

# General Dual Fuzzy Linear Systems

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## Abstract

This paper mainly intends to discuss the solution of fully fuzzy linear systems (FFLS)  $Ax + b = Cx + d$ , where  $A$  and  $C$  are fuzzy matrices,  $b$  and  $d$  are fuzzy vectors. We transform the systems by using fuzzy numbers with a new parametric form for finding a fuzzy vector  $x$  that satisfies in the system.

**Keywords:** Fuzzy number, Full fuzzy linear systems, Fuzzy arithmetic, Parametric representation

## 1 Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [26], Dubois and Prade [9]. We refer the reader to [19] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems ranging from fuzzy topological spaces [7] to control chaotic systems [15, 18], fuzzy metric spaces [23], fuzzy differential equations [3], fuzzy linear and nonlinear systems [1, 2, 5, 6] and particle physics [10, 11, 12, 13, 14, 22, 24].

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One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems and fully fuzzy linear systems, several problems in various areas such as economics, engineering and physics boil down to the solution of a linear system of equations. In many applications, at least some of the parameters of the system should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop numerical procedures that would appropriately treat fuzzy linear systems and solve them.

Friedman *et al.* [16] introduced a general model for solving a fuzzy  $n \times n$  linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy  $n \times n$  linear system by a crisp  $2n \times 2n$  linear system and studied duality in fuzzy linear systems  $Ax = Bx + y$  where  $A, B$  are real  $n \times n$  matrix, the unknown vector  $x$  is vector consisting of  $n$  fuzzy numbers and the constant  $y$  is vector consisting of  $n$  fuzzy numbers, in [17]. In [1, 2, 6] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. Also, Wang *et al.* [25] presented an iterative algorithm for solving dual linear system of the form  $x = Ax + u$ , where  $A$  is real  $n \times n$  matrix, the unknown vector  $x$  and the constant  $u$  are all vectors consisting of fuzzy numbers and Abbasbandy *et al.* [4] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form  $Ax + f = Bx + c$ , where  $A, B$  are real  $m \times n$  matrix, the unknown vector  $x$  is vector consisting of  $n$  fuzzy numbers and the constant  $f, c$  are vectors consisting of  $m$  fuzzy numbers. Recently, Muzziloi *et al.* [21] considered fully fuzzy linear systems of the form  $A_1x + b_1 = A_2x + b_2$  with  $A_1, A_2$  square matrices of fuzzy coefficients and  $b_1, b_2$  fuzzy number vectors and Dehghan *et al.* [8] considered fully fuzzy linear systems of the form  $Ax = b$  where  $A$  and  $b$  are a fuzzy matrix and a fuzzy vector, respectively and them discussed the iterative solution of fully fuzzy linear systems.

In this paper, we are finding solution of a general duality fully fuzzy linear system  $n \times n$   $Ax + b = Cx + d$  based on new arithmetic calculation, with  $A, C$  square matrices of fuzzy coefficients and  $b, d$  fuzzy number vectors and the unknown vector  $x$  is vector consisting of  $n$  fuzzy numbers. In Section 2, we recall some fundamental results on fuzzy numbers. The proposed model for solving the system  $Ax + b = Cx + d$  are discussed in Section 3. Numerical examples are given in Section 4 followed by a discussion and concluding in Section 5.

## 2 Preliminaries

A popular fuzzy number is the triangular fuzzy number  $u = (a, b, c)$  where the membership function is

$$\mu_u(x) = \begin{cases} \frac{x-a}{b-a} & a \leq x \leq b, \\ \frac{c-x}{c-b} & b \leq x \leq c, \\ 0 & \text{otherwise.} \end{cases}$$

Its parametric form is

$$\underline{u}(r) = (b - a)r + a, \quad \bar{u}(r) = c - (c - b)r.$$

Parametric form of an arbitrary fuzzy number is given in [20] as follows. A fuzzy number  $u$  in parametric form is a pair  $(\underline{u}, \bar{u})$  of functions  $\underline{u}(r), \bar{u}(r), 0 \leq r \leq 1$ , which satisfy the following requirements:

1.  $\underline{u}(r)$  is a bounded left continuous non-decreasing function over  $[0, 1]$ ,
2.  $\bar{u}(r)$  is a bounded left continuous non-increasing function over  $[0, 1]$ ,
3.  $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$ .

The set of all these fuzzy numbers is denoted by  $E$  which is a complete metric space with Hausdorff distance. A crisp number  $\alpha$  is simply represented by  $\underline{u}(r) = \bar{u}(r) = \alpha, 0 \leq r \leq 1$ .

For arbitrary fuzzy numbers  $u = (\underline{u}(r), \bar{u}(r)), v = (\underline{v}(r), \bar{v}(r))$  and real number  $k$ , we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [20]

- (a)  $u = v$  if and only if  $\underline{u}(r) = \underline{v}(r)$  and  $\bar{u}(r) = \bar{v}(r)$ ,
- (b)  $u + v = (\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r))$ ,
- (c)  $ku = \begin{cases} (k\underline{u}, k\bar{u}), & k \geq 0, \\ (k\bar{u}, k\underline{u}), & k < 0. \end{cases}$

The collection of all the fuzzy numbers with addition and scalar multiplication as defined by above equations is denoted by  $E^1$  and is a convex cone.

**Definition 1.** We introduce a lattice  $L$  as  $L = \{h|h : [0, 1] \rightarrow [0, +\infty)\}$  is non-decreasing and left continuous }.

The order in  $L$  is the natural order defined by  $h \leq g$  if and only if  $h(r) \leq g(r)$  for all  $r \in [0, 1]$ . It is easy to show that

$$\begin{aligned} [h \vee g](r) &= \max\{h(r), g(r)\}, \\ [h \wedge g](r) &= \min\{h(r), g(r)\}. \end{aligned} \tag{1}$$

**Definition 2.** [20] For arbitrary fuzzy number  $u = (\underline{u}, \bar{u})$ , the number  $u_0 = \frac{1}{2}(\underline{u}(1) + \bar{u}(1))$  is said to be a location index number of  $u$ , and two non-decreasing left continuous functions

$$u_* = u_0 - \underline{u}, \quad (2)$$

$$u^* = \bar{u} - u_0,$$

are called the left fuzziness index function and the right fuzziness index function, respectively.

According to definition 2, every fuzzy number can be represented by  $(u_0, u_*, u^*)$ . It is obvious that a fuzzy number  $u$  is symmetric if and only if  $u_* = u^*$ .

**Definition 3.** [20] For arbitrary fuzzy number  $u = (u_0, u_*, u^*)$ , and  $v = (v_0, v_*, v^*)$  the four arithmetic operations are defined by

$$u \odot v = (u_0 \odot v_0, u_* \vee v_*, u^* \vee v^*),$$

where  $u \odot v$  is either of  $u + v, u - v, u.v$  and  $u/v$ .

**Theorem 1.** For all fuzzy numbers  $u, v$  and  $w$ , we have

1.  $u + v = v + u$ ,
2.  $(u + v) + w = u + (v + w)$ ,
3.  $uv = vu$ ,
4.  $(uv)w = u(vw)$ ,
5. There exists  $1 \in E$  such that  $u.1 = u$  for all  $u \in E$ ,
6.  $u(v + w) = uv + uw$ .

**Proof.** It is trivial by definition 3. □

**Definition 4.** A matrix  $A = (a_{ij})$  is called a fuzzy matrix, if each element of  $A$  is a fuzzy number, we represent  $A = (a_{ij})$  that  $a_{ij} = ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*)$  where  $(a_{ij})_0$  is location index of  $a_{ij}$  and  $(a_{ij})_*, (a_{ij})^*$  are left fuzziness index and right fuzziness index.

**Definition 5.** A vector  $b = (b_i)$  is called a fuzzy vector, if each element of  $b$  is a fuzzy number, with new notation  $b = ((b_i)_0, (b_i)_*, (b_i)^*)$ , where  $(b_i)_0$  is location index of  $b_i$ ,  $(b_i)_*$  and  $(b_i)^*$  are left fuzziness index and right fuzziness index.

### 3 The solution of $Ax + b = Cx + d$ by a new fuzzy arithmetic

Consider the  $n \times n$  general dual full fuzzy linear system of equations:

$$\begin{cases} a_{11}\underline{x}_1 + a_{12}\underline{x}_2 + \dots + a_{1n}\underline{x}_n + b_1 = c_{11}\underline{x}_1 + c_{12}\underline{x}_2 + \dots + c_{1n}\underline{x}_n + d_1, \\ a_{21}\underline{x}_1 + a_{22}\underline{x}_2 + \dots + a_{2n}\underline{x}_n + b_2 = c_{21}\underline{x}_1 + c_{22}\underline{x}_2 + \dots + c_{2n}\underline{x}_n + d_2, \\ \vdots \\ a_{n1}\underline{x}_1 + a_{n2}\underline{x}_2 + \dots + a_{nn}\underline{x}_n + b_n = c_{n1}\underline{x}_1 + c_{n2}\underline{x}_n + \dots + c_{nn}\underline{x}_n + d_n, \end{cases} \tag{3}$$

the matrix form of above equation is

$$Ax + b = Cx + d,$$

where the coefficient matrices  $A = (a_{ij}) = ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*)$  and  $C = (c_{ij}) = ((c_{ij})_0, (c_{ij})_*, (c_{ij})^*)$ ,  $1 \leq i, j \leq n$  are  $n \times n$  fuzzy matrices,  $b = (b_i) = ((b_i)_0, (b_i)_*, (b_i)^*)$ ,  $d = (d_i) = ((d_i)_0, (d_i)_*, (d_i)^*)$  and  $x = (x_i) = ((x_i)_0, (x_i)_*, (x_i)^*)$ ,  $1 \leq i \leq n$  are fuzzy vectors.

Let  $x_i$  be a solution of (3), that is

$$\sum_{j=1}^n a_{ij}x_j + b_i = \sum_{j=1}^n c_{ij}x_j + d_i, \quad j = 1, 2, \dots, n, \tag{4}$$

therefore, we have with definition 2 and Eq. (4)

$$\sum_{j=1}^n ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*) \cdot ((x_j)_0, (x_j)_*, (x_j)^*) + ((b_i)_0, (b_i)_*, (b_i)^*) = \tag{5}$$

$$\sum_{j=1}^n ((c_{ij})_0, (c_{ij})_*, (c_{ij})^*) \cdot ((x_j)_0, (x_j)_*, (x_j)^*) + ((d_i)_0, (d_i)_*, (d_i)^*), \quad i = 1, 2, \dots, n,$$

this implies for  $i = 1, 2, \dots, n$

$$\sum_{j=1}^n ((a_{ij})_0(x_j)_0, \max\{(a_{ij})_*, (x_j)_*\}, \max\{(a_{ij})^*, (x_j)^*\}) + ((b_i)_0, (b_i)_*, (b_i)^*) = \tag{6}$$

$$\sum_{j=1}^n ((c_{ij})_0(x_j)_0, \max\{(c_{ij})_*, (x_j)_*\}, \max\{(c_{ij})^*, (x_j)^*\}) + ((d_i)_0, (d_i)_*, (d_i)^*),$$

let

$$\max\{(a_{ij})_*, (x_j)_*\} = (m_{ij})_*, \quad (7)$$

$$\max\{(a_{ij})^*, (x_j)^*\} = (m_{ij})^*,$$

$$\max\{(c_{ij})_*, (x_j)_*\} = (n_{ij})_*,$$

$$\max\{(c_{ij})^*, (x_j)^*\} = (n_{ij})^*,$$

then, we have for  $i = 1, 2, \dots, n$ :

$$\sum_{j=1}^n ((a_{ij})_0(x_j)_0 + (b_i)_0, \max\{(m_{ij})_*, (b_i)_*\}, \max\{(m_{ij})^*, (b_i)^*\}) = \quad (8)$$

$$\sum_{j=1}^n ((c_{ij})_0(x_j)_0 + (d_i)_0, \max\{(n_{ij})_*, (d_i)_*\}, \max\{(n_{ij})^*, (d_i)^*\}).$$

**Theorem 2.** If  $(x_j)_0$  for  $j = 1, 2, \dots, n$  are the solution of the crisp linear system

$$\sum_{j=1}^n (a_{ij})_0(x_j)_0 + (b_i)_0 = \sum_{j=1}^n (c_{ij})_0(x_j)_0 + (d_i)_0,$$

$$i = 1, 2, \dots, n,$$

and  $(x_j)_*$ ,  $(x_j)^*$  for  $j = 1, 2, \dots, n$  are obtained by

$$(x_j)_* = \max_{1 \leq i \leq n} \{(a_{ij})_*, (b_i)_*, (c_{ij})_*, (d_i)_*\}, \quad (9)$$

$$(x_j)^* = \max_{1 \leq i \leq n} \{(a_{ij})^*, (b_i)^*, (c_{ij})^*, (d_i)^*\}. \quad (10)$$

Then the fuzzy vector  $x = (x_j) = (\underline{x}_j, \bar{x}_j)$  obtained by

$$\underline{x}_j = (x_j)_0 + (x_j)_*,$$

$$\bar{x}_j = (x_j)_0 + (x_j)^*,$$

for  $j = 1, 2, \dots, n$  is a solution of (3).

**Proof.** The attention of the results, it is clear.  $\square$

## 4 Numerical examples and applications

**Example 1.** Consider the  $2 \times 2$  fully fuzzy linear system

$$\begin{cases} (1, 2, 3)x_1 + (4, 6, 9)x_2 + (1, 3, 4) = (0, 1, 3)x_1 + (5, 6, 8)x_2 + (0, 1, 7), \\ (5, 6, 8)x_1 + (3, 5, 6)x_2 + (0, 7, 8) = (1, 4, 5)x_1 + (2, 3, 4)x_2 + (8, 9, 12). \end{cases}$$

By simple calculations of the new arithmetic, we have following system for finding location index number of  $x_1$  and  $x_2$ :

$$2(x_1)_0 + 6(x_2)_0 + 3 = (x_1)_0 + 6(x_2)_0 + 1,$$

$$6(x_1)_0 + 5(x_2)_0 + 7 = 4(x_1)_0 + 3(x_2)_0 + 9,$$

therefore, we have

$$(x_1)_0 = -2, \quad (x_2)_0 = 3.$$

We obtain left fuzziness index and right fuzziness index function  $x_1$  and  $x_2$  by Eq.(9) and Eq. (10):

$$(x_1)_* = 1 - r, \quad (x_2)_* = 2 - 2r,$$

$$(x_1)^* = 1 - r, \quad (x_2)^* = 1 - r.$$

The parametric form of  $x_1$  and  $x_2$  are the following form:

$$\underline{x}_1(r) = r - 3, \quad \underline{x}_2(r) = 2r + 1,$$

$$\overline{x}_1(r) = -1 - r, \quad \overline{x}_2(r) = 4 - r.$$

**Example 2.** For production of a high quality chemical compound, we need about 0.4 ((0.3,0.4,0.5)) kg poly ethylene high density (PEHD) and about 0.3 ((0.1,0.3,0.45)) kg poly ethylene low density (PELD) and from poly propylen (PP), we need exactly 0.267 kg which its price is about 3 ((1.217,3,4.775)) dollar. Now from same chemical compound with lower quality with same cost so the products would have higher expansion. We need from the PEHD about 0.5 ((0.3,0.5,0.8)) kg and from PELD about 0.4 ((0.2,0.4,0.5)) kg and from PP exactly 0.1 kg which its price is about 3 ((1,3,3.5)) dollar and for production of second high quality chemical compound, we need about 0.2 ((0.15,0.2,0.3)) kg PEHD and about 0.7 ((0.6,0.7,0.95)) kg PELD and from poly estyrene (PE), we need exactly 0.1 kg which its price is about 5 ((2.625,5,6.75)) dollar. Now from same chemical compound with lower quality with same cost so the products would have higher expansion. We need from the PEHD about 0.3 ((0.2,0.3,0.5)) kg and from PELD about 0.3 ((0.15,0.3,0.4)) kg and from PE

exactly 0.3 kg which its price is about 5 ((4,5,7)) dollar. For obtaining this two chemical compound with different qualities how much would be about the price of PEHD and PELD ?

Let  $x_1$  and  $x_2$  show the price of PEHD and PELD. Together, these equations form a fully fuzzy linear system

$$\begin{cases} (0.3, 0.4, 0.5)x_1 + (0.1, 0.3, 0.45)x_2 + (0.325, 0.8, 1.275) = \\ \quad (0.3, 0.5, 0.8)x_1 + (0.2, 0.4, 0.5)x_2 + (0.1, 0.3, 0.35), \\ (0.15, 0.2, 0.3)x_1 + (0.6, 0.7, 0.95)x_2 + (0.2625, 0.5, 0.675) = \\ \quad (0.2, 0.3, 0.5)x_1 + (0.15, 0.3, 0.4)x_2 + (1.2, 1.5, 2.1). \end{cases}$$

By simple calculations of the new arithmetic, we have following system for finding location index number of  $x_1$  and  $x_2$ :

$$0.4(x_1)_0 + 0.3(x_2)_0 + 0.8 = 0.5(x_1)_0 + 0.4(x_2)_0 + 0.3,$$

$$0.2(x_1)_0 + 0.7(x_2)_0 + 0.5 = 0.3(x_1)_0 + 0.3(x_2)_0 + 1.5,$$

therefore, we have

$$(x_1)_0 = 2, \quad (x_2)_0 = 3.$$

We obtain left fuzziness index and right fuzziness index function  $x_1$  and  $x_2$  by Eq.(9) and Eq. (10):

$$(x_1)_* = 0.5 - 0.5r, \quad (x_2)_* = 0.75 - 0.75r,$$

$$(x_1)^* = 0.5 - 0.5r, \quad (x_2)^* = 0.5 - 0.5r.$$

The parametric form of  $x_1$  and  $x_2$  are the following form:

$$\underline{x}_1(r) = 1.5 + 0.5r, \quad \underline{x}_2(r) = 2.25 + 0.75r,$$

$$\bar{x}_1(r) = 2.5 - 0.5r, \quad \bar{x}_2(r) = 3.5 - 0.5r.$$

## 5 Summary and conclusions

In this paper, we propose a general model for solving a system of  $n$  fuzzy linear equations with  $n$  variables. The original system with coefficient matrix  $A$  and  $C$  are fuzzy matrices,  $b$  and  $d$  are fuzzy vectors. For finding the solution of general dual fully fuzzy linear system, we used of new arithmetic. Also, a condition for the existence of a fuzzy solution to the general dual fully fuzzy linear system, is presented. Initially researchers in the [1, 2, 4, 6, 16, 17] assumed



the coefficient matrix was crisp number and in the [8] researchers assumed the solution is positive fuzzy vector. This is too restrictive for applications, but in this paper we don't have these restrictions.

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