshpoints. Following this, we introduce an adaptive mesh to counter overshoot when the coefficients of the system are highly nonlinear. Finally, in Section 4 we summarise our results and suggest future developments of this research.

2 Nonlinear Equations and the Use of Adaptive Meshes

Suppose that the discretisation of (1.1) is

$$\begin{aligned} x_N(n+1) &= x_N(n) + h(N, x_N(n)) f(x_N(n)) \\ &\qquad + \sqrt{h(N, x_N(n))}\, g(x_N(n)) \xi_N(n+1), \\ x_N(0) &= \zeta > 0, \end{aligned} \tag{2.1}$$

for $n = 0, \dots, N-1$, where ξ_N is a sequence of random variables obeying (1.9), so that $x_N(n)$ can be viewed as an approximation to $X\left(\sum_{j=0}^{n-1} h_j\right)$. For each N there is a probability triple on which solutions of (1.8) are defined: $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$. Moreover the sequence ξ_N generates a natural discrete filtration $\{\mathcal{F}_N(n)\}_{n=1}^N$, where $\mathcal{F}_N(n) = \sigma\{\xi_N(j) : j \le n\}$ for $n = 1, \dots, N$. Notice that the stepsize function h is now dependent on the solution x_N at each step, and is therefore a random variable adapted to the filtration $\mathcal{F}_N$.

2.1 Strongly Zero-Reverting Drift Coefficient

We proceed by relaxing the restriction on the drift and diffusion coefficients given by (1.4) and (1.7) so that the drift coefficient dominates the diffusion, and is strongly directed towards zero. Suppose that we impose the conditions

$$f \text{ is locally Lipschitz continuous on } (0,\infty); \tag{2.2}$$
$$f(0) \geq 0; \tag{2.3}$$
$$\lim_{x\to\infty} \frac{f(x)}{x} = -\infty, \tag{2.4}$$

on the drift coefficient, and

$$g \text{ is locally Lipschitz continuous on } (0,\infty); \tag{2.5}$$
$$g(0) = 0, \quad g(x) > 0 \quad \text{for all } x > 0; \tag{2.6}$$
$$\sup_{x\geq 0} \frac{g^2(x)}{x(1+|f(x)|)} =: \Gamma_1 < +\infty, \tag{2.7}$$

on the diffusion coefficient. As discussed in the introduction, conditions (2.2), (2.3), (2.5), and (2.6) ensure the existence of a unique continuous adapted process X which obeys (1.1) and is a.s. positive up to a random explosion time τ_e^ζ. Imposing the constraint (2.4) on f ensures that the Feller's test criteria for the existence of a unique global solution on $(0,\infty)$ are satisfied (see Karatzas & Shreve [5, Theorem 5.5.29]).

2.1.1 Failure of a Uniform Mesh

Recall from [1, Remark 3.2] that the probability of positivity of solutions of the uniform Euler-discretisation of (1.1) with coefficients satisfying (1.2)–(1.4) and (1.5)–(1.7) on a fixed interval $[0,T]$ has lower bound

$$\mathbb{P}_N[P_N] \geq \left(1 - F\left(-\left(\frac{1-K_1T/N}{K_2\sqrt{T}}\right)\sqrt{N}\right)\right)^N,$$

which is independent of the initial value $X(0) = \zeta$, and which tends to limit one as $N \to \infty$, again independently of $X(0)$.

However when the constraints on the coefficients f and g are loosened to (2.2)–(2.4) and (2.5)–(2.7), this is no longer the case.

Proposition 2.1. *Let* $\{x_N(n)\}_{n=0}^{N-1}$ *be a solution of* (2.1) *with* f *satisfying* (2.2)–(2.4), g *satisfying* (2.5)–(2.7), *and* $h(N, x_N(n)) = h_N = T/N$. *Then*

$$\lim_{\zeta\to\infty} \mathbb{P}_N[x_N(1) > 0] = 0.$$

Proof.

$$\begin{aligned}
x_N(1,\omega) &= \zeta + h_N f(\zeta) + \sqrt{h_N} g(\zeta)\xi_N(1,\omega) \\
&= \zeta\left(1 + h_N \frac{f(\zeta)}{\zeta} + \sqrt{h_N}\frac{g(\zeta)}{\zeta}\xi_N(1,\omega)\right) \\
&= \zeta\sqrt{1+|f(\zeta)|}\left(\frac{1}{\sqrt{1+|f(\zeta)|}} + h_N \frac{f(\zeta)}{\zeta\sqrt{1+|f(\zeta)|}}\right. \\
&\qquad\left. + \sqrt{h_N}\sqrt{\frac{g^2(\zeta)}{\zeta^2(1+|f(\zeta)|)}}\xi_N(1,\omega)\right).
\end{aligned}$$

In order that $x_N(1,\omega) > 0$ it is necessary and sufficient to have

$$\frac{1}{\sqrt{1+|f(\zeta)|}} + \frac{h_N}{\sqrt{1+|f(\zeta)|}}\frac{f(\zeta)}{\zeta} + \sqrt{\frac{h_N}{\zeta}}\sqrt{\frac{g^2(\zeta)}{\zeta(1+|f(\zeta)|)}}\xi_N(1,\omega) > 0,$$

or equivalently

$$\begin{aligned}
\xi_N(1,\omega) &> -\left(\frac{\frac{1}{\sqrt{1+|f(\zeta)|}} + \frac{h_N}{\sqrt{1+|f(\zeta)|}}\frac{f(\zeta)}{\zeta}}{\sqrt{\frac{h_N}{\zeta}}\sqrt{\frac{g^2(\zeta)}{\zeta(1+|f(\zeta)|)}}}\right) \\
&> -\left(\frac{\frac{1}{\sqrt{1+|f(\zeta)|}} + \frac{h_N}{\sqrt{1+|f(\zeta)|}}\frac{f(\zeta)}{\zeta}}{\sqrt{\frac{h_N}{\zeta}}\Gamma_1}\right) > -\left(\frac{1 + h_N\frac{f(\zeta)}{\zeta}}{\sqrt{1+|f(\zeta)|}\sqrt{\frac{h_N}{\zeta}}\Gamma_1}\right) \\
&> -\frac{1}{\Gamma_1\sqrt{h_N}}\left(\frac{1 + h_N\frac{f(\zeta)}{\zeta}}{\sqrt{\frac{1}{\zeta} + \frac{|f(\zeta)|}{\zeta}}}\right) = \frac{\sqrt{h_N}}{\Gamma_1}\left(\frac{-\frac{1}{h_N} + \frac{|f(\zeta)|}{\zeta}}{\sqrt{\frac{1}{\zeta} + \frac{|f(\zeta)|}{\zeta}}}\right).
\end{aligned}$$

By (2.4) we have

$$\lim_{\zeta\to\infty}\mathbb{P}\left[\xi_N(1) > \frac{\sqrt{h_N}}{\Gamma_1}\left(\frac{-\frac{1}{h_N} + \frac{|f(\zeta)|}{\zeta}}{\sqrt{\frac{1}{\zeta} + \frac{|f(\zeta)|}{\zeta}}}\right)\right] = 0,$$

and the statement of the proposition follows. □

Remark 2.2. Proposition 2.1 is indicative of the tendency for large negative feedback from f to cause overshoot in solutions; without the use of an adaptive mesh for large values there is no guarantee that positivity will be preserved in the uniform discretisation.

2.1.2 Construction of an Adaptive Mesh

Since a uniform mesh is inadequate, we define a state-dependent mesh as follows: for fixed N let $d_N \in (0,1)$ be a constant density parameter and set

$$h\left(N, x_N(n)\right) := d_N \frac{x_N(n)}{1+|f(x_N(n))|}, \quad t_n := \sum_{j=0}^{n-1} h\left(N, x_N(j)\right), \quad n = 0, \ldots, N-1,$$

with $t_0 = 0$. Then (2.1) becomes

$$\begin{aligned} x_N(n+1) &= x_N(n) \left(1 + d_N \frac{f(x_N(n))}{1+|f(x_N(n))|}\right. \\ &\qquad \left. + \sqrt{d_N}\sqrt{\frac{g^2(x_N(n))}{x(n)(1+|f(x_N(n))|)}}\,\xi_N(n+1)\right), \qquad (2.8) \\ x_N(0) &= \zeta > 0, \end{aligned}$$

for $n = 0, \ldots, N-1$.

Remark 2.3. Condition (2.4) ensures that, for fixed n,

$$\lim_{N\to\infty} d_N \frac{x_N(n)}{1+|f(x_N(n))|} = 0, \quad a.s.$$

Moreover, when f is close to linear and $x_N(n)$ is large, $x_N(n)/1+|f(x_N(n))|$ will be close to 1, and the adaptive mesh $h(N, x_N(n))$ will be similar to a uniform discretisation.

2.1.3 Main Result

The adaptive mesh defined in Section 2.1.2 allows us to generate positive trajectories with arbitrarily high probability by adjusting the density parameter d_N.

Theorem 2.4. *Let* $\{x_N(n)\}_{n=0}^{\infty}$ *be the solution of* (2.8), *with* f *satisfying* (2.2)–(2.4) *and* g *satisfying* (2.5)–(2.7), *and let* $d_N = \tau/N$. *Define the event*

$$P_N = \{\omega \in \Omega_N \,:\, x_N(n,\omega) > 0 \text{ for } n = 0, \ldots, N\}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}_N[P_N] = 1.$$

Proof. Define

$$R_N(n) = \left\{\omega \,:\, \xi_N(n+1,\omega) > -\frac{1-d_N}{\sqrt{d_N \Gamma_1}}\right\}, \quad n = 0, \ldots, N-1.$$

First, we show that

$$P_N \supseteq \bigcap_{n=0}^{N-1} R_N(n). \tag{2.9}$$

Define the sequence of events $\{\widetilde{P_N}(n)\}_{n=1}^{N}$ so that

$$\widetilde{P_N}(n) = \{\omega \in \Omega_N \,:\, x_N(i,\omega) > 0,\, i \le n\}\,. \tag{2.10}$$

Now each $\bigcap_{i=1}^{n} \widetilde{P_N}(i) = \widetilde{P_N}(n)$, and in particular $\bigcap_{n=1}^{N} \widetilde{P_N}(n) = \widetilde{P_N}(N) = P_N$.

If we can show by induction that each $\bigcap_{i=0}^{n-1} R_N(i) \subseteq \widetilde{P_N}(n)$, then (2.9) holds. First, let $n = 0$. For $\omega \in R_N(0)$ we have

$$\begin{aligned}
x_N(1,\omega) &= \zeta\left(1 + d_N\frac{f(\zeta)}{1+|f(\zeta)|} + \sqrt{d_N}\sqrt{\frac{g^2(\zeta)}{\zeta(1+|f(\zeta)|)}}\xi_N(1,\omega)\right)\\
&> \zeta\left(1 + d_N\frac{f(\zeta)}{1+|f(\zeta)|} - \sqrt{\frac{g^2(\zeta)}{\zeta(1+|f(\zeta)|)}}\frac{1}{\sqrt{\Gamma_1}}(1-d_N)\right)\\
&> \zeta d_N\left(1 + \frac{f(\zeta)}{1+|f(\zeta)|}\right)\\
&> 0.
\end{aligned}$$

Hence $R_N(0) \subseteq \widetilde{P_N}(1)$, and we have demonstrated the base case. Next, we assume that $\bigcap_{i=0}^{k-1} R_N(i) \subseteq \widetilde{P_N}(k)$. Then, if $\omega \in R_N(k)$ we have $x_N(k,\omega) > 0$ and

$$\begin{aligned}
x_N(k+1,\omega) &= x_N(k,\omega)\left(1 + d_N\frac{f(x_N(k,\omega))}{1+|f(x_N(k,\omega))|}\right.\\
&\qquad\left. + \sqrt{d_N}\sqrt{\frac{g^2(x_N(k,\omega))}{x_N(k,\omega)(1+|f(x_N(k,\omega))|)}}\xi_N(k+1,\omega)\right)\\
&> x_N(k,\omega)\left(1 + d_N\frac{f(x_N(k,\omega))}{1+|f(x_N(k,\omega))|}\right.\\
&\qquad\left. - \sqrt{\frac{g^2(x_N(k,\omega))}{x_N(k,\omega)(1+|f(x_N(k,\omega))|)}}\frac{1}{\sqrt{\Gamma_1}}(1-d_N)\right)\\
&> x_N(k,\omega)\left(1 + \frac{f(x_N(k,\omega))}{1+|f(x_N(k,\omega))|}\right)\\
&> 0.
\end{aligned}$$

Hence, by induction $\bigcap_{i=0}^{n-1} R_N(i) \subseteq \widetilde{P_N}(n)$ and (2.9) holds.

Since each term in the sequence of events $\{R_N(n)\}_{n=0}^{N-1}$ is independent of all the others, and by (2.9), it follows that

$$\mathbb{P}_N[P_N] \geq \left(1 - F\left(-\frac{1-\tau/N}{\sqrt{\tau\Gamma_1}}\sqrt{N}\right)\right)^N .$$

The remainder of the proof is identical to that of [1, Theorem 3.1]. □

Remark 2.5. Note that we cannot ensure a prespecified probability of $\{x_N\}$ being positive on a given compact interval $[0,T]$ by choosing N arbitrarily large and choosing a fixed $\tau > T$, because the time $\tau \frac{1}{N}\sum_{j=0}^{N-1} x_N(j)/\{1+|f(x_N(j))|\}$ is random. Hence it is only possible to determine a lower bound on the probability of obtaining a positive solution over a given number of timesteps, rather than over a pre-determined interval of simulation.

2.2 Highly Variable Diffusion Coefficient

Next, we relax the requirement that f dominate the diffusion coefficient. The drift satisfies

$$f \text{ is locally Lipschitz continuous on } (0,\infty); \tag{2.11}$$
$$f(0) \geq 0, \tag{2.12}$$

and the diffusion is highly variable, and dominates f:

$$g \text{ is locally Lipschitz continuous on } (0,\infty); \tag{2.13}$$
$$g(0) = 0, \quad g(x) > 0 \quad \text{for all} \quad x > 0; \tag{2.14}$$
$$\lim_{x\to\infty} \frac{g(x)}{x} = \infty; \tag{2.15}$$
$$\sup_{x>0} \frac{x|f(x)|}{1+g^2(x)} =: \Gamma_2 < +\infty. \tag{2.16}$$

Once again $X(0) = \zeta > 0$, (2.11), (2.12), (2.13) and (2.14) ensure the existence of a unique adaptive process obeying (1.1) and which is a.s. positive up to an explosion time τ_e^ζ. Additionally imposing (2.15) and (2.16) ensures that the Feller's test criteria for the existence of a unique global solution are satisfied (see [5, Theorem 5.5.29]).

2.2.1 Failure of a Uniform Mesh

Conditions (2.14)–(2.16) describe a highly variable diffusion that dominates the drift. In this case, the variation due to the stochastic term cannot be offset by the drift, and is likely to cause overshoot on a uniform mesh when directed towards zero. Proposition 2.6 below illustrates this effect when the dominance of the diffusion is expressed by

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=0, \tag{2.17}$$

rather than (2.16).

Proposition 2.6. *Let* $\{x_N(n)\}_{n=0}^{N-1}$ *be a solution of* (2.1) *with* f *satisfying* (2.11) *and* (2.12), g *satisfying* (2.13)–(2.15) *and* (2.17), *and* $h(N,x_N(n))=h_N=T/N$. *Then*

$$\lim_{\zeta\to\infty}\mathbb{P}_N[x_N(1)>0]=\frac{1}{2}.$$

Proof. By (2.17),

$$\begin{aligned} x_N(1,\omega) &= \zeta+h_Nf(\zeta)+\sqrt{h_N}g(\zeta)\xi_N(1,\omega)\\ &= \zeta\left(1+h_N\frac{f(\zeta)}{\zeta}+\sqrt{h_N}\frac{g(\zeta)}{\zeta}\xi_N(1,\omega)\right). \end{aligned}$$

In order that $x_N(1,\omega)>0$ it is necessary and sufficient to show

$$1+h_N\left(\frac{f(\zeta)}{\zeta}\right)+\sqrt{h_N}\left(\frac{g(\zeta)}{\zeta}\right)\xi_N(1,\omega)>0,$$

or equivalently

$$\xi_N(1,\omega)>\frac{-1-h_N\left(\frac{f(\zeta)}{\zeta}\right)}{\sqrt{h_N\left(\frac{g(\zeta)}{\zeta}\right)}}=-\frac{\zeta}{\sqrt{h_N}g(\zeta)}-\sqrt{h_N}\left(\frac{f(\zeta)}{g(\zeta)}\right).$$

Since $\xi_N(1)$ satisfies (1.9b), by (2.15) and (2.17) we have

$$\lim_{\zeta\to\infty}\mathbb{P}_N\left[\xi_N(1)>-\frac{\zeta}{\sqrt{h_N}g(\zeta)}-\sqrt{h_N}\left(\frac{f(\zeta)}{g(\zeta)}\right)\right]=\frac{1}{2},$$

and the proof is complete. □

Remark 2.7. The statement of Proposition 2.6 is consistent with the fact that when the terms of the stochastic sequence $\{\xi_N\}$ are symmetrically distributed around zero, as required by Condition (1.9), the stochastic component of (2.1) will act towards zero exactly half the time: only then is overshoot possible. Note also that (2.17) is somewhat more restrictive than (2.16), and indeed the latter is implied by the former. Alternative dominance conditions may be chosen which illustrate the same principle.

2.2.2 Construction of an Adaptive Mesh

Let f satisfy (2.11) and (2.12), and let g satisfy (2.13)–(2.16). For fixed N let $d_N \in (0,1)$ be a constant density parameter and set

$$h\left(N, x_N(n)\right) := d_N \frac{x_N^2(n)}{1+g^2(x_N(n))}, \quad t_n := \sum_{j=0}^{n-1} h\left(N, x_N(j)\right), \tag{2.18}$$

for $n = 0, \ldots, N-1$, and with $t_0 = 0$. Then (2.1) becomes

$$\begin{aligned} x_N(n+1) = x_N(n) \Bigg(1 + d_N \frac{x_N(n) f(x_N(n))}{1+g^2(x_N(n))} \\ + \sqrt{d_N} \sqrt{\frac{g^2(x_N(n))}{1+g^2(x_N(n))}} \xi_N(n+1)\Bigg), \end{aligned} \tag{2.19}$$

$$x_N(0) = \zeta > 0,$$

for $n = 0, \ldots, N-1$. (2.19) is well defined since each $h(N, x_N(n))$ is an $\mathcal{F}_n$-measurable random variable.

2.2.3 Main Result

As before, the adaptive mesh (2.18) allows us to generate positive trajectories with arbitrarily high probability.

Theorem 2.8. *Let* $\{x_N(n)\}_{n=0}^{\infty}$ *be the solution of* (2.19)*, where* f *satisfies* (2.11) *and* (2.12)*,* g *satisfies* (2.13)–(2.16)*, and* $d_N = \tau/N$. *Define the event*

$$P_N = \{\omega \in \Omega_N \,:\, x_N(n,\omega) > 0 \textit{ for } n = 0, \ldots, N\}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}_N[P_N] = 1.$$

Proof. Define

$$Q_N(n) = \left\{\omega \,:\, \xi_N(n+1,\omega) > \frac{-1 + d_N \Gamma_2}{\sqrt{d_N}}\right\}, \quad n = 0, \ldots, N-1.$$

Just as in the proof of Theorem 2.4, we must prove by induction that, for each n,

$$\bigcap_{i=0}^{n-1} Q_N(i) \subseteq \widetilde{P_N}(n),$$

where $\widetilde{P_N}(n)$ is as defined as (2.10). We demonstrate here the induction step:

If $\omega \in \cap_{i=0}^{k} Q_N(i)$, then we have $x_N(k,\omega) > 0$ and

$$\begin{aligned} x_N(k+1,\omega) &= x_N(k,\omega)\left(1 + d_N \frac{x_N(k,\omega) f(x_N(k,\omega))}{1+g^2(x_N(k,\omega))}\right. \\ &\qquad \left. + \sqrt{d_N}\sqrt{\frac{g^2(x_N(k,\omega))}{1+g^2(x_N(k,\omega))}}\xi_N(n+1)\right) \\ &> x_N(k,\omega)\left(1 - d_N \frac{x_N(k,\omega)|f(x_N(k,\omega))|}{1+g^2(x_N(k,\omega))}\right. \\ &\qquad \left. - \sqrt{\frac{g^2(x_N(k,\omega))}{1+g^2(x_N(k,\omega))}}(1-d_N\Gamma_2)\right) \\ &> x_N(k,\omega)\left(1 - d_N \frac{x_N(k,\omega)|f(x_N(k,\omega))|}{1+g^2(x_N(k,\omega))} - (1-d_N\Gamma_2)\right) \\ &= x_N(k,\omega)\left(d_N\left(\Gamma_2 - \frac{x_N(k,\omega)|f(x_N(k,\omega))|}{1+g^2(x_N(k,\omega))}\right)\right) \\ &> 0. \end{aligned}$$

The remainder of the proof follows as before. □

3 Finite-Dimensional Equations

When the drift and diffusion coefficients of (1.1) have linear bounds, the probability of positivity of solutions of the uniform discretisation (1.8) on a finite interval tends to one as the number of equidistant mesh points increases. This was proved as [1, Theorem 3.1]. In this section we investigate the generalisation of this result for systems of nonlinear stochastic differential equations. For simplicity we consider only a system of two equations; the analysis generalises to systems of finite dimension. Following this, we weaken the linear bounds on the coefficients of the system, demonstrating that trajectories will overshoot zero, and introduce an adaptive mesh to counter this effect.

3.1 A Two-Dimensional System

Let $B = (B_1, B_2)$ be a two–dimensional standard Brownian motion and consider the stochastic differential equation

$$\begin{aligned} dX(t) &= f_1(X(t),Y(t))\,dt + g_1(X(t),Y(t))\,dB_1(t), \\ dY(t) &= f_2(X(t),Y(t))\,dt + g_2(X(t),Y(t))\,dB_2(t), \\ (X(0),Y(0)) &= (\zeta_1,\zeta_2) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{3.1}$$

for $t \geq 0$. We assume that there exist $K > 0$ and $\kappa > 0$ such that the following hold for $k = 1, 2$:

$$f_k,\ g_k \text{ are locally Lipschitz continuous on } [0,\infty)\times[0,\infty); \tag{3.2a}$$

$$g_k(0,0) = 0 \text{ and } 0 < g_k(x,y) \le Kx, \quad x,y > 0; \tag{3.2b}$$
$$f_k(x,y) \ge -\kappa x, \quad x,y \ge 0. \tag{3.2c}$$

Under these conditions, it was shown in Appleby, Mao & Rodkina [2] that there is a unique continuous adapted solution (X,Y) of (3.1) such that $X(t) > 0$ and $Y(t) > 0$ for all $t \ge 0$ a.s.

3.1.1 A Uniform Euler Discretisation of (3.1)

We now formulate the discretisation of (3.1). Let $T > 0$, $N \in \mathbb{N}$ and suppose $h(N) = T/N$. Let $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ be a probability triple. Let $\xi_N^{(k)} = \{\xi_N^{(k)}(n) : n = 1,\dots,N\}$ for $k = 1,2$ be sequences of mutually independent $\mathcal{F}_N$–measurable random variables each obeying (1.9). The pair of sequences $\xi_N := (\xi_N^{(1)}, \xi_N^{(2)})$ generates a discrete filtration $(\mathcal{F}_N(n))_{n=1}^N$, where $\mathcal{F}_N(n) = \sigma(\xi_N(j) : j \le n\}$ for $n = 1,\dots,N$. Let $(x_N, y_N) = \{(x_N(n), y_N(n)) : n = 0,\dots,N\}$ be the unique solution of the two–dimensional stochastic difference equation

$$\begin{aligned} x_N(n+1) &= x_N(n) + h(N) f_1(x_N(n), y_N(n)) \\ &\qquad + \sqrt{h(N)}\, g_1(x_N(n), y_N(n))\xi_N^{(1)}(n+1), \\ y_N(n+1) &= y_N(n) + h(N) f_2(x_N(n), y_N(n)) \\ &\qquad + \sqrt{h(N)}\, g_2(x_N(n), y_N(n))\xi_N^{(2)}(n+1), \\ (x_N(0), y_N(0)) &= (\zeta_1, \zeta_2) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{3.3}$$

where $n = 0,\dots,N-1$. Then $\{(x_N(n), y_N(n))\}_{n=0}^\infty$ is a $\mathcal{F}_N$–adapted process. We may interpret $(x_N(n), y_N(n))$ as the simulated approximation of $(X(nh(N)), Y(nh(N))$.

3.1.2 Positivity of Trajectories of (3.4)

Theorem 3.1. *Let $T > 0$, $N \in \mathbb{N}$ and suppose $h(N) = T/N$. Suppose that f_1, f_2, g_1, g_2 obey* (3.2)*, and let $\xi_N^{(1)}$ and $\xi_N^{(2)}$ be two sequences of mutually independent random variables obeying* (1.9)*. Let (x_N, y_N) be the solution of* (3.4)*. If $\tilde{P}_N$ is the event*

$$\tilde{P}_N = \{\omega \in \Omega_N : x_N(n,\omega) > 0,\ y_N(n,\omega) > 0 \ \text{ for } n = 0,\dots,N\},$$

then

$$\lim_{N\to\infty} \mathbb{P}[\tilde{P}_N] = 1.$$

Proof. Applying conditions (3.2b)–(3.2c) we estimate

$$\begin{aligned} \mathbb{P}\left\{ x_N(n+1) > 0 \middle| x_N(n), y_N(n) > 0 \right\} &= \mathbb{P}\left\{ x_N(n) + \frac{T}{N} f_1(x_N(n), y_N(n)) \right. \\ &\qquad \left. + \sqrt{\frac{T}{N}}\, g_1(x_N(n))\xi_N^{(1)}(n+1) > 0 \middle| x_N(n), y_N(n) > 0 \right\} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{\xi_N^{(1)}(n+1) > -\sqrt{\frac{N}{T}}\frac{x_N(n)}{g_1(x_N(n),y_N(n))}\right.\\
&\qquad\qquad\left. -\sqrt{\frac{T}{N}}\frac{f_1(x_N(n),y_N(n))}{g_1(x_N(n),y_N(n))}\right| x_N(n), y_N(n) > 0\Bigg\}\\
&= \mathbb{P}\left\{\xi_N^{(1)}(n+1) > -\sqrt{\frac{N}{T}}\frac{1}{K} + \sqrt{\frac{T}{N}}\frac{\kappa}{K}\right| x_N(n), y_N(n) > 0\Bigg\}\\
&= \mathbb{P}\left\{\xi_N^{(1)}(n+1) > \mathcal{Q}(N)\right\} = 1 - F(\mathcal{Q}(N))
\end{aligned}
$$

where

$$
\mathcal{Q}(N) = -\sqrt{\frac{N}{T}}\frac{1}{K} + \sqrt{\frac{T}{N}}\frac{\kappa}{K}.
$$

Similarly,

$$
\begin{aligned}
&\mathbb{P}\left\{y_N(n+1) > 0 \middle| x_N(n), y_N(n) > 0\right\}\\
&\qquad\geq \mathbb{P}\left\{\xi_N^{(2)}(n+1) > \mathcal{Q}(N) \middle| x_N(n), y_N(n) > 0\right\}\\
&\qquad\qquad = \mathbb{P}\left\{\xi_N^{(2)}(n+1) > \mathcal{Q}(N)\right\} = 1 - F(\mathcal{Q}(N)).
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\mathbb{P}\left\{x_N(n+1) > 0, y_N(n+1) > 0 \middle| x_N(n), y_N(n) > 0\right\}\\
&= \mathbb{P}\left\{x_N(n+1) > 0 \middle| x_N(n), y_N(n) > 0\right\}\\
&\qquad\times \mathbb{P}\left\{y_N(n+1) > 0 \middle| x_N(n), y_N(n) > 0, x(n+1) > 0\right\}\\
&= \mathbb{P}\left\{\xi_N^{(1)}(n+1) > \mathcal{Q}(N) \middle| x_N(n), y_N(n) > 0\right\}\\
&\qquad\times \mathbb{P}\left\{\xi_N^{(2)}(n+1) > \mathcal{Q}(N) \middle| x_N(n), y_N(n) > 0, x_N(n+1) > 0\right\}.
\end{aligned}
$$

Since $x_N(n+1)$ does not depend on $\xi_N^{(2)}(n+1)$, we have

$$
\begin{aligned}
&\mathbb{P}\left\{\xi_N^{(2)}(n+1) > \mathcal{Q}(N) \middle| x_N(n), y_N(n) > 0, x_N(n+1) > 0\right\}\\
&\qquad\qquad \geq \mathbb{P}\left\{\xi_N^{(2)}(n+1) > \mathcal{Q}(N)\right\} = 1 - \Phi(\mathcal{Q}(N)),
\end{aligned}
$$

which implies that

$$\mathbb{P}\left\{x_N(n+1)>0, y(n+1)>0 \middle| x_N(n), y_N(n)>0\right\} \geq (1-F(\mathcal{Q}(N)))^2 .$$

Then

$$\begin{aligned}\mathbb{P}\{\tilde{P}_N\} &= \mathbb{P}\{x_N(n)>0, y_N(n)>0, n=1,2,\ldots,N\} \\ &= \prod_{n=0}^{N-1} \mathbb{P}\left\{x_N(n+1)>0, y_N(n+1)>0 \middle| x_N(n), y_N(n)>0\right\} \\ &\geq (1-F(\mathcal{Q}(N)))^{2N} .\end{aligned}$$

We complete the proof in the same way as in [1, Theorem 3.1]. □

3.2 Use of an Adaptive Mesh for Systems of Stochastic Differential Equations

Consider again the system of stochastic differential equations given by (3.1), with coefficients satisfying (3.2a). In this section we loosen the linear bounds (3.2b) and (3.2c), instead assuming that

$$\lim_{x,y\to\infty} \frac{f_k(x,y)}{\sqrt{x^2+y^2}} = -\infty, \quad k=1,2, \tag{3.4}$$

and that there exists $\Gamma_3 \in \mathbb{R}^+$ such that

$$g_k(x,y) \leq \Gamma_3\sqrt{x^2+y^2}, \quad k=1,2. \tag{3.5}$$

Let $h_N = h(N, x_N(n), y_N(n))$ be a state-dependent mesh, and suppose that the discretisation of (3.1) is

$$\begin{aligned} x_N(n+1) &= x_N(n) + h_N f_1(x_N(n), y_N(n)) \\ &\qquad + \sqrt{h_N} g_1(x_N(n), y_N(n))\xi_N^{(1)}(n+1), \\ y_N(n+1) &= y_N(n) + h_N f_2(x_N(n), y_N(n)) \\ &\qquad + \sqrt{h_N} g_2(x_N(n), y_N(n))\xi_N^{(2)}(n+1), \\ (x_N(0), y_N(0)) &= (\zeta_1, \zeta_2) \in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{3.6}$$

where $n = 0, \ldots, N-1$.

3.2.1 Failure of a Uniform Mesh

First, we show that under conditions (3.2a), (3.4) and (3.5), a uniform mesh fails to preserve positivity, in the sense that as the initial values ζ_1 and ζ_2 grow large, the probability of positivity over N steps tends to zero.

Proposition 3.2. *Let* $\{x_N(n)\}_{n=0}^{N-1}$ *and* $\{y_N(n)\}_{n=0}^{N-1}$ *be solutions of* (3.7) *with* $h_N = h(N, x_N(n), y_N(n)) = T/N$*. Assume that* (3.2a)*,* (3.4) *and* (3.5) *hold. Then*

$$\lim_{\zeta_1,\zeta_2\to\infty} \mathbb{P}_N[y_N(1)>0]=0, \qquad \lim_{\zeta_1,\zeta_2\to\infty} \mathbb{P}_N[x_N(1)>0]=0.$$

Proof. We have for sufficiently large ζ_1, ζ_2,

$$\begin{aligned}\xi_N^{(1)}(1,\omega) &> -\frac{\zeta_1 + h_N f_1(\zeta_1,\zeta_2)}{\sqrt{h_N}g_1(\zeta_1,\zeta_2)} = -\frac{\frac{\zeta_1}{\sqrt{\zeta_1^2+\zeta_2^2}} + h_N\frac{f_1(\zeta_1,\zeta_2)}{\sqrt{\zeta_1^2+\zeta_2^2}}}{\sqrt{h_N}\frac{g_1(\zeta_1,\zeta_2)}{\sqrt{\zeta_1^2+\zeta_2^2}}}\\ &\geq -\frac{\frac{\zeta_1}{\sqrt{\zeta_2^2+\zeta_2^2}} + h_N\frac{f_1(\zeta_1,\zeta_2)}{\sqrt{\zeta_1^2+\zeta_2^2}}}{\sqrt{h_N}\Gamma_3} > \frac{-1 + h_N\frac{|f_1(\zeta_1,\zeta_2)|}{\sqrt{\zeta_1^2+\zeta_2^2}}}{\sqrt{h_N}\Gamma_3},\end{aligned}$$

and

$$\lim_{\zeta_1,\zeta_2\to\infty} P_N\left\{\xi_N^{(1)}(1,\omega) > \frac{-1 + h_N\frac{|f_1(\zeta_1,\zeta_2)|}{\sqrt{\zeta_1^2+\zeta_2^2}}}{\sqrt{h_N}\Gamma_3}\right\} = 0.$$

The proof that $\lim_{\zeta_1,\zeta_2\to\infty} \mathbb{P}_N[y_N(1)>0]=0$ is similar. □

3.2.2 Construction of a State-Dependent Mesh

Suppose that, for $x, y > 0$,

$$g_k^2(x,y) \leq \Gamma_3 \min\left\{x(1+|f_1(x,y)|), y(1+|f_2(x,y)|)\right\}, \quad k=1,2. \tag{3.7}$$

Remark 3.3. Note that (3.7) is consistent with (3.4) and (3.5) for large values of x and y. Since Proposition 3.2 indicates that large trajectory values lead to drift-induced overshoot when (3.4) and (3.5) hold, a uniform mesh will not be adequate to preserve positivity in solutions of (3.7) when g satisfies (3.7).

For fixed N let $d_N \in (0,1)$ be a constant density parameter and set, for $n = 0,\ldots,N-1$,

$$\begin{aligned}&h(N, x_N(n), y_N(n))\\ &\qquad := d_N \min\left\{\frac{x_N(n)}{1+|f_1(x_N(n),y_N(n))|}, \frac{y_N(n)}{1+|f_2(x_N(n)),y_N(n))|}\right\}.\end{aligned} \tag{3.8}$$

3.2.3 Main Result

We present the following as the natural extension of Theorem 2.4 to the system (3.7) with adaptive mesh (3.7), thus demonstrating that our results may be adapted to finite-dimensional systems.

Theorem 3.4. *Let* $\{x_N(n)\}_{n=0}^{\infty}$ *and* $\{y_N(n)\}_{n=0}^{\infty}$ *be solutions of* (3.7)*, where the state-dependent mesh* $h(N, x_N(n), y_N(n))$ *satisfies* (3.8) *with* $d_N = \tau/N$*, and* f_1, f_2, g_1, g_2 *satisfy* (3.2a), (3.4), (3.5)*, and* (3.7)*. Define the event*

$$P_N = \{\omega \in \Omega_N \,:\, x_N(n,\omega) > 0,\; y_N(n,\omega) > 0 \text{ for } n = 0, \ldots, N\}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}_N[P_N] = 1.$$

Proof. Define for each $n = 0, \ldots, N-1$,

$$\begin{aligned} R_{N,x}(n) &= \left\{\omega \,:\, \xi_N^{(1)}(n+1,\omega) > -\frac{1-d_N}{\sqrt{d_N\Gamma_3}}\right\}, \\ R_{N,y}(n) &= \left\{\omega \,:\, \xi_N^{(2)}(n+1,\omega) > -\frac{1-d_N}{\sqrt{d_N\Gamma_3}}\right\}. \end{aligned}$$

First, we show that

$$P_N \supseteq \bigcap_{n=0}^{N-1} \left(R_{N,x}(n) \bigcap R_{N,y}(n)\right). \tag{3.9}$$

Define the sequence of events $\{\widetilde{P_N}(n)\}_{n=1}^{N}$ so that

$$\widetilde{P_N}(n) = \{\omega \in \Omega_N \,:\, x_N(i,\omega) > 0, \;, y_N(i,\omega) > 0,\; i \le n\}.$$

Now each $\bigcap_{i=1}^{n} \widetilde{P_N}(i) = \widetilde{P_N}(n)$, and in particular $\bigcap_{n=1}^{N} \widetilde{P_N}(n) = \widetilde{P_N}(N) = P_N$. If we can show that each $\bigcap_{i=0}^{n-1} \left(R_{N,x}(i) \bigcap R_{N,y}(i)\right) \subseteq \widetilde{P_N}(n)$, then (3.9) holds.

We proceed by induction. If $\omega \in R_{N,x}(0)$, then

$$\begin{aligned} x_N(1,\omega) &= \zeta_1 + h(N,\zeta_1,\zeta_2) f_1(\zeta_1,\zeta_2) + \sqrt{h(N,\zeta_1,\zeta_2)}\, g_1(\zeta_1,\zeta_2)\xi_N^{(1)}(1,\omega) \\ &> \zeta_1 + d_N f_1(\zeta_1,\zeta_2)\frac{\zeta_1}{1+|f_1(\zeta_1,\zeta_2)|} - \sqrt{\frac{d_N\zeta_1 g_1^2(\zeta_1,\zeta_2)}{(1+|f_1(\zeta_1,\zeta_2)|)}}\,|\xi_N^{(1)}(1,\omega)| \\ &= \zeta_1\left(1 + d_N\frac{f_1(\zeta_1,\zeta_2)}{1+|f_1(\zeta_1,\zeta_2)|} - \sqrt{\frac{d_N g_1^2(\zeta_1,\zeta_2)}{\zeta_1(1+|f_1(\zeta_1,\zeta_2)|)}}\,|\xi_N^{(1)}(1,\omega)|\right) \\ &> \zeta_1\left(1 + d_N\frac{f_1(\zeta_1,\zeta_2)}{1+|f_1(\zeta_1,\zeta_2)|} - \sqrt{\frac{g_1^2(\zeta_1,\zeta_2)}{\zeta_1(1+|f_1(\zeta_1,\zeta_2)|)}}\,\frac{1}{\sqrt{\Gamma_3}}(1-d_N)\right) \\ &> \zeta_1 d_N\left(1 + \frac{f_1(\zeta_1,\zeta_2)}{1+|f_1(\zeta_1,\zeta_2)|}\right) > 0. \end{aligned}$$

A similar argument shows that, if $\omega \in R_{N,y}(0)$, $y_N(1,\omega) > 0$. Consequently $R_{N,x}(0) \cap R_{N,y}(0) \subseteq \widetilde{P_N}(1)$. Thus the base case is proved.

Next, assume that $\cap_{i=1}^{k} (R_{N,x}(i) \cap R_{N,y}(i)) = \widetilde{P_N}(k)$. Then if $\omega \in \widetilde{P_N}(k+1)$ we have $x_N(k,\omega) > 0$ and $y_N(k,\omega) > 0$, and

$$\begin{aligned}
x_N(k+1,\omega) &= x_N(k,\omega) + h(N, x_N(k,\omega), y_N(k,\omega)) f_1(x_N(k,\omega), y_N(k,\omega)) \\
&\quad + \sqrt{h(N, x_N(k,\omega), y_N(k,\omega))}\, g_1(x_N(k,\omega), y_N(k,\omega)) \xi_N^{(1)}(1,\omega) \\
&> x_N(k,\omega) + d_N f_1(x_N(k,\omega), y_N(k,\omega)) \frac{x_N(k,\omega)}{1 + |f_1(x_N(k,\omega), y_N(k,\omega))|} \\
&\quad - \sqrt{\frac{d_N x_N(k,\omega) g_1^2(x_N(k,\omega), y_N(k,\omega))}{(1 + |f_1(x_N(k,\omega), y_N(k,\omega))|)}} |\xi_N^{(1)}(1,\omega)| \\
&> x_N(k,\omega) \left(1 + d_N \frac{f_1(x_N(k,\omega), y_N(k,\omega))}{1 + |f_1(x_N(k,\omega), y_N(k,\omega))|} \right. \\
&\quad \left. - \sqrt{\frac{g_1^2(x_N(k,\omega), y_N(k,\omega))}{x_N(k,\omega)(1 + |f_1(x_N(k,\omega), y_N(k,\omega))|)}} \frac{1}{\sqrt{\Gamma_3}} (1 - d_N) \right) \\
&> x_N(k,\omega) \left(d_N \left(1 + \frac{f_1(x_N(k,\omega), y_N(k,\omega))}{1 + |f_1(x_N(k,\omega), y_N(k,\omega))|} \right) \right) > 0.
\end{aligned}$$

Hence, by induction $\bigcap_{i=0}^{n-1} \left(R_{N,x}(i) \bigcap R_{N,y}(i) \right) \subseteq \widetilde{P_N}(n)$ and (3.9) holds.

Since each term in the sequence of events $\{R_{N,x}(n) \cap R_{N,y}(n)\}_{n=0}^{N-1}$ is independent of all the others, and by (3.9), it follows that

$$\mathbb{P}_N[P_N] \geq \left(1 - F\left(-\frac{1 - \tau/N}{\sqrt{\tau \Gamma_3}} \sqrt{N} \right) \right)^{2N}.$$

The remainder of the proof follows as before. □

4 Conclusions and Further Work

We have continued the investigation of positivity in strong and weak Euler discretisations of stochastic differential equations that was begun in [1]. Moving beyond the near-linear analysis of that paper, we find that a dominant zero-reverting drift coefficient, or a dominant and highly variable diffusion coefficient, will cause overshoot in a uniform discretisation. However, positive trajectories may be generated with arbitrarily high probability by choosing an appropriate adaptive mesh. We also extend the analysis, both in [1] and here, to Euler discretisations of finite-dimensional systems of stochastic differential equations.

All equations examined in this paper satisfy local Lipschitz conditions on both drift and diffusion. However, adaptive meshes may be applied more generally. As discussed in [1, Section 6], for equations with a zero equilibrium solution, relaxing one or other of these Lipschitz conditions can cause solutions to strike equilibrium in finite time. An important example where this happens is the square-root diffusion stochastic differential equation

$$dX(t) = \lambda(\mu - X(t))dt + \sigma\sqrt{|X(t)|}dB(t), \quad t \geq 0,$$

with $X(0) > 0$. Such equations arise in mathematical finance and population dynamics, and more details may be found in [8]. In future work, we investigate the usefulness of an adaptive mesh in preserving the dynamics of this equation.

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