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The free boundary problem describing information diffusion in online social networks ☆

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ABSTRACT

In this paper we consider a free boundary problem for a reaction–diffusion logistic equation with a time-dependent growth rate. Such a problem arises in the modeling of information diffusion in online social networks, with the free boundary representing the spreading front of news among users. We present several sharp thresholds for information diffusion that either lasts forever or suspends in finite time. In the former case, we give the asymptotic spreading speed which is determined by a corresponding elliptic equation.

1. Introduction

Online social networks have recently become important media for spreading information and facilitating the building of social relations among a huge number of people. Research efforts on understanding information diffusion have a significant impact on real life applications such as product marketing, political online campaign, etc. Extensive investigations have been made to understand network structure, user interactions, and traffic properties [1,9,13,14,17] and to study the characteristics

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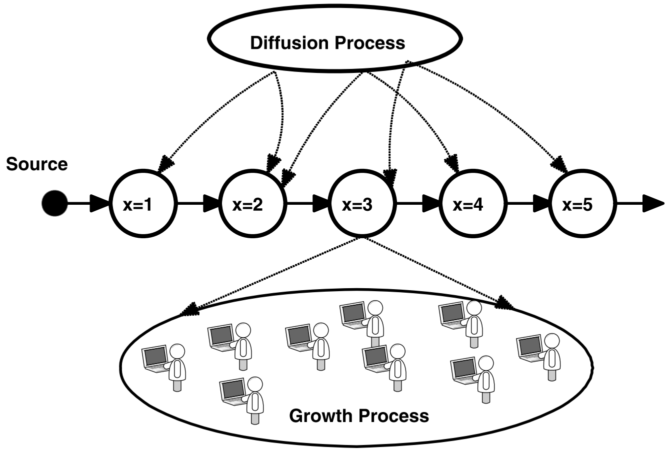

Fig. 1. Conceptual illustration of the information spreading in online social networks (adapted from [19]).

of information diffusion [4,10,18,21]. Mathematical modeling has played an increasingly important role in understanding information diffusion in online social networks (see [8,20] and references therein).

Most existing models of information diffusion on online social networks have concentrated on the temporal dimension. Recently a Diffusive Logistic (DL) model was proposed in [19] to study the process of information diffusion process in online social networks over time and position. To describe distance in online social networks, they defined a natural metric called *friendship hops*. The metric is measured by the number of hops from one user to another in social network graphs, and is different from the physical distances between users. From the definition of the distance metric, users in an online social network can be classified into groups according to their distance x from the source s. Let U_x represent the users with distance x to the source. The total user population is $U = \{U_1, U_2, \ldots, U_x, \ldots, U_m\}$, where m is the maximum distance from the source s to the users and its value varies according to the definition of the distance (see Fig. 1).

Let $u(t,x)$ denote the density of influenced users at time t and distance x. The density of influenced users $u(t,x)$ depends on two major factors. First, the users in U_y $(y \neq x)$ can influence those in U_x through direct or indirect friendship links that are usually bidirectional or reciprocal in a manner of random walk. Secondly, the users in U_x could influence each other. Then information diffusion process in online social networks can be divided into the following two parts: growth part and diffusion part. As widely used in spatial biology and epidemiology, the diffusion process is modeled by the Laplacian and the growth part by a logistic equation [12]. They proposed in [19] a model described by the following partial differential equation:

$$\begin{cases} u_t - du_{xx} = r(t)u\left(1 - \dfrac{u}{K}\right), & t > 1,\ l < x < L, \\ u(1,x) = u_0(x), & l \leqslant x \leqslant L, \\ u_x(t,l) = 0, \quad u_x(t,L) = 0, & t > 1, \end{cases} \tag{1.1}$$

where r, K and d indicate the intrinsic growth rate, the carrying capacity, and the diffusion rate, respectively. L and l represent the upper and lower bounds of the distances between the source s and other social networks users. In [19] the diffusive logistic model in online social networks was validated against a real dataset collected from a popular social news site, http://www.digg.com/. The experiment results show that the DL model is indeed able to characterize and predict the process of information propagation in online social networks. For example, let $K = 25$, $d = 0.01$ and

$$r(t) = 1.4e^{-1.5(t-1)} + 0.25,$$

for the most popular news with 24,099 votes in Digg, the average prediction accuracy of the DL model over all distances during the first 6 hours is 92.08%.

In online social networks, most of the users are clustered in a few number of friendship hops and the growth process plays a major rule. The intrinsic growth rate $r(t)$ is often dependent on time t. More specifically, $r(t)$ is a decreasing function of t, reflecting the fact that users gradually lose their interest to news.

In above system, l and L are fixed, which means the distance between the source s and the users in the social networks is constant over time. But in reality, the distance changes as time proceeds. To describe such scenario, recently, Du and Lin [6] proposed the following free boundary model to describe the spreading of a new or invasive species:

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0, \quad u\big(t,h(t)\big) = 0, & t > 0, \\ h'(t) = -\mu u_x\big(t,h(t)\big), & t > 0, \\ h(0) = h_0, \quad u(0,x) = u_0(x), & 0 \leqslant x \leqslant h_0, \end{cases} \tag{1.2}$$

where a and b are constants. They showed that problem (1.2) has a unique solution $(u(t,x), h(t))$ defined for all $t > 0$, with $u(t,x) > 0$ and $h'(t) > 0$. They gave a spreading–vanishing dichotomy of the model. In addition, they proved expanding front that moves at a constant speed for enough long time if spreading occurs.

Motivated by [6], in this paper we investigate the DL model with a free boundary for online social networks. The system is given in the following form:

$$\begin{cases} u_t - du_{xx} = r(t)u\left(1 - \dfrac{u}{K}\right), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0, \quad u\big(t,h(t)\big) = 0, & t > 0, \\ h'(t) = -\mu u_x\big(t,h(t)\big), & t > 0, \\ h(0) = h_0, \quad u(0,x) = u_0(x), & 0 \leqslant x \leqslant h_0. \end{cases} \tag{1.3}$$

The initial function $u_0(x)$ belongs to $\Sigma(h_0)$ for some $h_0 > 0$, where

$$\Sigma(h_0) = \big\{\varphi \in C^2\big([0,h_0]\big):\ \varphi'(0) = \varphi(h_0) = 0,\ \varphi(x) > 0 \text{ in } [0,h_0)\big\}.$$

$u(t,x)$ represents the density of influenced users with distance x at time t. $x = h(t)$ is the moving boundary to be determined and represents the spreading front of news (such as movie recommendation) among users. $u_x(t,0) = 0$ means no news traveling in the left part. Hence we only need consider the diffusion in the right part. K is the carrying capacity, d is the diffusion rate, and $r(t)$ is intrinsic growth rate.

As we discussed above, in the following, we always assume that

(A) $r(t)$ is a decreasing function of time t with a positive lower bound, i.e., $0 < r_\infty \leqslant r(t) \leqslant r(0)$.

In general, we call $h'(t) = -\mu u_x(t,h(t))$ Stefan condition, where μ represents the diffusion ability of the information in the new area. As we all known, Stefan condition has been used in many areas. For example, it was used to model the wound healing [3], the melting of ice [16], the spreading of species [5,6,11].

The paper is structured as follows. In Section 2, we first show that the solution of (1.3) is global and unique, and the free boundary $x = h(t)$ is increasing. Then we present the comparison principle. Finally, we use the comparison principle to give the upper bound of the solution.

In Section 3, we show that the information traveling either lasts forever or suspends in finite time, that is, if $h_0 > \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, then $h_\infty = \infty$ and $\lim_{t\to+\infty} u(t,x) = K$ in any bounded domain, which means the information diffusion lasts forever. On the other hand, if $h_\infty < \infty$, then $h_\infty \leqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$ and $\lim_{t\to+\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$ uniformly. In the other words, the information vanishes in finite time.

In Section 4, by constructing an upper solution we prove that if λ is sufficiently small, the information vanishing must occur. Then we show that there exists a threshold λ^* which is dependent on $\varphi \in \Sigma(h_0)$ such that when $\lambda > \lambda^*$, the information with the initial data $u_0 = \lambda\varphi$ travels in the whole distance. Otherwise, the information vanishing happens.

In Section 5, we demonstrate that if the information spreading happens, the expanding front $x = h(t)$ moves at a constant speed k_0, which is determined by an elliptic equation derived from the free boundary problem (1.3). Finally, we show that spreading speed k_0 is continuously dependent on r_∞, K, μ, d, and satisfies

$$\lim_{\frac{\mu K}{d}\to+\infty} \frac{k_0}{\sqrt{r_\infty d}} = 2, \qquad \lim_{\frac{\mu K}{d}\to 0} \frac{k_0}{\sqrt{r_\infty d}}\frac{d}{\mu K} = \frac{1}{\sqrt{3}}.$$

The paper ends with a brief discussion.

2. Existence and uniqueness

We first present the local existence and uniqueness of the solution by the contraction mapping theorem. The global existence is then given by a priori estimate of the solution.

Theorem 2.1. *Let $D = \{(t,x) \in \mathbb{R}^2\colon\ t \in [0,\infty),\ x \in [0,h(t)]\}$. For any $\alpha \in (0,1)$, problem (1.3) admits a unique solution*

$$(u,h) \in C^{(1+\alpha)/2,1+\alpha}(D) \times C^{1+\alpha/2}\big([0,\infty)\big).$$

Furthermore,

$$0 < u(t,x) \leqslant M, \qquad 0 < h'(t) \leqslant C$$

for all $t > 0$, $0 < x < h(t)$, where C and M are positive constants.

Proof. This theorem can be proved by using the same methods in [6] (see Section 2). For brevity, the detailed proof is omitted. □

Lemma 2.2 *(Comparison principle).*

(i) *Suppose that $\bar u \in C(\overline{D^*_{T_0}}) \cap C^{1,2}(D^*_{T_0})$, $\bar h \in C^1([T_0,T])$ with $D^*_{T_0} = \{(t,x) \in \mathbb{R}^2\colon\ 0 \leqslant T_0 < t \leqslant T,\ 0 < x < \bar h(t)\}$, and*

$$\begin{cases} \bar u_t - d\bar u_{xx} \geqslant r(t)\bar u\left(1 - \dfrac{\bar u}{K}\right), & T_0 < t \leqslant T,\ 0 < x < \bar h(t), \\ \bar u = 0, \quad \bar h'(t) \geqslant -\mu \bar u_x, & T_0 < t \leqslant T,\ x = \bar h(t), \\ \bar u_x(t,0) \leqslant 0, & T_0 < t \leqslant T, \\ \bar h(T_0) \geqslant h(T_0), \quad \bar u(T_0,x) \geqslant u(T_0,x), & 0 \leqslant x \leqslant h(T_0). \end{cases} \tag{2.1}$$

Then the solution (u,h) *of the free boundary problem* (1.3) *satisfies*

$$h(t) \leqslant \bar{h}(t) \quad \textit{and} \quad u(t,x) \leqslant \bar{u}(t,x)$$

for $(t,x) \in [T_0,T] \times [0,h(t)]$.

(ii) *Assuming that* $\underline{u} \in C(\overline{D^*_{T_0}}) \cap C^{1,2}(D^*_{T_0})$, $\underline{h}(t) \in C^1([T_0,T])$ *with* $D^*_{T_0} = \{(t,x) \in \mathbb{R}^2\colon\ 0 \leqslant T_0 < t \leqslant T,\ 0 < x < \underline{h}(t)\}$, *and*

$$\begin{cases} \underline{u}_t - d\underline{u}_{xx} \leqslant r(t)\underline{u}\left(1 - \dfrac{\underline{u}}{K}\right), & T_0 < t \leqslant T,\ 0 < x < \underline{h}(t), \\ \underline{u} = 0, \quad \underline{h}'(t) \leqslant -\mu \underline{u}_x, & T_0 < t \leqslant T,\ x = \underline{h}(t), \\ \underline{u}_x(t,0) \geqslant 0, & T_0 < t \leqslant T, \\ \underline{h}(T_0) \leqslant h(T_0), \quad \underline{u}(T_0,x) \leqslant u(T_0,x), & 0 \leqslant x \leqslant \underline{h}(T_0). \end{cases} \tag{2.2}$$

Then the solution (u,h) *of the free boundary problem* (1.3) *satisfies*

$$h(t) \geqslant \underline{h}(t) \quad \textit{and} \quad u(t,x) \geqslant \underline{u}(t,x)$$

for $(t,x) \in [T_0,T] \times [0,\underline{h}(t)]$.

It follows from Lemma 2.2 that

Corollary 2.3. $u \leqslant \bar{u}$, *where*

$$\bar{u} = K e^{\int_0^t r(\tau)\,d\tau} \left(\frac{K}{\|u_0\|_\infty} - 1 + e^{\int_0^t r(\tau)\,d\tau} \right)^{-1}$$

is the solution of

$$\begin{cases} \dfrac{d\bar{u}}{dt} = r(t)\bar{u}\left(1 - \dfrac{\bar{u}}{K}\right), & t > 0, \\ \bar{u}(0) = \|u_0\|_\infty. \end{cases} \tag{2.3}$$

3. Spreading and vanishing

It follows from Theorem 2.1 that $x = h(t)$ is monotonically increasing. We then have that $\lim_{t\to+\infty} h(t) := h_\infty \in (0,+\infty]$.

Definition 3.1. The information is vanishing if

$$h_\infty < \infty \quad \text{and} \quad \lim_{t\to+\infty} \left\| u(t,\cdot) \right\|_{C([0,h(t)])} = 0,$$

while the information is spreading if

$$h_\infty = \infty \quad \text{and} \quad \lim_{t\to+\infty} u(t,x) = K$$

uniformly for x in any bounded set of $[0,\infty)$.

Lemma 3.2. *Suppose that* (A) *hold.*

(i) *If* $h_\infty < \infty$, *then* $h_\infty \leqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$.
(ii) *If* $h_\infty < \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, *then* $\lim_{t\to+\infty}\|u(t,\cdot)\|_{C([0,h(t)])} = 0$.

Proof. (i) We prove that $h_\infty \leqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$ by contradiction. If $h_\infty > \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, namely there exists T such that $l := h(T) > \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$. That is, $r_\infty > \lambda_1 := d(\frac{\pi}{2l})^2$, where λ_1 denotes the first eigenvalue of the problem

$$\begin{cases} -d\psi'' = \lambda\psi & \text{in } (-l,l), \\ \psi(-l) = \psi(l) = 0, \end{cases} \tag{3.1}$$

the correspondence eigenfunction $\psi(x) = \cos(\frac{\pi}{2l}x)$. For any small $\varepsilon > 0$, it follows from the above results and the continuity that $\lambda_1^\varepsilon < r_\infty$, where λ_1^ε is the first eigenvalue of the problem

$$\begin{cases} -d\psi'' - \varepsilon\psi' = \lambda\psi & \text{in } (-l,l), \\ \psi(-l) = \psi(l) = 0. \end{cases} \tag{3.2}$$

For any given small $\varepsilon > 0$, we consider the problem

$$\begin{cases} -dw'' - \varepsilon w' = r_\infty w\left(1 - \dfrac{w}{K}\right) & \text{in } (-l,l), \\ w(\pm l) = 0. \end{cases} \tag{3.3}$$

For this logistic problem, using Proposition 3.3 in [2] yields that problem (3.3) admits a unique positive solution $w = w_\varepsilon$. Obviously, $w(x)$ is symmetric about $x = 0$, and then it is easy to get $w'(x) < 0$ for $x \in (0,l]$. Furthermore, $w < K$ for $x \in [-l,l]$ by using the comparison principle. Motivated by [6], we define $v(t,x) = w(\frac{l}{h(t)}x)$. Direct calculations yield that

$$v_t = -\frac{lx}{h^2(t)}h'(t)w', \qquad v_{xx} = \frac{l^2}{h^2(t)}w'',$$
$$v_t - dv_{xx} = \frac{l^2}{h^2(t)}\left[-\frac{x}{l}h'(t)w' - dw''\right].$$

From Theorem 2.1, $h'(t) \to 0$ as $t \to +\infty$. Hence we can find a $T' > T$ such that $h'(t) < \varepsilon\frac{l}{h_\infty}$ for $t \geqslant T'$. That is, $\frac{h_\infty}{l}h'(t) < \varepsilon$. We then have

$$v_t - dv_{xx} \leqslant \frac{l^2}{h^2(t)}\left[-\varepsilon w' - dw''\right] = \frac{l^2}{h^2(t)}r_\infty w\left(1 - \frac{w}{K}\right)$$
$$\leqslant r_\infty w\left(1 - \frac{w}{K}\right) = r_\infty v\left(1 - \frac{v}{K}\right).$$

We choose a sufficiently small δ such that $\underline{u}(T',x)=\delta v(T',x)\leqslant u(T',x)$, and then have

$$\begin{cases} \underline{u}_t - d\underline{u}_{xx} \leqslant r_\infty \underline{u}\left(1-\dfrac{\underline{u}}{K}\right), & t>T',\ 0<x<h(t), \\ \underline{u}_x(t,0)=0, \quad \underline{u}\big(t,h(t)\big)=0, & t\geqslant T', \\ \underline{u}\big(T',x\big)\leqslant u\big(T',x\big), & 0\leqslant x\leqslant h\big(T'\big). \end{cases} \tag{3.4}$$

According to Lemma 2.2, we obtain $\underline{u}(t,x)\leqslant u(t,x)$ for $t\geqslant T'$, $0\leqslant x\leqslant h(t)$. Using the comparison principle, we have

$$u_x\big(t,h(t)\big)\leqslant \underline{u}_x\big(t,h(t)\big)=\delta\frac{l}{h(t)}w'(l)\to \delta\frac{l}{h_\infty}w'(l)<0.$$

But the free boundary implies that

$$u_x\big(t,h(t)\big)=-\frac{1}{\mu}h'(t)\to 0 \quad \text{as } t\to+\infty.$$

Thus the proof is completed.

The proof of (ii) can be performed by constructing a vanishing upper solution. We omit the proof here, but a strong conclusion (Theorem 3.4) is given instead. □

Corollary 3.3. *If $h_0\geqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, then $h_\infty=\infty$.*

Theorem 3.4. *If $h_\infty<\infty$, then $\lim_{t\to+\infty}\|u(t,\cdot)\|_{C([0,h(t)])}=0$.*

Proof. Suppose $\limsup_{t\to+\infty}\|u(t,\cdot)\|_{C([0,h(t)])}=\varepsilon>0$ by contradiction. We first show that for any $0<\alpha<1$

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times[0,h(t)])}+\|h\|_{C^{1+\alpha/2}([0,\infty))}\leqslant C, \tag{3.5}$$

where C depends on h_0, α, $\|u_0\|_{C^2([0,h_0])}$ and h_∞.

In fact, we consider a transformation $y=\frac{h_0x}{h(t)}$, which straightens the free boundary $x=h(t)$ to the line $y=h_0$.

Let $u(t,x)=v(t,y)$, and direct calculations show that

$$u_t=v_t+v_y\frac{\partial y}{\partial t}=v_t-\frac{h'(t)}{h(t)}yv_y, \qquad u_{xx}=\frac{h_0^2}{h^2(t)}v_{yy}.$$

Hence the free boundary problem (1.3) becomes

$$\begin{cases} v_t-\dfrac{h'(t)y}{h(t)}v_y-d\dfrac{h_0^2}{h^2(t)}v_{yy}=r(t)v\left(1-\dfrac{v}{K}\right), & t>0,\ 0<y<h_0, \\ v_y(t,0)=v(t,h_0)=0, & t>0, \\ v(0,y)=u_0(y), & 0\leqslant y\leqslant h_0. \end{cases} \tag{3.6}$$

It follows from Theorem 2.1 that

$$\left\| r(t)v\left(1-\frac{v}{K}\right)\right\|_{L^\infty}\leqslant C_1,\qquad \left\|\frac{h'(t)y}{h(t)}\right\|_{L^\infty}\leqslant C_2,\qquad \left\|\frac{h_0^2}{h^2(t)}\right\|_{L^\infty}\leqslant C_3,$$

where C_1, C_2, C_3 are constants.

Using the L^p estimates for parabolic equation and Sobolev imbedding theorem gives that

$$\|v\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times[0,h_0])}\leqslant C_4,$$

where C_4 is a constant depending on α, h_0, C_1, C_2, C_3, and $\|u_0\|_{C^2([0,h_0])}$. We immediately obtain (3.5).

It follows from the assumption $\limsup_{t\to+\infty}\|u(t,\cdot)\|_{C([0,h(t)])}=\varepsilon>0$ that there exists a sequence (t_k,x_k) in $(0,+\infty)\times[0,h(t))$, such that $u(t_k,x_k)\geqslant\varepsilon/2$ for $k\in N$, and $t_k\to+\infty$ as $k\to+\infty$. Since x_k is bounded, there exists a subsequence $\{x_{k_n}\}$ such that $x_{k_n}\to x_0\in[0,h_\infty)$ as $n\to+\infty$.

Define $u_n(t,x)=u(t_{k_n}+t,x)$ for $t\in(-t_{k_n},\infty)$, $x\in[0,h(t_{k_n}+t))$. It follows from the parabolic regularity that $\{u_n\}$ has a subsequence $\{u_{n_i}\}$ satisfying $u_{n_i}\to\widetilde{u}$ as $i\to+\infty$, where $\widetilde{u}$ satisfies the following problem

$$\widetilde{u}_t-d\widetilde{u}_{xx}=r_\infty\widetilde{u}\left(1-\frac{\widetilde{u}}{K}\right),\quad t\in\mathbb{R},\ 0<x<h_\infty.$$

Since that $\widetilde{u}(0,x_0)\geqslant\varepsilon/2$, we then have $\widetilde{u}>0$ in $(-\infty,\infty)\times[0,h_\infty)$. Note that $r_\infty(1-\frac{\widetilde{u}}{K})$ is bounded by $N:=\|r_\infty(1-\frac{\widetilde{u}}{K})\|_{L^\infty}$. Using the Hopf lemma to the equation $\widetilde{u}_t-d\widetilde{u}_{xx}\geqslant-N\widetilde{u}$ at the point $(0,h_\infty)$ yields that $\widetilde{u}_x(0,h_\infty)\leqslant-\sigma_0$, where σ_0 is a positive constant.

On the other hand, since $\|h\|_{C^{1+\alpha/2}([0,\infty))}\leqslant C$ and $h'(t)>0$, we have $h'(t)\to0$ as $t\to+\infty$. Therefore,

$$u_x\big(t_{k_n},h(t_{k_n})\big)=-\frac{1}{\mu}h'(t_{k_n})\to0,\quad n\to\infty.$$

But the fact $\|u\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times[0,h(t)])}\leqslant C$ implies

$$u_x\big(t_{k_n},h(t_{k_n})\big)=(u_n)_x\big(0,h(t_{k_n})\big)\to\widetilde{u}_x(0,h_\infty)\leqslant-\sigma_0,\quad n\to\infty,$$

which leads to a contradiction. □

Lemma 3.5. *Suppose that* (*A*) *hold. If* $h_\infty=\infty$, *then* $\lim_{t\to+\infty}u(t,x)=K$ *uniformly for* x *in any bounded set of* $[0,\infty)$.

Proof. It follows from Corollary 2.3 that $u(t,x)\leqslant\bar{u}(t)$ for $t>0$, $0\leqslant x\leqslant h(t)$, where

$$\bar{u}=Ke^{\int_0^{h_0}r(\tau)\,d\tau}\left(\frac{K}{\|u_0\|_\infty}-1+e^{\int_0^{h_0}r(\tau)\,d\tau}\right)^{-1}$$

is the solution of the problem (2.3).

Clearly we obtain $\lim_{t\to+\infty}\bar{u}(t)=K$, and then $\limsup_{t\to+\infty}u(t,x)\leqslant K$ uniformly for $x\in[0,\infty)$. Since $h_\infty=\infty$, there exists $t_l>0$ such that $l:=h(t_l)>\frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$. Let $\underline{u}_l$ be the solution of the following problem

$$\begin{cases}(\underline{u}_l)_t-d(\underline{u}_l)_{xx}=r_\infty\underline{u}_l\left(1-\dfrac{\underline{u}_l}{K}\right), & t>t_l,\ 0<x<l,\\ (\underline{u}_l)_x(t,0)=0,\quad \underline{u}_l(t,l)=0, & t\geqslant t_l,\\ \underline{u}_l(t_l,x)=u(t_l,x), & 0\leqslant x\leqslant l.\end{cases}\tag{3.7}$$

Using Lemma 2.2 gives that $\underline{u}_l(t,x) \leqslant u(t,x)$ for $t \geqslant t_l$ and $0 \leqslant x \leqslant l$. Since $r_\infty > (\frac{\pi}{2l})^2 d$, it follows from a well-known result that $\underline{u}_l(t,x) \to \underline{u}_l^*(x)$ as $t \to +\infty$ uniformly in the compact subset of $[0,l)$, where $\underline{u}_l^*$ is the solution of the following problem

$$\begin{cases} -d(\underline{u}_l^*)_{xx} = r_\infty \underline{u}_l^* \left(1 - \dfrac{\underline{u}_l^*}{K}\right), & -l < x < l, \\ \underline{u}_l^*(-l) = \underline{u}_l^*(l) = 0. \end{cases} \tag{3.8}$$

Applying Lemma 2.2 of [7], we know $\underline{u}_l^*(x) \to K$ as $l \to +\infty$ uniformly in any compact subset of $[0,\infty)$. So $\liminf_{t\to+\infty} u(t,x) \geqslant K$ and then $\lim_{t\to+\infty} u(t,x) = K$ uniformly in any compact subset of $[0,\infty)$. □

4. Sharp threshold given by the initial value

In [6], a threshold value of μ was constructed for the species spreading or vanishing. But here we consider the effect of the initial users' size and let $u_0(x) = \lambda\varphi(x)$ with $\varphi \in \Sigma(h_0)$. It follows from Corollary 3.3 that if $h_0 \geqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, then $h_\infty = \infty$ for any $\lambda > 0$, the other cases will be given in the following lemma.

Lemma 4.1. *Suppose that* (A) *hold.*

(i) *Assume that* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{r(0)}}$. *If* λ *is sufficiently small, then* $h_\infty < \infty$ *and* $\lim_{t\to+\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$.

(ii) *Assume that* $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$. *If* λ *is big enough, then* $h_\infty = \infty$ *and* $\lim_{t\to+\infty} u(t,x) = K$ *uniformly in any compact subset of* $[0,\infty)$.

Proof. (i) Motivated by [15], we set

$$\xi(t) := h_0\left(1 + \delta - \frac{\delta}{2}e^{-\alpha t}\right), \quad t \geqslant 0,$$

$$v(y) := \cos\left(\frac{\pi}{2}y\right), \quad 0 \leqslant y \leqslant 1,$$

$$w(t,x) := \varepsilon e^{-\beta t} v\left(\frac{x}{\xi(t)}\right), \quad t \geqslant 0,\ 0 \leqslant x \leqslant \xi(t),$$

where $\varepsilon > 0$ is small such that $\pi\mu\varepsilon \leqslant \alpha\delta h_0^2$. Moreover, α, β and δ are positive constants.

Direct calculations yield that

$$w_t - dw_{xx} - r(t)w\left(1 - \frac{w}{K}\right) \geqslant \varepsilon e^{-\beta t} v\left[-\beta + \frac{d\pi^2}{4\xi^2(t)} - r(0)\right].$$

Since that $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{r(0)}}$, there exists a unique $\delta > 0$ such that

$$\left(\frac{\pi}{2}\right)^2 \frac{d}{(1+\delta)^2 h_0^2} - r(0) = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 \frac{d}{h_0^2} - r(0)\right].$$

Taking $\alpha = \beta = \frac{1}{4}(\frac{d\pi^2}{4h_0^2} - r(0))$, we then have

$$w_t - dw_{xx} - r(t)w\left(1 - \frac{w}{K}\right) \geqslant 0 \quad \text{for } t > 0,\ 0 < x < \xi(t).$$

In addition,

$$-\mu w_x\bigl(t, \xi(t)\bigr) = \varepsilon e^{-\beta t}\frac{\mu\pi}{2\xi(t)} \leqslant \varepsilon e^{-\beta t}\frac{\mu\pi}{2h_0} \leqslant \xi'(t) \quad \text{for } t > 0.$$

Moreover, when $\lambda \leqslant \frac{\varepsilon}{\|\varphi\|_{L^\infty}}\cos\frac{\pi}{2+\delta}$, we have

$$u_0(x) \leqslant \varepsilon\cos\frac{\pi}{2+\delta} \leqslant w(0, x) \quad \text{for } 0 < x < h_0.$$

Namely,

$$\begin{cases} w_t - dw_{xx} \geqslant r(t)w\left(1 - \dfrac{w}{K}\right), & t > 0,\ 0 < x < \xi(t), \\ w = 0, \quad \xi'(t) \geqslant -\mu w_x, & t > 0,\ x = \xi(t), \\ w_x(t, 0) = 0, & t > 0, \\ \xi(0) = h_0\left(1 + \dfrac{\delta}{2}\right), \quad u_0(x) \leqslant w(0, x), & 0 \leqslant x \leqslant h_0. \end{cases} \tag{4.1}$$

Using Lemma 2.2 gives that

$$h(t) \leqslant \xi(t) \quad \text{and} \quad u(t, x) \leqslant w(t, x) \quad \text{for } t > 0,\ 0 \leqslant x \leqslant h(t).$$

Thus

$$h_\infty \leqslant \lim_{t\to+\infty}\xi(t) = h_0(1 + \delta) < \infty,$$

which together with Theorem 3.4 shows that

$$\lim_{t\to+\infty}\bigl\|u(t, \cdot)\bigr\|_{C([0,h(t)])} = 0.$$

(ii) Similarly as in [6], let λ_1 be the first eigenvalue of this problem

$$\begin{cases} -d\varphi_1'' - \dfrac{1}{2}\varphi_1' = \lambda_1\varphi_1, \quad 0 < x < 1, \\ \varphi_1'(0) = \varphi_1(1) = 0 \end{cases} \tag{4.2}$$

and corresponding eigenfunction φ_1 can be chosen positive in $[0, 1)$ and $\|\varphi_1\|_{L^\infty} = 1$. Furthermore, it is easy to see that $\lambda_1 > \frac{1}{16d}$ and $\varphi_1' < 0$ in $[0, 1)$.

We now construct a suitable lower solution to (1.3). Firstly, we take $0 < \alpha \leqslant \min\{1, h_0^2\}$, $X = 1 + \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, $T_0 > X^2$. Define

$$\underline{U}(t,x)=\frac{K_0}{(t+\alpha)^m}\varphi_1\left(\frac{x}{\sqrt{t+\alpha}}\right)\quad \text{for } 0\leqslant t\leqslant T_0,\ 0\leqslant x\leqslant\sqrt{t+\alpha},$$
$$\underline{h}(t)=\sqrt{t+\alpha},\quad 0\leqslant t\leqslant T_0.$$

Let λ be big enough such that

$$\underline{U}(0,x)=\frac{K_0}{\alpha^m}\varphi_1\left(\frac{x}{\sqrt{\alpha}}\right)\leqslant u_0(x)=\lambda\varphi(x).$$

Choosing K_0 and m such that

$$-2\mu K_0\varphi_1'(1)>(T_0+1)^m,\quad m>\lambda_1+(T_0+1)\frac{\|r(t)\underline{U}\|_{L^\infty}}{K},$$

we then have

$$\underline{U}_t-d\underline{U}_{xx}-r(t)\underline{U}\left(1-\frac{\underline{U}}{K}\right)\leqslant-\frac{K_0}{(t+\alpha)^{m+1}}\left(d\varphi_1''+\frac{1}{2}\varphi_1'+\lambda_1\varphi_1\right).$$

Namely,

$$\underline{U}_t-d\underline{U}_{xx}\leqslant r(t)\underline{U}\left(1-\frac{\underline{U}}{K}\right)\quad \text{for } 0<x<\underline{h}(t),\ 0<t\leqslant T_0.$$

Moreover,

$$\underline{h}'+\mu\underline{U}_x\big(t,\underline{h}(t)\big)=\frac{1}{2\sqrt{t+\alpha}}+\frac{\mu K_0}{(t+\alpha)^m\sqrt{t+\alpha}}\varphi_1'(1)<0.$$

Then we have

$$\begin{cases}\underline{U}_t-d\underline{U}_{xx}\leqslant r(t)\underline{U}\left(1-\frac{\underline{U}}{K}\right), & 0<t\leqslant T_0,\ 0<x<\underline{h}(t),\\ \underline{U}=0,\quad \underline{h}'\leqslant-\mu\underline{U}_x, & 0<t\leqslant T_0,\ x=\underline{h}(t),\\ \underline{U}_x(t,0)=0, & 0<t\leqslant T_0.\end{cases}\tag{4.3}$$

In addition, $\underline{h}(0)=\sqrt{\alpha}\leqslant h_0$, applying Lemma 2.2 immediate gives that

$$\underline{h}(t)\leqslant h(t)\quad \text{in } [0,T_0].$$

Hence,

$$h(T_0)\geqslant\underline{h}(T_0)=\sqrt{T_0+\alpha}\geqslant\sqrt{T_0}>\frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}.$$

It follows from Corollary 3.3 and Lemma 3.5 that $h_\infty=\infty$ and spreading happens. □

Theorem 4.2 *(Threshold result). Suppose that (A) hold. Assuming that (u,h) is the solution of (1.3) with the initial value $u_0(x)=\lambda\varphi(x)$ for some $\lambda>0$. Then there exists a $\lambda^*=\lambda^*(h_0,\varphi)\in[0,\infty)$ such that vanishing happens when $0<\lambda\leqslant\lambda^*$, and spreading happens when $\lambda>\lambda^*$.*

Proof. It follows from Corollary 3.3 that spreading always happens if $h_0 \geqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$. Hence in this case we have $\lambda^*(h_0, \varphi) = 0$ for any $\varphi \in \Sigma(h_0)$.

For the remaining case that $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, define

$$\lambda^* = \sup\{\lambda\colon h_\infty(\lambda\varphi) < +\infty\} \in [0, +\infty].$$

In the view of Lemma 4.1(ii), the spreading must happen when λ is large enough. Thus we get $\lambda^* \in [0, +\infty)$. It follows from Lemma 3.2 that $h_\infty \leqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$ when $0 < \lambda < \lambda^*$ and $h_\infty = \infty$ when $\lambda > \lambda^*$.

It remains to show that $h_\infty \leqslant \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$ if $\lambda = \lambda^*$. Next we use the contradiction argument to demonstrate the conclusion. Suppose that $h_\infty = \infty$ when $\lambda = \lambda^*$. Hence we can find $T > 0$ such that $h(T) > \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}} + 1$. Since the solution of (1.3) continuously depends on the initial value, we can find a sufficiently small $\varepsilon > 0$ such that the solution with $u_0 = (\lambda^* - \varepsilon)\varphi$, denoted by $(u_\varepsilon, h_\varepsilon)$, satisfying $h_\varepsilon(T) > \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$. This implies that the information spreading happens to the case that $\lambda = \lambda^* - \varepsilon$, which contradicts to the definition of λ^*. □

5. Spreading speed

This section deals with the asymptotic spreading speed. We will show that when the information spreading happens, the free boundary moves at a constant speed for large time, that is, $\lim_{t\to+\infty} \frac{h(t)}{t} = k_0$, where k_0 is determined by the corresponding elliptic problem of (1.3) and satisfies $k_0 = \mu V'_{k_0}(0)$ with $V_k(x)$ satisfying

$$\begin{cases} -dV'' + kV' = r_\infty V\left(1 - \dfrac{V}{K}\right), & x > 0, \\ V(0) = 0. \end{cases} \tag{5.1}$$

The next result shows that k_0 is well defined, and the proof is similar to Proposition 4.1 in [6] with slight modifications.

Proposition 5.1. *For any $0 < k < 2\sqrt{r_\infty d}$, problem (5.1) admits a unique positive solution $V = V_k$. In addition, $V'_k(x) > 0$ for $x \geqslant 0$, $V'_{k_1}(0) > V'_{k_2}(0)$, $V_{k_1}(x) > V_{k_2}(x)$ for $x > 0$ and $k_1 < k_2$. And for each μ there exists a unique $k_0 = k_0(\mu) \in (0, 2\sqrt{r_\infty d})$ such that $\mu V'_{k_0}(0) = k_0$.*

Lemma 5.2. *Suppose that (A) hold. If $h_\infty = +\infty$, then $\liminf_{t\to+\infty} \frac{h(t)}{t} \geqslant k_0$.*

Proof. As in [6], first we consider

$$\begin{cases} -d\omega'' + k_0\omega' = r_\infty\omega\left(1 - \dfrac{\omega}{K}\right), & 0 < x < l, \\ \omega(0) = \omega(l) = 0. \end{cases} \tag{5.2}$$

Then we define

$$\omega_0(x) = \begin{cases} \omega_{l_0}(x), & 0 \leqslant x \leqslant a_0, \\ \omega_{l_0}(a_0), & x > a_0, \end{cases}$$

where $a_0 \in (0, l_0)$, such that $\omega_{l_0}(a_0) = \max_{[0,l_0]} \omega_{l_0}$. We can verify that ω_0 satisfies

$$\begin{cases} -d\omega_0'' + k_0\omega_0' \leqslant r_\infty \omega_0\left(1 - \dfrac{\omega_0}{K}\right), & 0 \leqslant x < +\infty, \\ \omega_0(0) = 0, \quad \omega_0(x) < K, & x \geqslant 0. \end{cases} \tag{5.3}$$

According to Lemma 3.5, we know that for any $\varepsilon > 0$ there exists $T := T_{\varepsilon,a_0}$ such that

$$h(T) > a_0 \quad \text{and} \quad u(T, x) \geqslant K\sqrt{1-\varepsilon}, \quad \forall x \in [0, a_0].$$

We define

$$w = \sqrt{1-\varepsilon}\,\omega_0\big(\eta(t) - x\big), \quad t > 0,\ 0 \leqslant x \leqslant \eta(t),\ \eta(t) = (1-\varepsilon)k_0 t + a_0,\ t > 0.$$

Direct computation yields that

$$\begin{aligned} w_t - dw_{xx} &\leqslant \sqrt{1-\varepsilon}\big(k_0\omega_0' - d\omega_0''\big) \\ &\leqslant r_\infty \sqrt{1-\varepsilon}\,\omega_0\left(1 - \frac{\omega_0}{K}\right) \\ &\leqslant r_\infty w\left(1 - \frac{w}{K}\right) \end{aligned}$$

for $t > 0$ and $0 < x < \eta(t)$. Similarly as in the proof of Theorem 4.2 in [6], we get

$$\begin{aligned} \eta'(t) = (1-\varepsilon)k_0 < \sqrt{1-\varepsilon}\,\mu\omega_{l_0}'(0) = \sqrt{1-\varepsilon}\,\mu\omega_0'(0) = -\mu w_x\big(t, \eta(t)\big), \\ w(0, x) = \sqrt{1-\varepsilon}\,\omega_0(a_0 - x) \leqslant \sqrt{1-\varepsilon}K \leqslant u(T, x), \quad \forall x \in [0, a_0], \\ w_x(t, 0) = -\sqrt{1-\varepsilon}\,\omega_0'\big(\eta(t)\big) = 0. \end{aligned}$$

Hence, it follows from Lemma 2.2 that

$$h(t+T) \geqslant \eta(t) \quad \text{for } t > 0.$$

Then

$$\liminf_{t\to+\infty} \frac{h(t)}{t} \geqslant \lim_{t\to+\infty} \frac{\eta(t-T)}{t} = (1-\varepsilon)k_0.$$

As ε is an arbitrarily positive number, we have $\liminf_{t\to+\infty} \frac{h(t)}{t} \geqslant k_0$. $\square$

Lemma 5.3. *Suppose that (A) hold. If $h_\infty = +\infty$, then $\limsup_{t\to+\infty} \frac{h(t)}{t} \leqslant k_0$.*

Proof. For any sufficiently small ϵ_0, since $\lim_{t\to+\infty} r(t) = r_\infty$, there exists $T > 0$ such that

$$r(t) < r_\infty + \epsilon_0, \quad t \geqslant T.$$

From the proof of Lemma 3.5, we know $\limsup_{t\to+\infty} u(t,x) \leqslant K$, uniformly for $0 \leqslant x < \infty$. Hence, there exists $T_0 > T$ such that

$$u(t,x) \leqslant K(1-\epsilon_0)^{-1}, \quad t \geqslant T_0,\ x \geqslant 0.$$

Next we use the similar methods with Theorem 4.2 in [6] and construct a suitable upper solution of problem (1.3). We first consider the following problem with the solution ν_{ϵ_0}

$$\begin{cases} -d\nu'' + k(\epsilon_0)\nu' = (r_\infty + \epsilon_0)\nu\left(1 + \delta(\epsilon_0) - \dfrac{\nu}{K}\right), & x > 0, \\ \nu(0) = 0, \end{cases} \tag{5.4}$$

where $k(\epsilon_0) = \mu\nu'_{\epsilon_0}(0)$. If $\epsilon_0 \to 0$, then $\delta(\epsilon_0) \to 0$. With Proposition 5.1, $\nu'_{\epsilon_0}(x) > 0$ for $x \geqslant 0$, and $\nu_{\epsilon_0} \to K(1+\delta(\epsilon_0))$, as $x \to +\infty$. Hence there exists X which is a positive large number, such that

$$\nu_{\epsilon_0}(x) > K\big(1+\delta(\epsilon_0)\big)(1-\epsilon_0) \quad \text{for } x \geqslant X.$$

We define

$$\bar{h}(t) = (1-\epsilon_0)^{-2}k(\epsilon_0)t + X + h(T_0), \quad t \geqslant T_0,$$
$$\bar{u}(t,x) = (1-\epsilon_0)^{-2}\nu_{\epsilon_0}\big(\bar{h}(t) - x\big), \quad t \geqslant T_0,\ 0 \leqslant x \leqslant \bar{h}(t).$$

Clearly, we obtain

$$\bar{u}\big(t,\bar{h}(t)\big) = 0, \qquad \bar{u}_x(t,0) \leqslant 0, \quad t \geqslant T_0,$$
$$\bar{h}'(t) = -\mu\bar{u}_x\big(t,\bar{h}(t)\big), \quad t \geqslant T_0,$$
$$\bar{h}(T_0) \geqslant h(T_0), \qquad \bar{u}(T_0,x) > 0, \qquad h(T_0) < x < \bar{h}(T_0),$$

and for $0 < x < h(T_0)$,

$$\bar{u}(T_0,x) = (1-\epsilon_0)^{-2}\nu_{\epsilon_0}\big(\bar{h}(T_0) - x\big) \geqslant (1-\epsilon_0)^{-1}K\big(1+\delta(\epsilon_0)\big) \geqslant u(T_0,x).$$

Using Lemma 2.2, we get

$$u(t,x) \leqslant \bar{u}(t,x) \quad \text{and} \quad h(t) \leqslant \bar{h}(t), \quad t \geqslant T_0,\ 0 < x < h(t).$$

Then

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \leqslant \lim_{t\to+\infty} \frac{\bar{h}(t)}{t} = k(\epsilon_0)(1-\epsilon_0)^{-2}.$$

It is well known that ν_{ϵ_0} continuously depends on ϵ_0, that is, as $\epsilon_0 \to 0$, $\nu_{\epsilon_0} \to V_{k_0}$ and $k(\epsilon_0) \to k_0$. Let $\epsilon_0 \to 0$, we easily obtain

$$\limsup_{t\to+\infty} \frac{h(t)}{t} \leqslant k_0. \qquad \square$$

Combining Lemmas 5.2 and 5.3 gives the following main result.

Theorem 5.4. *Suppose that* (A) *hold. If* $h_\infty = +\infty$, *then* $\lim_{t\to+\infty} \frac{h(t)}{t} = k_0$.

Next we show how the asymptotic spreading speed k_0 changes as the parameters in (1.3) vary, see Proposition 4.3 in [6].

Theorem 5.5. *Let* k_0 *be the asymptotic spreading speed determined by Proposition* 5.1. *Then we have*

$$\lim_{\frac{\mu K}{d}\to+\infty} \frac{k_0}{\sqrt{r_\infty d}} = 2 \quad \text{and} \quad \lim_{\frac{\mu K}{d}\to 0} \frac{k_0}{\sqrt{r_\infty d}} \frac{d}{\mu K} = \frac{1}{\sqrt{3}}.$$

6. Discussion

In this paper, we have examined a DL model with a free boundary $x = h(t)$, which describes the information diffusion in online social networks. The dynamic behavior of information diffusion with spreading front $x = h(t)$ are discussed. It was proved that if $h_0 > \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, then $h_\infty = \infty$ and $\lim_{t\to+\infty} u(t,x) = K$ uniformly on any compact subset of $\mathbb{R}^1$, that is, the information diffuses in the whole distance. However, if $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{r_\infty}}$, whether or not the information is spreading depends on the initial users' size. For $u_0(x) = \lambda\varphi(x)$, there exists a threshold λ^* such that if $\lambda \leqslant \lambda^*$, then $h_\infty < \infty$ and $\lim_{t\to+\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$, that is, the information spreads in a finite portion of population over finite time and vanishing happens. If $\lambda > \lambda^*$, the information spreading happens (Theorem 4.2). Moreover, if the information is spreading, we showed that for large time, the information spreading front $x = h(t)$ moves at a constant speed k_0, which is less than the minimal wave speed, $2\sqrt{r_\infty d}$ (see Theorem 5.5).

To our best knowledge, this paper is the first try to model information diffusion by using the free boundary to describe the moving front. We feel that it is reasonable to conclude that the free boundary problem is able to capture several important characteristics of information diffusion in online networks.

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