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The Modified Benjamin-Bona-Mahony Equation via the Extended Generalized Riccati Equation Mapping Method

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Abstract

The generalized Riccati equation mapping is extended together with the (G'/G)-expansion method and is a powerful mathematical tool for solving nonlinear partial differential equations. In this article, we construct twenty seven new exact traveling wave solutions including solitons and periodic solutions of the modified Benjamin-Bona-Mahony equation by applying the extended generalized Riccati equation mapping method. In this method, $G'(\mu)=p+rG(\mu)+sG^2(\mu)$ is implemented as the auxiliary equation, where r,s and p are arbitrary constants and called the generalized Riccati equation. The obtained solutions are described in four different families including the hyperbolic functions, the trigonometric functions and the rational functions. In addition, it is worth mentioning that one of newly obtained solutions is identical for a special case with already published result which validates our other solutions.

Mathematics Subject Classification: 35K99, 35P99, 35P05

Keywords: The modified Benjamin-Bona-Mahony equation, the generalized Riccati equation, the (G'/G)-expansion method, traveling wave solutions, nonlinear evolution equations.

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1. Introduction

The investigation of traveling wave solutions of the nonlinear partial differential equations (PDEs) plays the most important role in the study of nonlinear physical phenomena which arise in mathematical physics, engineering sciences and other technical arena [1-43]. In recent times, many researchers established various methods to construct exact traveling wave solutions of the nonlinear PDEs, such as, the inverse scattereing method [1], the variational iteration method [2,3], the Hirota's bilinear transformation method [4], the Jacobi elliptic function expansion method [5], the tanh-coth method [6,7], the Backlund transformation method [8], the direct algebraic method [9], the Cole-Hopf transformation method [10], the sine-cosine method [11], the Exp-function method [12-14], the Adomian decomposition method [15] and others [16-22].

Recently, Wang *et al.* [23] presented a method, called the $\left(G'/G\right)$-expansion method and they established traveling wave solutions for some nonlinear PDEs. In this method, they employed the second order linear ordinary differential equation with constant coefficients $G''(\theta)+\lambda G'(\theta)+\mu G(\theta)=0,$ as an auxiliary equation. Subsequently, many researchers implemented this powerful $\left(G'/G\right)$-expansion method to investigate different nonlinear PDEs for obtaining exact traveling wave solutions. For example, Feng *et al*. [24] studied the Kolmogorov-Petrovskii-Piskunov equation to construct exact solutions via this method. Naher *et al*. [25] applied the same method to obtain traveling wave solutions of the higher-order Caudrey-Dodd-Gibbon equation. In Ref. [26], Abazari investigated the Zoomeron equation for establishing solitary wave solutions by using this method. Zayed and Al-Joudi [27] concerned about this method for constructing analytical solutions of some nonlinear partial differential equations whilst Gepreel [28] studied nonlinear PDEs with variable coefficients in mathematical physics by using this method and found exact solutions. Ozis and Aslan [29] applied the $\left(G'/G\right)$-expansion method to establish traveling wave solutions for the Kawahara type equations using symbolic computation while Aslan [30] investigated three nonlinear evolution equations to construct exact solutions by applying this method. Naher *et al*. [31] concerned about the improved $\left(G'/G\right)$-expansion method to obtain traveling wave solutions of the higher dimensional nonlinear evolution equation while Naher and Abdullah [32] constructed some new traveling wave solutions of the nonlinear reaction diffusion equation via this method and so on.

Zhu [33] introduced the generalized Riccati equation mapping with the extended tanh-function method to investigate the (2+1)-dimensional Boiti-Leon-Pempinelle equation. In this generalized Riccati equation mapping, the auxiliary equation $G'(\mu)=p+r\,G(\mu)+s\,G^2(\mu)$ is used, where r,s and p are arbitrary constants. Bekir and Cevikel [34] implemented the tanh-coth method combined with the

Riccati equation to solve nonlinear coupled equation in mathematical physics. In Ref. [35], Guo *et al.* studied the diffusion-reaction and the mKdV equation with variable coefficient via the extended Riccati equation mapping method while Li *et al.* [36] used the generalized Riccati equation expansion method to study the (3+1)-dimensional Jimbo-Miwa equation. Salas [37] utilized the projective Riccati equation method to obtain some exact solutions for the Caudrey-Dodd-Gibbon equation. Li and Dai [38] executed the generalized Riccati equation mapping method to construct traveling wave solutions for the higher dimensional nonlinear evolution equation and so on.

Many researchers implemented different methods to obtain traveling wave solutions of the modified Benjamin-Bona-Mahony equation. For example, Taghizadeh and Mirzazadeh [39] used modified extended tanh method to establish traveling wave solutions of this equation. Yusufoglu [40] applied the Exp-function method to construct traveling wave solutions of the same equation while Yusufoglu and Bekir [41] investigated this equation to seek exact solutions via the tanh and sine-cosine methods. In Ref. [42], Abbasbandy and Shirzadi executed the first integral method to obtain analytical solutions of the same equation. Gao [43] studied this equation by using the algebraic method to establish exact solutions. Aslan [30] studied the same equation for constructing traveling wave solutions by applying the basic $\left(G'/G\right)$-expansion method. In the basic $\left(G'/G\right)$-expansion method, the second order linear ordinary differential equation (LODE) with constant coefficients is considered, as an auxiliary equation. To the best of our knowledge, no body studied the modified Benjamin-Bona-Mahony equation to construct exact traveling wave solutions by applying the extended generalized Riccati equation mapping method.

In this article, we investigate the modified Benjamin-Bona-Mahony equation to construct exact traveling wave solutions including solitons, periodic, and rational solutions via the extended generalized Riccati equation mapping method.

2. The extended generalized Riccati equation mapping method

Suppose the general nonlinear partial differential equation:

$$A\left(v, v_t, v_x, v_{xt}, v_{tt}, v_{xx}, \ldots\right) = 0, \tag{1}$$

where $v = v\left(x,t\right)$ is an unknown function, A is a polynomial in $v = v\left(x,t\right)$ and the subscripts indicate the partial derivatives.

The most important steps of the generalized Riccati equation mapping together with the $\left(G'/G\right)$-expansion method [23,33] are as follows:

Step 1. Consider the traveling wave variable:

$$v\left(x,t\right) = p\left(\mu\right), \qquad \mu = Kx + H\,t, \tag{2}$$

Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $p(\mu)$:

$$B\left(p, p', p'', p''',...\right)=0, \tag{3}$$

where the superscripts stand for the ordinary derivatives with respect to μ.

Step 2. Eq. (3) integrates term by term one or more times according to possibility, yields constant(s) of integration. The integral constant(s) may be zero for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [23,33]:

$$p(\mu)=\sum_{j=0}^{n} a_j\left(\frac{G'}{G}\right)^j \tag{4}$$

where $a_j\left(j=0,1,2,...,n\right)$ and $a_n \neq 0$, with $G=G(\mu)$ is the solution of the generalized Riccati equation:

$$G' = p + r\,G + s\,G^2, \tag{5}$$

where r, s, p are arbitrary constants and $s \neq 0.$

Step 4. To decide the positive integer n, consider the homogeneous balance between the nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substitute Eq. (4) along with Eq. (5) into the Eq. (3), then collect all the coefficients with the same order, the left hand side of Eq. (3) converts into polynomials in $G^m(\mu)$ and $G^{-m}(\mu),(m=0,1,2,...)$. Then equating each coefficient of the polynomials to zero, yield a set of algebraic equations for $a_j\ \left(j=0,1,2,...,n\right), r, s, p, K$ and H.

Step 6. Solve the system of algebraic equations which are found in Step 5 with the aid of algebraic software Maple and we obtain values for $a_j\ \left(j=0,1,2,...,n\right)$ and H. Then, substitute obtained values in Eq. (4) along with Eq. (5) with the value of n, we obtain exact solutions of Eq. (1).
In the following, we have twenty seven solutions including four different families of Eq. (5).

Family 1: When $r^2-4sp>0$ and $rs \neq 0$ or $sp \neq 0$, the solutions of Eq. (5) are:

$$W_1=\frac{-1}{2s}\left(r+\sqrt{r^2-4sp}\ \tanh\left(\frac{\sqrt{r^2-4sp}}{2}\mu\right)\right),$$

$$W_2 = \frac{-1}{2s}\left(r + \sqrt{r^2 - 4sp}\, \coth\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right) \right),$$

$$W_3 = \frac{-1}{2s}\left(r + \sqrt{r^2 - 4sp}\left(\tanh\left(\sqrt{r^2 - 4sp}\,\mu \right) \pm i\,\operatorname{sec}h\left(\sqrt{r^2 - 4sp}\,\mu \right) \right) \right),$$

$$W_4 = \frac{-1}{2s}\left(r + \sqrt{r^2 - 4sp}\left(\coth\left(\sqrt{r^2 - 4sp}\,\mu \right) \pm \operatorname{csc}h\left(\sqrt{r^2 - 4sp}\,\mu \right) \right) \right),$$

$$W_5 = \frac{-1}{4s}\left(2r + \sqrt{r^2 - 4sp}\left(\tanh\left(\frac{\sqrt{r^2 - 4sp}}{4}\mu \right) + \cot h\left(\frac{\sqrt{r^2 - 4sp}}{4}\mu \right) \right) \right),$$

$$W_6 = \frac{1}{2s}\left(-r + \frac{\sqrt{\left(Q^2 + R^2\right)\left(r^2 - 4sp\right)} - Q\sqrt{r^2 - 4sp}\,\cosh\left(\sqrt{r^2 - 4sp}\,\mu \right)}{Q\sinh\left(\sqrt{r^2 - 4sp}\,\mu \right) + R} \right),$$

$$W_7 = \frac{1}{2s}\left(-r - \frac{\sqrt{\left(R^2 - Q^2\right)\left(r^2 - 4sp\right)} + Q\sqrt{r^2 - 4sp}\,\sinh\left(\sqrt{r^2 - 4sp}\,\mu \right)}{Q\cosh\left(\sqrt{r^2 - 4sp}\,\mu \right) + R} \right),$$

where Q and R are two non-zero real constants and satisfies $R^2 - Q^2 > 0$.

$$W_8 = \frac{2p\cosh\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right)}{\sqrt{r^2 - 4sp}\,\sinh\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right) - r\cosh\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right)},$$

$$W_9 = \frac{-2p\sinh\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right)}{r\sinh\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right) - \sqrt{r^2 - 4sp}\,\cosh\left(\frac{\sqrt{r^2 - 4sp}}{2}\mu \right)},$$

$$W_{10} = \frac{2p\cosh\left(\sqrt{r^2 - 4sp}\,\mu \right)}{\sqrt{r^2 - 4sp}\,\sinh\left(\sqrt{r^2 - 4sp}\,\mu \right) - r\cosh\left(\sqrt{r^2 - 4sp}\,\mu \right) \pm i\sqrt{r^2 - 4sp}},$$

$$W_{11} = \frac{2p\sinh\left(\sqrt{r^2 - 4sp}\,\mu \right)}{-r\sinh\left(\sqrt{r^2 - 4sp}\,\mu \right) + \sqrt{r^2 - 4sp}\,\cosh\left(\sqrt{r^2 - 4sp}\,\mu \right) \pm \sqrt{r^2 - 4sp}},$$

$$W_{12}=\frac{4p\sinh\left(\frac{\sqrt{r^2-4sp}}{4}\mu\right)\cosh\left(\frac{\sqrt{r^2-4sp}}{4}\mu\right)}{-2r\sinh\left(\frac{\sqrt{r^2-4sp}}{4}\mu\right)\cosh\left(\frac{\sqrt{r^2-4sp}}{4}\mu\right)+2\sqrt{r^2-4sp}\cosh^2\left(\frac{\sqrt{r^2-4sp}}{4}\mu\right)-\sqrt{r^2-4sp}},$$

Family 2: When $r^2-4sp<0$ and $rs\neq 0$ or $sp\neq 0$, the solutions of Eq. (5) are:

$$W_{13}=\frac{1}{2s}\left(-r+\sqrt{4sp-r^2}\,\tan\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)\right),$$

$$W_{14}=\frac{-1}{2s}\left(r+\sqrt{4sp-r^2}\,\cot\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)\right),$$

$$W_{15}=\frac{1}{2s}\left(-r+\sqrt{4sp-r^2}\left(\tan\left(\sqrt{4sp-r^2}\,\mu\right)\pm\sec\left(\sqrt{4sp-r^2}\,\mu\right)\right)\right),$$

$$W_{16}=\frac{-1}{2s}\left(r+\sqrt{4sp-r^2}\left(\cot\left(\sqrt{4sp-r^2}\,\mu\right)\pm\csc\left(\sqrt{4sp-r^2}\,\mu\right)\right)\right),$$

$$W_{17}=\frac{1}{4s}\left(-2r+\sqrt{4sp-r^2}\left(\tan\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)-\cot\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)\right)\right),$$

$$W_{18}=\frac{1}{2s}\left(-r+\frac{\pm\sqrt{\left(Q^2-R^2\right)\left(4sp-r^2\right)}-Q\sqrt{4sp-r^2}\,\cos\left(\sqrt{4sp-r^2}\,\mu\right)}{Q\sin\left(\sqrt{4sp-r^2}\,\mu\right)+R}\right),$$

$$W_{19}=\frac{1}{2s}\left(-r-\frac{\pm\sqrt{\left(Q^2-R^2\right)\left(4sp-r^2\right)}+Q\sqrt{4sp-r^2}\cos\left(\sqrt{4sp-r^2}\,\mu\right)}{Q\sin\left(\sqrt{4sp-r^2}\,\mu\right)+R}\right),$$

where Q and R are two non-zero real constants and satisfies $Q^2-R^2>0$.

$$W_{20}=\frac{-2p\cos\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)}{\sqrt{4sp-r^2}\sin\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)+r\cos\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)},$$

$$W_{21}=\frac{2p\sin\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)}{-r\sin\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)+\sqrt{4sp-r^2}\cos\left(\frac{\sqrt{4sp-r^2}}{2}\mu\right)},$$

$$W_{22}=\frac{-2p\cos\left(\sqrt{4sp-r^2}\,\mu\right)}{\sqrt{4sp-r^2}\sin\left(\sqrt{4sp-r^2}\,\mu\right)+r\cos\left(\sqrt{4sp-r^2}\,\mu\right)\pm\sqrt{4sp-r^2}},$$

$$W_{23}=\frac{2p\sin\left(\sqrt{4sp-r^2}\,\mu\right)}{-r\sin\left(\sqrt{4sp-r^2}\,\mu\right)+\sqrt{4sp-r^2}\cos\left(\sqrt{4sp-r^2}\,\mu\right)\pm\sqrt{4sp-r^2}},$$

$$W_{24}=\frac{4p\sin\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)\cos\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)}{-2r\sin\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)\cos\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)+2\sqrt{4sp-r^2}\cos^2\left(\frac{\sqrt{4sp-r^2}}{4}\mu\right)-\sqrt{4sp-r^2}},$$

Family 3: when $p=0$ and $rs\neq 0,$ the solution Eq. (5) becomes:

$$W_{25}=\frac{-r\,g_1}{s\left(g_1+\cosh(r\mu)-\sinh(r\mu)\right)},$$

$$W_{26}=\frac{-r\left(\cosh(r\mu)+\sinh(r\mu)\right)}{s\left(g_1+\cosh(r\mu)+\sinh(r\mu)\right)},$$

where g_1 is an arbitrary constant.

Family 4: when $s\neq 0$ and $p=r=0,$ the solution of Eq. (5) becomes:

$$W_{27}=\frac{-1}{s\mu+d_1},$$

where d_1 is an arbitrary constant.

3. Applications of the method

We construct twenty seven exact traveling wave solutions including solitons, periodic, and rational solutions of the modified Benjamin-Bona-Mahony equation in this section.

3.1 The Modified Benjamin-Bona-Mahony equation

We consider the modified Benjamin-Bona-Mahony equation with parameters followed by Aslan [30]:

$$u_t + \alpha u_x + \beta u^2 u_x - \gamma u_{xxt} = 0, \qquad (6)$$

where α, γ are free parameters and $\beta \neq 0$.
Now, we use the transformation Eq. (2) into the Eq. (6), which yields:

$$\left(H + K\alpha\right) p' + K\beta p^2 p' - HK^2\gamma p''' = 0, \qquad (7)$$

Eq. (7) is integrable, therefore, integrating with respect μ once yields:

$$\left(H + K\alpha\right) p + \frac{K\beta}{3} p^3 - HK^2\gamma p'' + C = 0, \qquad (8)$$

where C is an integral constant which is to be determined later.
Taking the homogeneous balance between p^3 and p'' in Eq. (8), we obtain $n = 1$.
Therefore, the solution of Eq. (8) is of the form:

$$p\left(\mu\right) = a_1\left(G'/G\right) + a_0,\ \ a_1 \neq 0. \qquad (9)$$

Using Eq. (5), Eq. (9) can be re-written as:

$$p\left(\mu\right) = a_1\left(r + pG^{-1} + sG\right) + a_0, \qquad (10)$$

where r, s and p are free parameters.
By substituting Eq. (10) into Eq. (8), collecting all coefficients of G^k and G^{-k} $\left(k = 0, 1, 2, \ldots\right)$ and setting them equal to zero, we obtain a set of algebraic equations for a_0, a_1, r, s, p, C and H (algebraic equations are not shown, for simplicity). Solving the system of algebraic equations with the help of algebraic software Maple, we obtain

$$a_0 = \mp Kr\sqrt{\frac{-3\alpha\gamma}{\beta\left(2 + K^2\gamma\left(r^2 + 8sp\right)\right)}},\ a_1 = \pm 2K\sqrt{\frac{-3\alpha\gamma}{\beta\left(2 + K^2\gamma\left(r^2 + 8sp\right)\right)}}, H = \frac{-2\alpha K}{2 + K^2\gamma\left(r^2 + 8sp\right)},$$

$$C = \pm\frac{8K^4\alpha\gamma rst}{2 + K^2\gamma\left(r^2 + 8sp\right)}\sqrt{\frac{-3\alpha\gamma}{\beta\left(2 + K^2\gamma\left(r^2 + 8sp\right)\right)}},$$

where α, γ are free parameters and $\beta \neq 0$.

Family 1: The soliton and soliton-like solutions of Eq. (6) (when $r^2 - 4sp > 0$ and $rs \neq 0$ or $sp \neq 0$) are:

$$p_1 = a_1 \frac{\Phi^2 \sec h^2\left(\frac{\Phi}{2}\mu\right)}{2\left(r + \Phi \tanh\left(\frac{\Phi}{2}\mu\right)\right)} + a_0,$$

where $\Phi = \sqrt{r^2 - 4sp}$, $\Phi^2 = r^2 - 4sp$, $a_0 = \mp Kr\sqrt{\frac{-3\alpha\gamma}{\beta\left(2 + K^2\gamma\left(r^2 + 8sp\right)\right)}}$,

$a_1 = \pm 2K\sqrt{\frac{-3\alpha\gamma}{\beta\left(2 + K^2\gamma\left(r^2 + 8sp\right)\right)}}$ and $\mu = Kx - \frac{2\alpha K}{2 + K^2\gamma\left(r^2 + 8sp\right)}t.$

$$p_2 = a_1 \frac{-\Phi^2 \csc h^2\left(\frac{\Phi}{2}\mu\right)}{2\left(r + \Phi \coth\left(\frac{\Phi}{2}\mu\right)\right)} + a_0,$$

$$p_3 = a_1 \frac{\Phi^2\left(\sec h^2(\Phi\mu) \mp i \tanh(\Phi\mu)\sec h(\Phi\mu)\right)}{r + \Phi\left(\tanh(\Phi\mu) \pm i \sec h(\Phi\mu)\right)} + a_0,$$

$$p_4 = a_1 \frac{-\Phi^2\left(\csc h^2(\Phi\mu) \pm \coth(\Phi\mu)\csc h(\Phi\mu)\right)}{r + \Phi\left(\coth(\Phi\mu) \pm \csc h(\Phi\mu)\right)} + a_0,$$

$$p_5 = a_1 \frac{\Phi^2\left(\sec h^2\left(\frac{\Phi}{4}\mu\right) - \csc h^2\left(\frac{\Phi}{4}\mu\right)\right)}{8r + 4\Phi\left(\tanh\left(\frac{\Phi}{4}\mu\right) + \coth\left(\frac{\Phi}{4}\mu\right)\right)} + a_0,$$

$$p_6 = a_1 \frac{-Q\left(r^2 Q - \sinh(\Phi\mu) r^2 R - 4spQ + 4spR\sinh(\Phi\mu) - \Phi^2\sqrt{\left(Q^2 + R^2\right)}\cosh(\Phi\mu)\right)}{\left(Q\sinh(\Phi\mu) + R\right)\left(rQ\sinh(\Phi\mu) + rR - \Phi\sqrt{\left(Q^2 + R^2\right)} + Q\Phi\cosh(\Phi\mu)\right)} + a_0,$$

$$p_7 = a_1 \frac{-Q\left(r^2 Q - \sinh(\Phi\mu) r^2 R - 4spQ + 4spR\sinh(\Phi\mu) + \Phi^2\sqrt{\left(Q^2 + R^2\right)}\cosh(\Phi\mu)\right)}{\left(Q\sinh(\Phi\mu) + R\right)\left(rQ\sinh(\Phi\mu) + rR + \Phi\sqrt{\left(Q^2 + R^2\right)} + Q\Phi\cosh(\Phi\mu)\right)} + a_0,$$

where Q and R are two non-zero real constants and satisfies $R^2 - Q^2 > 0$.

$$p_8 = a_1 \frac{-\Phi^2}{2\cosh\left(\frac{\Phi}{2}\mu\right)\left(\Phi\sinh\left(\frac{\Phi}{2}\mu\right)-r\cosh\left(\frac{\Phi}{2}\mu\right)\right)}+a_0,$$

$$p_9 = a_1 \frac{\Phi^2}{2\sinh\left(\frac{\Phi}{2}\mu\right)\left(-r\sinh\left(\frac{\Phi}{2}\mu\right)+\Phi\cosh\left(\frac{\Phi}{2}\mu\right)\right)}+a_0,$$

$$p_{10} = a_1 \frac{-\Phi^2+i\,r^2\sinh(\Phi\mu)-i4sp\sinh(\Phi\mu)}{\Phi\sinh(\Phi\mu)-r\cosh(\Phi\mu)+i\Phi\cosh(\Phi\mu)}+a_0,$$

$$p_{11} = a_1 \frac{\Phi^2+r^2\cosh(\Phi\mu)-4sp\cosh(\Phi\mu)}{(-r\sinh(\Phi\mu)+\Phi\cosh(\Phi\mu)+\Phi)\sinh(\Phi\mu)}+a_0,$$

$$p_{12} = a_1 \frac{\Phi^2}{4\sinh\left(\frac{\Phi}{4}\mu\right)\cosh\left(\frac{\Phi}{4}\mu\right)\left(-2r\sinh\left(\frac{\Phi}{4}\mu\right)\cosh\left(\frac{\Phi}{4}\mu\right)+2\Phi\cosh^2\left(\frac{\Phi}{4}\mu\right)-\Phi\right)}+a_0,$$

Family 2: The periodic form solutions of Eq. (6) (when $r^2-4sp<0$ and $rs\neq 0$ or $sp\neq 0$) are:

$$p_{13} = a_1 \frac{\Pi^2}{2\cos\left(\frac{\Pi}{2}\mu\right)\left(-r\cos\left(\frac{\Pi}{2}\mu\right)+\Pi\sin\left(\frac{\Pi}{2}\mu\right)\right)}+a_0,$$

where $\Pi=\sqrt{-r^2+4sp}$, $\Pi^2=4sp-r^2$, $a_0=\mp Kr\sqrt{\frac{-3\alpha\gamma}{\beta\left(2+K^2\gamma\left(r^2+8sp\right)\right)}}$,

$a_1=\pm 2K\sqrt{\frac{-3\alpha\gamma}{\beta\left(2+K^2\gamma\left(r^2+8sp\right)\right)}}$ and $\mu=Kx-\frac{2\alpha K}{2+K^2\gamma\left(r^2+8sp\right)}t.$

$$p_{14} = a_1 \frac{\Pi^2}{2\left(-1+\cos^2\left(\frac{\Pi}{2}\mu\right)\right)\left(r+\Pi\cot\left(\frac{\Pi}{2}\mu\right)\right)}+a_0,$$

$$p_{15} = a_1 \frac{\Pi^2\left(1+\sin(\Pi\mu)\right)}{\cos(\Pi\mu)\left(-r\cos(\Pi\mu)+\Pi\sin(\Pi\mu)+\Pi\right)}+a_0,$$

$$p_{16} = a_1 \frac{\Pi^2\sin(\Pi\mu)}{\cos(\Pi\mu)r\sin(\Pi\mu)+\Pi\cos^2(\Pi\mu)-r\sin(\Pi\mu)-\Pi}+a_0,$$

$$p_{17} = a_1 \frac{-\Pi^2}{4\cos^2\left(\frac{\Pi}{4}\mu\right)\left(-1+\cos^2\left(\frac{\Pi}{4}\mu\right)\right)\left(-2r+\Pi\left(\tan\left(\frac{\Pi}{4}\mu\right)-\cot\left(\frac{\Pi}{4}\mu\right)\right)\right)}+a_0,$$

$$p_{18}=a_1\frac{Q\left(-4spQ-4spR\sin(\Pi\mu)+r^2Q+r^2R\sin(\Pi\mu)+4sp\sqrt{(Q^2-R^2)}\cos(\Pi\mu)-r^2\sqrt{(Q^2-R^2)}\cos(\Pi\mu)\right)}{\left(-Q^2+Q^2\cos^2(\Pi\mu)-2QR\sin(\Pi\mu)-R^2\right)\left(-r+\dfrac{\Pi\sqrt{(Q^2-R^2)}}{Q\sin(\Pi\mu)+R}-Q\Pi\cos(\Pi\mu)\right)}+a_0,$$

$$p_{19}=a_1\frac{Q\left(-4spQ-4spR\sin(\Pi\mu)+r^2Q+r^2R\sin(\Pi\mu)-4sp\sqrt{(Q^2-R^2)}\cos(\Pi\mu)+r^2\sqrt{(Q^2-R^2)}\cos(\Pi\mu)\right)}{\left(-Q^2+Q^2\cos^2(\Pi\mu)-2QR\sin(\Pi\mu)-R^2\right)\left(-r-\dfrac{\Pi\sqrt{(Q^2-R^2)}}{Q\sin(\Pi\mu)+R}+Q\Pi\cos(\Pi\mu)\right)}+a_0,$$

where Q and R are two non-zero real constants and satisfies $Q^2-R^2>0.$

$$p_{20}=a_1\frac{-\Pi^2\sec\left(\frac{\Pi}{2}\mu\right)\left(\Pi\sin\left(\frac{\Pi}{2}\mu\right)+r\cos\left(\frac{\Pi}{2}\mu\right)\right)}{2\left(4sp-4sp\cos^2\left(\frac{\Pi}{2}\mu\right)-r^2+2r^2\cos^2\left(\frac{\Pi}{2}\mu\right)+2r\Pi\sin\left(\frac{\Pi}{2}\mu\right)\cos\left(\frac{\Pi}{2}\mu\right)\right)}+a_0,$$

$$p_{21}=a_1\frac{-\Pi^2\left(-r\sin\left(\frac{\Pi}{2}\mu\right)+\Pi\cos\left(\frac{\Pi}{2}\mu\right)\right)}{2\sin\left(\frac{\Pi}{2}\mu\right)\left(-r^2+2r^2\cos^2\left(\frac{\Pi}{2}\mu\right)+2r\Pi\sin\left(\frac{\Pi}{2}\mu\right)\cos\left(\frac{\Pi}{2}\mu\right)-4sp\cos^2\left(\frac{\Pi}{2}\mu\right)\right)}+a_0,$$

$$p_{22}=\frac{\frac{1}{2}a_1\sec(\Pi\mu)\left(-\Pi^2-4sp\sin(\Pi\mu)+r^2\sin(\Pi\mu)\right)\left(\Pi\sin(\Pi\mu)+r\cos(\Pi\mu)+\Pi\right)}{\left(4sp-2sp\cos^2(\Pi\mu)-r^2+r^2\cos^2(\Pi\mu)+\Pi r\sin(\Pi\mu)\cos(\Pi\mu)+4sp\sin(\Pi\mu)-r^2\sin(\Pi\mu)+r\Pi\cos(\Pi\mu)\right)}+a_0,$$

$$p_{23}=a_1\frac{-\Pi^2\left(-r\sin(\Pi\mu)+\Pi\cos(\Pi\mu)+\Pi\right)}{2\sin(\Pi\mu)\left(-2sp\cos(\Pi\mu)+r^2\cos(\Pi\mu)+r\Pi\sin(\Pi\mu)-2sp\right)}+a_0,$$

$$p_{24}=\frac{\frac{-\Pi^2}{4}a_1\csc\left(\frac{\Pi}{4}\mu\right)\sec\left(\frac{\Pi}{4}\mu\right)\left(-2r\sin\left(\frac{\Pi}{4}\mu\right)\cos\left(\frac{\Pi}{4}\mu\right)+2\Pi\cos^2\left(\frac{\Pi}{4}\mu\right)-\Pi\right)}{\left(-8r^2\cos^2\left(\frac{\Pi}{4}\mu\right)+8r^2\cos^4\left(\frac{\Pi}{4}\mu\right)+8\Pi r\cos^3\left(\frac{\Pi}{4}\mu\right)\sin\left(\frac{\Pi}{4}\mu\right)-4r\Pi\sin\left(\frac{\Pi}{4}\mu\right)\cos\left(\frac{\Pi}{4}\mu\right)-16sp\cos^4\left(\frac{\Pi}{4}\mu\right)+16sp\cos^2\left(\frac{\Pi}{4}\mu\right)-\Pi^2\right)}+a_0,$$

Family 3: The soliton and soliton-like solutions of Eq. (6) (when $p=0$ and $rs\neq 0$) are:

$$p_{25}=a_1\frac{r\left(\cosh(r\mu)-\sinh(r\mu)\right)}{g_1+\cosh(r\mu)-\sinh(r\mu)}+a_0,$$

$$p_{26} = a_1 \frac{r\, g_1}{g_1 + \cosh(r\mu) + \sinh(r\mu)} + a_0,$$

where g_1 is an arbitrary constant, $a_0 = \mp Kr\sqrt{\dfrac{-3\alpha\gamma}{\beta\left(2+K^2\gamma\left(r^2+8sp\right)\right)}}$,

$a_1 = \pm 2K\sqrt{\dfrac{-3\alpha\gamma}{\beta\left(2+K^2\gamma\left(r^2+8sp\right)\right)}}$ and $\mu = Kx - \dfrac{2\alpha K}{2+K^2\gamma\left(r^2+8sp\right)}t.$

Family 4: The rational function solution **(**when $s \neq 0$ and $p = r = 0$) is:

$$p_{27} = \frac{-a_1 s}{s\,\mu + d_1},$$

where d_1 is an arbitrary constant , $a_1 = \pm 2K\sqrt{\dfrac{-3\alpha\gamma}{\beta\left(2+K^2\gamma\left(r^2+8sp\right)\right)}}$ and

$$\mu = Kx - \frac{2\alpha K}{2+K^2\gamma\left(r^2+8sp\right)}t.$$

4. Results and discussion

It is important to point out that our solution p_{27} is identical for a special case with $u_{5,6}(x,t)$ in subsection 3.3 of section 3 of Aslan [30]. If we substitute $C_3 = 1, C_2 = 0, \lambda^2 - 4\mu = 0, \alpha = \gamma = 1, \beta = -6$ and $k = 1$ in solution $u_{5,6}(x,t)$ of Aslan [40] and becomes $u_{5,6}(x,t) = \mp\dfrac{1}{x-t}$. Also, if we substitute $p_{27} = u_{5,6}(x,t)$ and $K = 1, \alpha = \gamma = 1, p = r = 0, s = 1, \beta = -6, d_1 = 0$, in our solution p_{27} and becomes $u_{5,6}(x,t) = \mp\dfrac{1}{x-t}$. Beyond this, we obtain new traveling wave solutions p_1 to p_{26}, and to the best of our knowledge, which have not been reported in the previous literature. Furthermore, the graphical demonstrations of some obtained solutions are shown in figure 1 to figure 12 in the following subsection.

4.1 Graphical illustrations of the solutions

Some of our obtained traveling wave solutions are represented in the following figures with the aid of commercial software Maple:

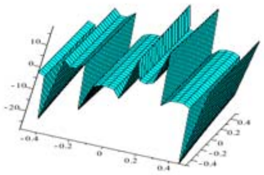

Fig. 1: Periodic solution for $r=1, s=2, p=3, \alpha=1, \beta=1, \gamma=2, K=3$

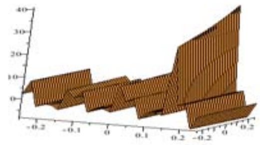

Fig. 2: Periodic solution for $r=3, s=4, p=5, \alpha=2, \beta=3, \gamma=1, K=5$

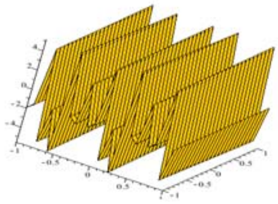

Fig. 3: Periodic solution for $r=2, s=1, p=3, \alpha=2, \beta=3, \gamma=5, K=8$

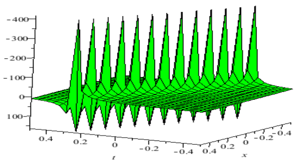

Fig. 4: Solitons solution for $r=0, s=17, p=0, \alpha=2, \beta=0.5, \gamma=1, K=25, g_1=5$

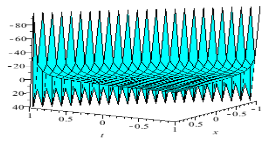

Fig. 5: Solitons solution for $r=0, s=12, p=0, \alpha=1, \beta=\gamma=2, K=10, g_1=3$

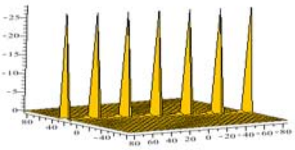

Fig. 6: Solitons solution for $r=0, s=7, p=0, \alpha=4, \beta=3, \gamma=4, K=5, g_1=7$

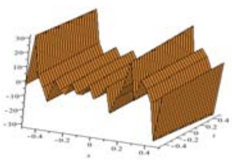

Fig. 7: Periodic solutions for $r=3, s=5, p=7, \alpha=4, \beta=3, \gamma=3, K=9$

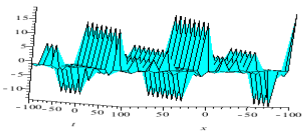

Fig. 8: Solitons solution for $r=5, s=1, p=7, \alpha=4, \beta=4, \gamma=0.5, K=1$

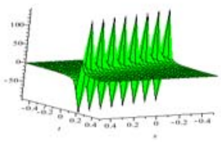

Fig. 9: Solitons solution for $r=0, s=5, p=0, \alpha=3, \beta=3, \gamma=1, K=3, g_1=1$

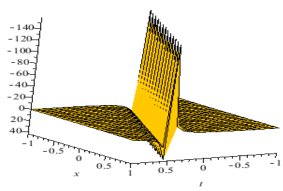

Fig. 10: Solitons solution for $r=0, s=25, p=0, \alpha=2, \gamma=2, \beta=3, K=9, g_1=4$

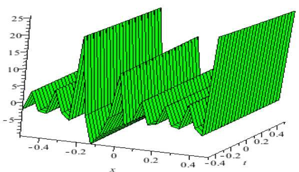

Fig. 11: Periodic solution for $r=5, s=3, p=5, \alpha=4, \beta=3, \gamma=4, K=9$

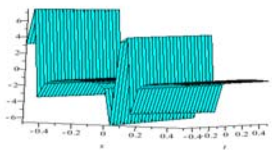

Fig. 12: Periodic solution for $r=5, s=2, p=4, \alpha=3, \beta=3, \gamma=4, K=4$

5. Conclusions

By applying the extended generalized Riccati equation mapping method, abundant new exact traveling wave solutions of the modified Benjamin-Bona-Mahony equation are constructed in this article. In this method, the auxiliary equation $G'(\mu)=p+rG(\mu)+sG^2(\mu)$ is used with constant coefficients, instead of the second order linear ordinary differential equation with constant coefficients. The obtained solutions disclose noteworthy properties of the shapes in that the solutions come as solitons and periodic solutions. Further, it is imperative to mention out that one of our solutions are identical with the solutions available in the literature and some are new. We hope that this straightforward method can be more successfully used to investigate nonlinear evolution equations which arise in mathematical physics, engineering sciences and other technical arena.

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