Methods of Functional Analysis and Topology
Vol. 18 (2012), no. 1, pp. 90–100

RECOVERING ARBITRARY ORDER DIFFERENTIAL OPERATORS ON NONCOMPACT STAR-TYPE GRAPHS

V. YURKO

To the memory of A.G. Kostjuchenko.

Abstract. We study an inverse spectral problem for arbitrary order ordinary differential equations on noncompact star-type graphs. As the main spectral characteristics we introduce and study the so-called Weyl-type matrices which are a generalization of the Weyl function for the classical Sturm-Liouville operator. We provide a procedure for constructing the solution of the inverse problem and prove its uniqueness.

1. Introduction

We study arbitrary order ordinary differential operators on noncompact star-type graphs with boundary conditions in boundary vertices and with matching conditions in the internal vertex. Boundary value problems on graphs (networks, trees) often appear in natural sciences and engineering (see [1]–[6]). Most of the works in the spectral theory on graphs are devoted to the so-called direct problems of studying properties of the spectrum and the root functions. Inverse spectral problems (which consist in recovering operators from their spectral characteristics) because of their nonlinearity, are more difficult for investigating. For *Sturm-Liouville operators on compact graphs* inverse problems of recovering *potentials* from various spectral characteristics were studied in [7]–[13] and other works. The noncompact case for Sturm-Liouville operators was considered in [14]–[16]. Inverse problems for *higher-order* differential operators on *compact graphs* were investigated in [17]–[18]. Inverse problems for *higher-order* differential operators on *noncompact graphs* have not been studied yet. We note that inverse spectral problems for second-order and for higher-order ordinary differential operators on *an interval* have been studied fairly completely by many authors (see the monographs [19]–[24] and the references therein). Inverse problems for the matrix Sturm-Liouville equation were investigated in [25]–[28].

In this paper we study the inverse spectral problem for arbitrary order differential operators on noncompact star-type graphs. As the main spectral characteristics we introduce and study the so-called Weyl-type matrices which are a generalization of the Weyl function (m-function) for the classical Sturm-Liouville operator (see [29]) and a generalization of the Weyl matrix for higher-order differential operators on the half-line introduced in [23]–[24]. Properties of the Weyl-type matrices for noncompact graphs are different from the compact case. In the compact case, the spectrum is discrete, and the Weyl-type matrices are meromorphic with respect to the spectral parameter. In the noncompact case, the spectrum consists of unbounded continuous and bounded discrete parts, and the Weyl-type matrices have cuts, poles and spectral singularities in the complex plane of the spectral parameter. We study analytical and asymptotical properties

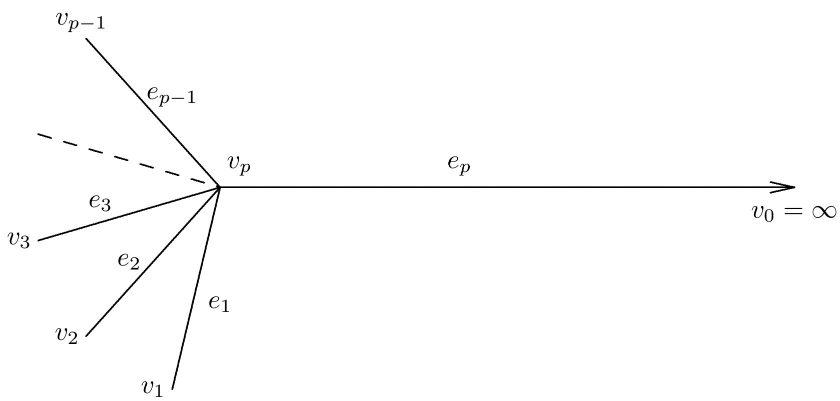
1991 *Mathematics Subject Classification.* Primary 34A55; Secondary 34L05, 47E05.
Key words and phrases. Noncompact graphs, inverse spectral problems, Weyl-type matrices.

of the Weyl-type matrices for noncompact graphs.We show that the specification of the Weyl-type matrices uniquely determines the coefficients of the differential equation on the graph, and we provide a constructive procedure for the solution of the inverse problem from the given Weyl-type matrices. For studying this inverse problem we develope the ideas of the method of spectral mappings [23]–[24]. The obtained results are natural generalizations of the well-known results on inverse problems for the differential operators on the half-line. We note that the results and the methods of the theory of inverse spectral problems can be useful for investigating inverse problems for partial differential equations. Inverse problems for partial differential equations are reflected in [30–35] and others works.

Consider a noncompact star-type graph T in $\mathbf{R}^N$ with the set of vertices $V = \{v_1, \dots, v_p\}$, and the set of edges $\mathcal{E} = \{e_1, \dots, e_p\}$, where $e_j = [v_j, v_p]$, $j = \overline{1, p-1}$, are finite segments, and $e_p = [v_p, v_0)$ is an infinite ray, $v_0 := \infty$ (see fig. 1). The vertices $v_1, \dots, v_{p-1}$ are the boundary vertices, v_p is the internal vertex, and

$$\bigcap_{j=1}^{p} e_j = \{v_p\}.$$

Let l_j be the length of the edge e_j, $j = \overline{1, p-1}$. Each edge e_j, $j = \overline{1, p-1}$, is parameterized by the parameter $x_j \in [0, l_j]$ such that the initial point v_j corresponds to $x_j = 0$, and the terminal point v_p corresponds to $x_j = l_j$. The ray $e_p = [v_p, \infty)$ is parameterized by the parameter $x_p \in [0, \infty)$ such that $x_p = 0$ corresponds to the internal vertex v_p.



A function Y on T may be represented as $Y = \{y_j\}_{j=\overline{1,p}}$, where the function $y_j(x_j)$, is defined on the edge e_j.

Fix $n \geq 2$. Let $q_\nu = \{q_{\nu j}\}_{j=\overline{1,p}}$, $\nu = \overline{0, n-2}$ be integrable complex-valued functions on T. Consider the following n-th order differential equation on T:

$$(1) \qquad y_j^{(n)}(x_j) + \sum_{\nu=0}^{n-2} q_{\nu j}(x_j) y_j^{(\nu)}(x_j) = \lambda y_j(x_j), \quad j = \overline{1, p},$$

where λ is the spectral parameter, $q_{\nu j}(x_j)$ are complex-valued integrable functions, and $y_j^{(\nu)}(x_j) \in AC[0, l_j]$, $j = \overline{1, p}$, $\nu = \overline{0, n-1}$, for all $l_p > 0$. Denote by $q = \{q_\nu\}_{\nu=\overline{0,n-2}}$ the set of the coefficients of equation (1); q is called the potential.

Consider the linear forms

$$U_{j\nu}(y_j) = \sum_{\mu=0}^{\nu} \gamma_{j\nu\mu} y_j^{(\mu)}(l_j), \quad j = \overline{1, p-1}, \quad \nu = \overline{0, n-1},$$

where $\gamma_{j\nu\mu}$ are complex numbers, and $\gamma_{j\nu} := \gamma_{j\nu\nu} \neq 0$. The linear forms $U_{j\nu}$ will be used in matching conditions in the internal vertex v_p for special solutions of equation (1).

Let $\lambda = \rho^n$. The ρ - plane is partitioned into sectors S of angle $\frac{\pi}{n}$ $\Big(\arg\rho \in \Big(\frac{\nu\pi}{n}, \frac{(\nu+1)\pi}{n}\Big),$ $\nu = \overline{0, 2n-1}\Big)$ in which the roots $R_1, R_2, \dots, R_n$ of the equation $R^n - 1 = 0$ can be numbered in such a way that

$$\text{(2)} \qquad \mathrm{Re}(\rho R_1) < \mathrm{Re}(\rho R_2) < \cdots < \mathrm{Re}(\rho R_n), \quad \rho \in S.$$

Let $\rho^* := 2n \max\limits_{\nu,j} \|q_{\nu j}\|_{L(e_j)}$. It is known [36, Ch. 1] that for each fixed $j = \overline{1,p}$, on the edge e_j there exists a fundamental system of solutions of equation (1) $\{E_{kj}(x_j, \rho)\}_{k=\overline{1,n}}$ having the properties:
1) for each sector S with the property (2), the functions $E_{kj}^{(\nu-1)}(x_j, \rho),\ k, \nu = \overline{1,n}$ are holomorphic in $\rho \in S,\ |\rho| > \rho^*$, and are continuous for $\rho \in \overline{S},\ |\rho| \geq \rho^*$;
2) as $|\rho| \to \infty,\ \rho \in \overline{S}$,

$$\text{(3)} \qquad E_{kj}^{(\nu-1)}(x_j, \rho) = (\rho R_k)^{\nu-1} \exp(\rho R_k x_j)[1],$$

where $k, \nu = \overline{1,n},\ j = \overline{1,p},\ [1] = 1 + O(\rho^{-1})$.

Let $\Psi_{sk} = \{\psi_{skj}\}_{j=\overline{1,p}}$, $s = \overline{1, p-1}$, $k = \overline{1,n}$, be solutions of equation (1) satisfying the matching conditions

$$\text{(4)} \qquad \left.\begin{aligned} &\psi_{skp}^{(\nu)}(0,\lambda) + U_{j\nu}(\psi_{skj}(x_j,\lambda)) = 0, \quad j = \overline{1,p-1}, \quad \nu = \overline{0,k-1}, \\ &\psi_{skp}^{(\nu)}(0,\lambda) + \sum_{j=1}^{p-1} U_{j\nu}(\psi_{skj}(x_j,\lambda)) = 0, \quad \nu = \overline{k, n-1} \end{aligned}\right\}$$

and the boundary conditions

$$\text{(5)} \qquad \left.\begin{aligned} &\psi_{sks}^{(\nu-1)}(0,\lambda) = \delta_{k\nu}, \quad \nu = \overline{1,k}, \\ &\psi_{skj}^{(\xi-1)}(0,\lambda) = 0, \quad \xi = \overline{1, n-k}, \quad j = \overline{1,p-1} \setminus s, \\ &\psi_{skp}(x_p,\lambda) = O(\exp(\rho R_k x_p)), \quad \rho \in S, \quad x_p \to \infty \end{aligned}\right\}$$

for each sector S with property (2). Here and below, $\delta_{k\nu}$ is the Kronecker symbol. The function Ψ_{sk} is called the Weyl-type solution of order k with respect to the boundary vertex v_s. We introduce the matrices

$$M_s(\lambda) = [M_{sk\nu}(\lambda)]_{k,\nu=\overline{1,n}}, \quad s = \overline{1,p-1},$$

where $M_{sk\nu}(\lambda) := \psi_{sks}^{(\nu-1)}(0,\lambda)$. It follows from the definition of ψ_{skj} that $M_{sk\nu}(\lambda) = \delta_{k\nu}$ for $k \geq \nu$, and $\det M_s(\lambda) \equiv 1$. The matrix $M_s(\lambda)$ is called the Weyl-type matrix with respect to the boundary vertex v_s. Denote by $M = \{M_s\}_{s=\overline{1,p-1}}$ the set of the Weyl-type matrices. The inverse problem is formulated as follows.

Inverse Problem 1. Given M, construct q on T.

We note that the notion of the Weyl-type matrices M is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator ([21], [29]) and is a generalization of the notion of Weyl matrix introduced in [23]–[24] for higher-order differential operators on the half-line. Thus, Inverse Problem 1 is a generalization of the classical inverse problems for differential operators on the half-line.

In section 2 asymptotical and analytical properties of the Weyl-type solutions and Weyl-type matrices are studied. Section 3 is devoted to the solution of auxiliary inverse problems of recovering the potential on a fixed edge. In section 4 we study Inverse problem 1. For this inverse problem we provide a constructive procedure for the solution and prove its uniqueness.

2. Auxiliary propositions

For the existence and a "regular behavior" of the Weyl-type solutions and the Weyl-type matrices, one needs certain restrictions on the coefficients $\gamma_{j\nu}$ from the matching conditions (see [17] for more details). In order to formulate regularity conditions for matching we introduce the numbers Δ^0_{sk}, $s=\overline{1,p-1}$, $k=\overline{1,n-1}$, as follows:

$$\Delta^0_{1k} := \det[d_{kj\nu}]_{j,\nu=\overline{1,p}}, \quad k=\overline{1,n-1},$$

where $d_{kj\nu}=[d^{\mu\xi}_{kj\nu}]$ are matrices of the form

$$\begin{aligned}
d_{k11} &= [d^{\mu\xi}_{k11}]_{\mu=\overline{1,k},\,\xi=\overline{1,n-k}}, \quad d^{\mu\xi}_{k11}=\gamma_{1,\mu-1}R^{\mu-1}_{k+\xi},\\
d_{kjp} &= [d^{\mu\xi}_{kjp}]_{\mu,\xi=\overline{1,k}}, \quad d^{\mu\xi}_{kjp}=R^{\mu-1}_{\xi}, \quad j=\overline{1,p-1},\\
d_{k1\nu} &= [d^{\mu\xi}_{k1\nu}]_{\mu,\xi=\overline{1,k}}, \quad d^{\mu\xi}_{k1\nu}=0, \quad \nu=\overline{2,p-1},\\
d_{kj1} &= [d^{\mu\xi}_{kj1}]_{\mu=\overline{1,k},\,\xi=\overline{1,n-k}}, \quad d^{\mu\xi}_{kj1}=0, \quad j=\overline{2,p-1},\\
d_{kj\nu} &= [d^{\mu\xi}_{kj\nu}]_{\mu,\xi=\overline{1,k}}, \quad d^{\mu\xi}_{kj\nu}=\delta_{j\nu}\gamma_{j,\mu-1}R^{\mu-1}_{n-k+\xi}, \quad j,\nu=\overline{2,p-1},\\
d_{kp1} &= [d^{\mu\xi}_{kp1}]_{\mu,\xi=\overline{1,n-k}}, \quad d^{\mu\xi}_{kp1}=\gamma_{1,k+\mu-1}R^{k+\mu-1}_{k+\xi},\\
d_{kpp} &= [d^{\mu\xi}_{kpp}]_{\mu=\overline{1,n-k},\,\xi=\overline{1,k}}, \quad d^{\mu\xi}_{kpp}=R^{k+\mu-1}_{\xi},\\
d_{kp\nu} &= [d^{\mu\xi}_{kp\nu}]_{\mu=\overline{1,n-k},\,\xi=\overline{1,k}}, \quad d^{\mu\xi}_{kp\nu}=\gamma_{\nu,k+\mu-1}R^{k+\mu-1}_{n-k+\xi}, \quad \nu=\overline{2,p-1}.
\end{aligned}$$

The numbers Δ^0_{sk}, $s=\overline{2,p-1}$, $k=\overline{1,n-1}$, are obtained from Δ^0_{1k} by interchanging places of $\gamma_{s\nu}$ and $\gamma_{1\nu}$.

We assume that

$$\Delta^0_{sk}\neq 0, \quad s=\overline{1,p-1}, \quad k=\overline{1,n-1}. \tag{6}$$

Condition (6) is called the regularity condition for matching. Differential operators on T which do not satisfy the regularity condition for matching (6), possess qualitatively different properties for the formulation and the investigation of inverse problems, and are not considered in this paper; they require a separate investigation. We note that (6) is a generalization of the regularity condition for *Sturm-Liouville operators* on graphs (see [37]). In particular, for the so-called standard matching conditions [8], (6) is satisfied obviously. We also note that the regularity condition (6) play a similar role in inverse problems as the Birkhoff regularity conditions [36] in direct problems.

Now we are going to study the asymptotic behavior of the Weyl-type solutions. Denote

$$\omega_k := \frac{\Omega_{k-1}}{\Omega_k}, \quad k=\overline{1,n}, \quad \Omega_k := \det[R^{\nu-1}_{\xi}]_{\xi,\nu=\overline{1,k}}, \quad \Omega_0 := 1.$$

Lemma 1. *Fix $j=\overline{1,p-1}$ and a sector S with property (2).*

1) Let $k=\overline{1,n-1}$, and let $y_j(x_j,\rho)$ be a solution of equation (1) on the edge e_j under the conditions

$$y_j(0,\rho)=\cdots=y_j^{(k-1)}(0,\rho)=0. \tag{7}$$

Then for $x_j\in(0,l_j]$, $\nu=\overline{0,n-1}$, $\rho\in S, |\rho|\to\infty$,

$$y_j^{(\nu)}(x_j,\rho)=\sum_{\mu=k+1}^{n} A_{\mu j}(\rho)(\rho R_\mu)^\nu \exp(\rho R_\mu x_j)[1], \tag{8}$$

where the coefficients $A_{\mu j}(\rho)$ do not depend on x_j. Here and below we assume that $\arg\rho=\text{const}$, *when* $|\rho|\to\infty$.

2) Let $k=\overline{1,n}$, and let $y_j(x_j,\rho)$ be a solution of equation (1) on the edge e_j under the conditions

$$y_j(0,\rho)=\cdots=y_j^{(k-2)}(0,\rho)=0,\quad y_j^{(k-1)}(0,\rho)=1.$$

Then for $x_j\in(0,l_j],\ \nu=\overline{0,n-1},\ \rho\in S,\ |\rho|\to\infty$,

$$y_j^{(\nu)}(x,\rho)=\frac{\omega_k}{\rho^{k-1}}(\rho R_k)^\nu\exp(\rho R_k x_j)[1]+\sum_{\mu=k+1}^{n}B_{\mu j}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_j)[1], \tag{9}$$

where the coefficients $B_{\mu j}(\rho)$ do not depend on x_j.

Proof. Using the fundamental system of solutions $\{E_{\mu j}(x_j,\rho)\}_{\mu=\overline{1,n}}$, one can write

$$y_j(x_j,\rho)=\sum_{\mu=1}^{n}A_{\mu j}(\rho)E_{\mu j}(x_j,\rho). \tag{10}$$

Substituting (10) into (7) we obtain a linear algebraic system with respect to $A_{1j}(\rho),\dots,A_{kj}(\rho)$. The determinant $A(\rho)$ of this system has the asymptotics $A(\rho)=\Omega_k+O(\rho^{-1})$ as $|\rho|\to\infty$. Solving the system by Cramer's rule and taking (3) into account we get

$$A_{\xi j}(\rho)=\sum_{\mu=k+1}^{n}(c_{\xi\mu j}+O(\rho^{-1}))A_{\mu j}(\rho),\quad \xi=\overline{1,k}, \tag{11}$$

where $c_{\xi\mu j}$ are constants. Substituting (11) into (10) and using (3) we arrive at (8). Relations (9) are proved analogously. □

Lemma 2. *Fix a sector S with property (2). For $\nu=\overline{0,n-1},\ s=\overline{1,p-1},\ k=\overline{1,n},\ x_s\in(0,l_s)$, the following asymptotical formula holds:*

$$\psi_{sks}^{(\nu)}(x_s,\lambda)=\frac{\omega_k}{\rho^{k-1}}\,(\rho R_k)^\nu\exp(\rho R_k x_s)[1],\quad \rho\in S,\quad |\rho|\to\infty. \tag{12}$$

Proof. For $k=n$, the assertion of the lemma follows from Lemma 1. Fix $k=\overline{1,n-1}$, $s=\overline{1,p-1}$. By virtue of (5) and Lemma 1, one gets for $x_j\in(0,l_j),\ \nu=\overline{0,n-1}$, $\rho\in S,\ |\rho|\to\infty$,

$$\left.\begin{aligned}
\psi_{sks}^{(\nu)}(x_s,\lambda)&=\frac{\omega_k}{\rho^{k-1}}\,(\rho R_k)^\nu\exp(\rho R_k x_s)[1]+\sum_{\mu=k+1}^{n}A_{\mu s}^{sk}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_s)[1],\\
\psi_{skj}^{(\nu)}(x_j,\lambda)&=\sum_{\mu=n-k+1}^{n}A_{\mu j}^{sk}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_j)[1],\quad j=\overline{1,p-1}\setminus s,\\
\psi_{skp}^{(\nu)}(x_p,\lambda)&=\sum_{\mu=1}^{k}A_{\mu p}^{sk}(\rho)(\rho R_\mu)^\nu\exp(\rho R_\mu x_p)[1],
\end{aligned}\right\} \tag{13}$$

where the coefficients $A_{\mu j}^{sk}(\rho)$ do not depend on x_j and ν. Substituting (13) into (4) we obtain a linear algebraic system σ_{sk} with respect to $A_{\mu j}^{sk}$. The determinant $\Delta_{sk}(\rho)$ of this system has the asymptotics

$$\Delta_{sk}(\rho)=\Delta_{sk}^{0}\exp\left(\rho\Big(\sum_{\mu=k+1}^{n}R_\mu\Big)l_s+\rho\Big(\sum_{\mu=n-k+1}^{n}R_\mu\Big)\Big(\sum_{\xi=1}^{p-1}l_\xi-l_s\Big)\right)[1],$$
$$\rho\in S,\quad |\rho|\to\infty.$$

Solving the system σ_{sk} by Cramer's rule we get

$$
\left.
\begin{aligned}
A^{sk}_{\mu s}(\rho) &= \frac{1}{\rho^{k-1}}\Big(a^{sk}_{\mu s} + O\Big(\frac{1}{\rho}\Big)\Big)\exp(\rho(R_k - R_\mu)l_s), \quad \mu = \overline{k+1, n}, \\
A^{sk}_{\mu p}(\rho) &= \frac{1}{\rho^{k-1}}\Big(a^{sk}_{\mu p} + O\Big(\frac{1}{\rho}\Big)\Big)\exp(\rho R_k l_s), \quad \mu = \overline{1, k}, \\
A^{sk}_{\mu j}(\rho) &= \frac{1}{\rho^{k-1}}\Big(a^{sk}_{\mu j} + O\Big(\frac{1}{\rho}\Big)\Big)\exp(\rho R_k l_s)\exp(-\rho R_\mu l_j), \\
& j = \overline{1, p-1} \setminus s, \quad \mu = \overline{n-k+1, n},
\end{aligned}
\right\} \tag{14}
$$

where $a^{sk}_{\mu j}$ are constants. Substituting (14) into (13) we arrive at (12). □

It follows from the proof of Lemma 2 that one can also get the asymptotics for $\psi^{(\nu)}_{skj}(x_j, \lambda)$, $j \neq s$; but for our purposes only (12) is needed. Denote $\Pi_\pm := \{\lambda :\ \pm\mathrm{Im}\,\lambda > 0\}$. Using the standard technique (see [23–24]) one gets that the following assertion holds.

Lemma 3. *The Weyl-type functions $M_{sk\nu}(\lambda)$, $\nu > k$, are holomorphic in Π_+ and Π_- with the exception of at most countable bounded sets of poles $\Lambda^+_{sk\nu}$ and $\Lambda^-_{sk\nu}$, respectively, and $M_{sk\nu}(\lambda)$ are continuous in $\overline{\Pi_+}$ and $\overline{\Pi_-}$ with the exception of the bounded sets $\hat\Lambda^+_{sk\nu}$ and $\hat\Lambda^-_{sk\nu}$, respectively. In other words, for real λ (with the exception of the bounded sets $\hat\Lambda^+_{sk\nu}$ and $\hat\Lambda^-_{sk\nu}$) there exist the finite limits*

$$M^\pm_{sk\nu}(\lambda) = \lim_{z\to 0,\ \mathrm{Re}\, z > 0} M_{sk\nu}(\lambda \pm iz).$$

3. Auxiliary inverse problems

In this section we study auxiliary inverse problems of recovering the potential on each fixed edge. Fix $s = \overline{1, p-1}$, and consider the following inverse problem on the edge e_s which is called IP(s): *Given the Weyl-type matrix M_s, construct the functions $q_{\nu s}$, $\nu = \overline{0, n-2}$ on the edge e_s.*

In the inverse problem IP(s) we construct the potential only on the edge e_s, but the Weyl-type matrix M_s brings a global information from the whole graph. In other words, IP(s) is not a local inverse problem related to the edge e_s.

Let us prove the uniqueness theorem for the solution of IP(s). For this purpose together with q we consider a potential $\tilde q$. Everywhere below if a symbol α denotes an object related to q, then $\tilde\alpha$ will denote the analogous object related to $\tilde q$.

Theorem 1. *Fix $s = \overline{1, p-1}$. If $M_s = \tilde M_s$, then $q_{\nu s} = \tilde q_{\nu s}$, $\nu = \overline{0, n-2}$. Thus, the specification of the Weyl-type matrix M_s uniquely determines the potential on the edge e_s.*

Proof. Let $\{C_{kj}(x_j, \lambda)\}_{k=\overline{1,n}}$, $j = \overline{1, p}$ be the fundamental system of solutions of equation (1) on the edge e_j under the initial conditions $C^{(\nu-1)}_{kj}(0, \lambda) = \delta_{k\nu}$, $k, \nu = \overline{1, n}$. For each fixed $x_j \in [0, l_j]$, the functions $C^{(\nu-1)}_{kj}(x_j, \lambda)$, $k, \nu = \overline{1, n}$, $j = \overline{1, p}$, are entire in λ of order $1/n$. Moreover,

$$\det[C^{(\nu-1)}_{kj}(x_j, \lambda)]_{k,\nu=\overline{1,n}} \equiv 1. \tag{15}$$

Using the fundamental system of solutions $\{C_{kj}(x_j, \lambda)\}_{k=\overline{1,n}}$, one can write

$$\psi_{skj}(x_j, \lambda) = \sum_{\mu=1}^{n} M_{skj\mu}(\lambda) C_{\mu j}(x_j, \lambda), \quad j = \overline{1, p}, \quad s = \overline{1, p-1}, \quad k = \overline{1, n}, \tag{16}$$

where the coefficients $M_{skj\mu}(\lambda)$ do not depend on x_j. In particular, $M_{sks\mu}(\lambda) = M_{sk\mu}(\lambda)$, and

$$\psi_{sks}(x_s,\lambda) = C_{ks}(x_s,\lambda) + \sum_{\mu=k+1}^{n} M_{sk\mu}(\lambda) C_{\mu s}(x_s,\lambda). \tag{17}$$

It follows from (15) and (17) that

$$\det[\psi_{sks}^{(\nu-1)}(x_s,\lambda)]_{k,\nu=\overline{1,n}} \equiv 1. \tag{18}$$

Denote $\psi_s(x_s,\lambda) := [\psi_{sks}^{(\nu-1)}(x_s,\lambda)]_{\nu,k=\overline{1,n}}$, $C_s(x_s,\lambda) := [C_{ks}^{(\nu-1)}(x_s,\lambda)]_{\nu,k=\overline{1,n}}$. Then relation (17) can be written in the form

$$\psi_s(x_s,\lambda) = C_s(x_s,\lambda) M_s^T(\lambda), \tag{19}$$

where T is the sign for the transposition. According to (15) and (18),

$$\det \psi_s(x_s,\lambda) = \det C_s(x_s,\lambda) \equiv 1. \tag{20}$$

We define the matrix $\mathcal{P}_s(x_s,\lambda) = [\mathcal{P}_{sjk}(x_s,\lambda)]_{j,k=\overline{1,n}}$ by the formula

$$\mathcal{P}_s(x_s,\lambda) = \psi_s(x_s,\lambda)(\tilde{\psi}_s(x_s,\lambda))^{-1}.$$

Taking (20) into account we calculate

$$\begin{aligned} \mathcal{P}_{sjk}(x_s,\lambda) &= \det[\tilde{\psi}_{s\nu s}^{(n-1)}(x_s,\lambda), \ldots, \tilde{\psi}_{s\nu s}^{(k)}(x_s,\lambda), \psi_{s\nu s}^{(j-1)}(x_s,\lambda), \\ &\tilde{\psi}_{s\nu s}^{(k-2)}(x_s,\lambda), \ldots, \tilde{\psi}_{s\nu s}(x_s,\lambda)]_{\nu=\overline{1,n}} = \sum_{\nu=1}^{n} (-1)^{\nu+k-n-1} \psi_{s\nu s}^{(j-1)}(x_s,\lambda) \\ &\times \det\Big[\tilde{\psi}_{sns}^{(\xi)}(x_s,\lambda), \ldots, \tilde{\psi}_{s,\nu+1,s}^{(\xi)}(x_s,\lambda), \\ &\tilde{\psi}_{s,\nu-1,s}^{(\xi)}(x_s,\lambda), \ldots, \tilde{\psi}_{s1s}^{(\xi)}(x_s,\lambda)\Big]_{\xi=\overline{0,n-1}\setminus k-1}. \end{aligned} \tag{21}$$

It follows from (21) and Lemma 2 that for $x_s \in (0, l_s)$, $k = \overline{1,n}$, $|\lambda| \to \infty$, $\arg\lambda = \theta \neq 0, \pi$, one has

$$\mathcal{P}_{s1k}(x_s,\lambda) - \delta_{1k} = O(\rho^{-1}). \tag{22}$$

We transform the matrix $\mathcal{P}_s(x_s,\lambda)$. For this purpose we use (19)

$$\begin{aligned} \mathcal{P}_s(x_s,\lambda) &= \psi_s(x_s,\lambda)(\tilde{\psi}_s(x_s,\lambda))^{-1} \\ &= C_s(x_s,\lambda) M_s^T(\lambda)(\tilde{M}_s^T(\lambda))^{-1}(\tilde{C}_s(x_s,\lambda))^{-1} = C_s(x_s,\lambda)(\tilde{C}_s(x_s,\lambda))^{-1}. \end{aligned}$$

In view of (20) we conclude that for each fixed x_s, the matrix-valued function $\mathcal{P}_s(x_s,\lambda)$ is an entire function in λ of order $1/n$. Together with (22) this yields $\mathcal{P}_{s11}(x_s,\lambda) \equiv 1$, $\mathcal{P}_{s1k}(x_s,\lambda) \equiv 0$, $k = \overline{2,n}$. Since

$$\mathcal{P}_s(x_s,\lambda)\tilde{\psi}_s(x_s,\lambda) = \psi_s(x_s,\lambda),$$

it follows that $\psi_{sks}(x_s,\lambda) \equiv \tilde{\psi}_{sks}(x_s,\lambda)$ for all x_s, λ, k, and hence $q_{\nu s} = \tilde{q}_{\nu s}$, $\nu = \overline{0, n-2}$. Theorem 1 is proved. □

Using the method of spectral mappings one can get a constructive procedure for the solution of the inverse problem IP(s). It can be obtained by the same arguments as for n-th order differential operators on the half-line (see [24, Ch. 2] for detais).

Now we define an auxiliary Weyl-type matrix with respect to the internal vertex v_p. Let $\psi_{pk}(x_p,\lambda)$, $k = \overline{1,n}$, be solutions of equation (1) on the edge e_p under the conditions

$$\left.\begin{aligned} \psi_{pk}^{(\nu-1)}(0,\lambda) = \delta_{k\nu}, \quad \nu = \overline{1,k}, \\ \psi_{pk}(x_p,\lambda) = O(\exp(\rho R_k x_p)), \quad x_p \to \infty, \quad \rho \in S \end{aligned}\right\} \tag{23}$$

in each sector S with property (2). We introduce the matrix $M_p(\lambda) = [M_{pk\nu}(\lambda)]_{k,\nu=\overline{1,n}}$, where $M_{pk\nu}(\lambda) := \psi_{pk}^{(\nu-1)}(0,\lambda)$. Clearly, $M_{pk\nu}(\lambda) = \delta_{k\nu}$ for $k \geq \nu$, and $\det M_p(\lambda) \equiv 1$. The matrix $M_p(\lambda)$ is called the Weyl-type matrix with respect to the internal vertex v_p. Consider the following inverse problem on the edge e_p which is called IP(p): *Given the Weyl-type matrix M_p, construct the functions $q_{\nu p}$, $\nu = \overline{0, n-2}$ on the edge e_p.*

This inverse problem is the classical one, since it is the inverse problem of recovering n-th order differential equation on the half-line from its Weyl-type matrix. This inverse problem has been solved in [24]. In particular, the following uniqueness theorem has been proved in [24].

Theorem 2. *If $M_p = \tilde{M}_p$, then $q_{\nu p} = \tilde{q}_{\nu p}$, $\nu = \overline{0, n-2}$. Thus, the specification of the Weyl-type matrix M_p uniquely determines the potential on the edge e_p.*

Moreover, in [24] an algorithm for the solution of the inverse problem IP(p) is given, and necessary and sufficient conditions for the solvability of this inverse problem are provided.

4. Solution of inverse problem 1

In this section we obtain a constructive procedure for the solution of Inverse Problem 1 and prove its uniqueness. Our plan is the following.

Step 1. Solving the inverse problem IP(s) for each fixed $s = \overline{1, p-1}$, we find the functions $q_{\nu s}$, $\nu = \overline{0, n-2}$, $s = \overline{1, p-1}$, i.e. we find the potential q on the edges $e_1, \dots, e_{p-1}$.

Step 2. Using the knowledge of the potential on the edges $e_1, \dots, e_{p-1}$, we construct the Weyl-type matrix M_p.

Step 3. Solving the inverse problem IP(p) we find the functions $q_{\nu p}$, $\nu = \overline{0, n-2}$, i.e. we find the potential on the edge e_p.

Steps 1 and 3 have been already studied in Section 3. It remains to fulfil Step 2.

Suppose that Step 1 was already made, and we found the functions $q_{\nu s}$, $\nu = \overline{0, n-2}$, $s = \overline{1, p-1}$, i.e. we found the potential q on the edges $e_1, \dots, e_{p-1}$. Fix $s = \overline{1, p-1}$. All calculations below will be made for this fixed s. Using the knowledge of the potential on the edge e_s, we calculate the functions $C_{ks}(x_s, \lambda)$, $k = \overline{1, n}$, and the functions $\psi_{sks}(x_s, \lambda)$, $k = \overline{1, n}$, by (17). Now we are going to construct the Weyl-type matrix M_p using $\psi_{sks}(x_s, \lambda)$, $k = \overline{1, n}$. First we prove an auxiliary assertion.

Lemma 4. *Fix $s = \overline{1, p-1}$. Then the following relations hold:*

$$M_{p1\nu}(\lambda) = \frac{\psi_{s1p}^{(\nu-1)}(0,\lambda)}{\psi_{s1p}(0,\lambda)}, \quad \nu = \overline{2, n}, \tag{24}$$

$$M_{pk\nu}(\lambda) = \frac{\det[\psi_{s\mu p}(0,\lambda), \dots, \psi_{s\mu p}^{(k-2)}(0,\lambda), \psi_{s\mu p}^{(\nu-1)}(0,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu p}^{(\xi-1)}(0,\lambda)]_{\xi,\mu=\overline{1,k}}}, \tag{25}$$
$$k = \overline{2, n-1}, \quad \nu = \overline{k+1, n}.$$

Proof. Denote $z_{p1}(x_p, \lambda) := \psi_{s1p}(x_p, \lambda)(\psi_{s1p}(0, \lambda))^{-1}$. The function $z_{p1}(x_p, \lambda)$ is a solution of equation (1) on the edge e_p, and $z_{p1}(0, \lambda) = 1$. Taking (5) and (23) into account we conclude that the solutions $z_{p1}(x_p, \lambda)$ and $\psi_{p1}(x_p, \lambda)$ satisfy the same boundary conditions, and consequently, $z_{p1}(x_p, \lambda) \equiv \psi_{p1}(x_p, \lambda)$. Thus,

$$\psi_{p1}(x_p, \lambda) = \frac{\psi_{s1p}(x_p, \lambda)}{\psi_{s1p}(0, \lambda)}. \tag{26}$$

Similarly, we calculate

$$\psi_{pk}(x_p,\lambda)=\frac{\det[\psi_{s\mu p}(0,\lambda),\ldots,\psi_{s\mu p}^{(k-2)}(0,\lambda),\psi_{s\mu p}(x_p,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu p}^{(\xi-1)}(0,\lambda)]_{\xi,\mu=\overline{1,k}}},\quad k=\overline{2,n-1}. \tag{27}$$

Since $M_{pk\nu}(\lambda)=\psi_{pk}^{(\nu-1)}(0,\lambda)$, it follows from (26)–(27) that (24)–(25) hold. □

Using the matching conditions (4) we get

$$U_{j\nu}(\psi_{skj})=U_{s\nu}(\psi_{sks}),\quad 0\le\nu<k\le n-1. \tag{28}$$

Since the functions ψ_{sks} were already calculated, the right-hand sides in (28) are known. For each fixed $k=\overline{1,n-1}$, we successively use (28) for $\nu=0,1,\ldots,k-1$, and calculate reccurently the functions

$$\psi_{skj}^{(\nu)}(l_j,\lambda),\quad k=\overline{1,n-1},\quad \nu=\overline{0,k-1},\quad j=\overline{1,p-1}\setminus s. \tag{29}$$

Furthermore, it follows from (16) and (5) that $M_{skj\mu}(\lambda)=0$ for $\mu=\overline{1,n-k}$, $j=\overline{1,p-1}\setminus s$, and consequently,

$$\psi_{skj}(x_j,\lambda)=\sum_{\mu=n-k+1}^{n}M_{skj\mu}(\lambda)C_{\mu j}(x_j,\lambda),\quad k=\overline{1,n-1},\quad j=\overline{1,p-1}\setminus s.$$

This yields

$$\psi_{skj}^{(\nu)}(l_j,\lambda)=\sum_{\mu=n-k+1}^{n}M_{skj\mu}(\lambda)C_{\mu j}^{(\nu)}(l_j,\lambda), \tag{30}$$
$$\nu=\overline{0,n-1},\quad k=\overline{1,n-1},\quad j=\overline{1,p-1}\setminus s.$$

Fix $k=\overline{1,n-1}$, $j=\overline{1,p-1}\setminus s$, and consider a part of relations (30), namely, for $\nu=\overline{0,k-1}$. They form a linear algebraic system with respect to the functions $M_{skj\mu}(\lambda)$, $\mu=\overline{n-k+1,n}$. Solving this system by Cramer's rule we find these functions. Substituting them into (30) for $\nu\ge k$, we calculate the functions

$$\psi_{skj}^{(\nu)}(l_j,\lambda),\quad k=\overline{1,n-1},\quad \nu=\overline{k,n-1},\quad j=\overline{1,p-1}\setminus s. \tag{31}$$

Substituting now the functions (29) and (31) into (4) we find

$$\psi_{skp}^{(\nu)}(0,\lambda),\quad k=\overline{1,n-1},\quad \nu=\overline{0,n-1}. \tag{32}$$

Since the functions (32) are known, one can calculate the Weyl-type matrix M_p via (24)–(25).

Thus, we have obtained the solution of Inverse Problem 1 and proved its uniqueness, i.e. the following assertion holds.

Theorem 3. *The specification of the Weyl-type matrices $M=\{M_s\}_{s=\overline{1,p-1}}$ uniquely determines the potential q on T. The solution of Inverse Problem 1 can be obtained by the following algorithm.*

Algorithm 1. *Given the Weyl-type matrices $M=\{M_s\}_{s=\overline{1,p-1}}$.*

1) Find the functions $q_{\nu s}$, $\nu=\overline{0,n-2}$, $s=\overline{1,p-1}$, by solving the inverse problem IP(s) for each $s=\overline{1,p-1}$.

2) Fix $s=\overline{1,p-1}$, and calculate $C_{ks}^{(\nu)}(l_j,\lambda)$ for $k=\overline{1,n}$, $\nu=\overline{0,n-1}$.

3) Construct the functions $\psi_{sks}^{(\nu)}(l_s,\lambda)$, $k=\overline{1,n-1}$, $\nu=\overline{0,n-1}$ by the formula

$$\psi_{sks}^{(\nu)}(l_s,\lambda)=C_{ks}^{(\nu)}(l_s,\lambda)+\sum_{\mu=k+1}^{n}M_{sk\mu}(\lambda)C_{\mu s}^{(\nu)}(l_s,\lambda).$$

4) Find the functions $\psi_{skj}^{(\nu)}(l_j,\lambda)$, $k=\overline{1,n-1}$, $\nu=\overline{0,k-1}$, $j=\overline{1,p-1}\setminus s$, *by using the recurrent formulae (28).*

5) Calculate $M_{skj\mu}(\lambda)$, $k=\overline{1,n-1}$, $\mu=\overline{n-k+1,n}$, $j=\overline{1,p-1}\setminus s$, *by solving the linear algebraic systems*

$$\sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda)C_{\mu j}^{(\nu)}(l_j,\lambda)=\psi_{skj}^{(\nu)}(l_j,\lambda),\quad \nu=\overline{0,k-1},$$

for each fixed $k=\overline{1,n-1}$, $j=\overline{1,p-1}\setminus s$.

6) Construct the functions $\psi_{skj}^{(\nu)}(l_j,\lambda)$, $k=\overline{1,n-1}$, $\nu=\overline{k,n-1}$, $j=\overline{1,p-1}\setminus s$, *by the formula*

$$\psi_{skj}^{(\nu)}(l_j,\lambda)=\sum_{\mu=n-k+1}^{n} M_{skj\mu}(\lambda)C_{\mu j}^{(\nu)}(l_j,\lambda),\quad \nu\ge k.$$

7) Find the functions $\psi_{skp}^{(\nu)}(0,\lambda)$, $k=\overline{1,n-1}$, $\nu=\overline{0,n-1}$, *by (4).*

8) Calculate the Weyl-type matrix M_p *via (24)–(25).*

9) Construct the functions $q_{\nu p}$, $\nu=\overline{0,n-2}$, *by solving the inverse problem IP(p).*

Acknowledgments. This research was supported in part by Grants 10-01-00099 and 10-01-92001-NSC of Russian Foundation for Basic Research and Taiwan National Science Council.

References

1. E. Montrol, *Quantum theory on a network*, J. Math. Phys. **11** (1970), no. 2, 635–648.
2. J. Langese, G. Leugering, and J. Schmidt, *Modelling, Analysis and Control of Dynamic Elastic Multi-Link Structures*, Birkhäuser, Boston, 1994.
3. T. Kottos and U. Smilansky, *Quantum chaos on graphs*, Phys. Rev. Lett. **79** (1997), 4794–4797.
4. P. Kuchment, *Quantum graphs. Some basic structures*, Waves Random Media **14** (2004), S107–S128.
5. Yu. Pokornyi and A. Borovskikh, *Differential equations on networks (geometric graphs)*, J. Math. Sci. (N.Y.) **119** (2004), no. 6, 691–718.
6. Yu. Pokornyi and V. Pryadiev, *The qualitative Sturm-Liouville theory on spatial networks*, J. Math. Sci. (N.Y.) **119** (2004), no. 6, 788–835.
7. M. I. Belishev, *Boundary spectral inverse problem on a class of graphs (trees) by the BC method*, Inverse Problems **20** (2004), 647–672.
8. V. A. Yurko, *Inverse spectral problems for Sturm-Liouville operators on graphs*, Inverse Problems **21** (2005), 1075–1086.
9. B. M. Brown and R. Weikard, *A Borg-Levinson theorem for trees*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2062, 3231–3243.
10. V. Pivovarchik, *Inverse problem for the Sturm-Liouville equation on a star-shaped graph*, Math. Nachr. **280** (2007), no. 13–14, 1595–1619.
11. V. A. Yurko, *Inverse problems for Sturm-Liouville operators on bush-type graphs*, Inverse Problems **25** (2009), no. 10, 105008, 14 pp.
12. V. A. Yurko, *An inverse problem for Sturm-Liouville operators on A-graphs*, Applied Math. Lett. **23** (2010), no. 8, 875–879.
13. V. A. Yurko, *Inverse spectral problems for differential operators on arbitrary compact graphs*, Journal of Inverse and Ill-Posed Proplems **18** (2010), no. 3, 245–261.
14. N. I. Gerasimenko, *Inverse scattering problem on a noncompact graph*, Teoret. Mat. Fiz. **74** (1988), no. 2, 187-200. (Russian); English transl. Theor. Math. Phys. **75** (1988), 460–470.
15. G. Freiling and V. A. Yurko, *Inverse spectral problems for Sturm-Liouville operators on noncompact trees*, Results in Math. **50** (2007), 195–212.
16. V. Pivovarchik and Y. Latushkin, *Scattering in a forked-shaped waveguide*, Integr. Equ. Oper. Theory **61** (2008), 365–399.
17. V. A. Yurko, *An inverse problem for higher-order differential operators on star-type graphs*, Inverse Problems **23** (2007), 893–903.
18. V. A. Yurko, *Inverse problems for differential of any order on trees*, Mat. Zametki **83** (2008), no. 1, 139–152. (Russian); English transl. Math. Notes **83** (2008), no. 1, 125–137.

19. V. A. Marchenko, *Sturm-Liouville Operators and their Applications*, Naukova Dumka, Kiev, 1977. (Russian); English transl. Birkhäuser, Basel, 1986.
20. B. M. Levitan, *Inverse Sturm-Liouville Problems*, Nauka, Moscow, 1984. (Russian); English transl. VNU Sci. Press, Utrecht, 1987.
21. G. Freiling and V. A. Yurko, *Inverse Sturm-Liouville Problems and their Applications*, NOVA Science Publishers, New York, 2001.
22. R. Beals, P. Deift, and C. Tomei, *Direct and Inverse Scattering on the Line*, Math. Surveys and Monographs, Vol. 28, Amer. Math. Soc., Providence, RI, 1988.
23. V. A. Yurko, *Inverse Spectral Problems for Differential Operators and their Applications*, Gordon and Breach, Amsterdam, 2000.
24. V. A. Yurko, *Method of Spectral Mappings in the Inverse Problem Theory*, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.
25. M. M. Malamud, *Uniqueness of the matrix Sturm-Liouville equation given a part of the monodromy matrix, and Borg type results*, Sturm-Liouville Theory. Past and Present, Birkhäuser, Basel, 2005, pp. 237–270.
26. V. A. Yurko, *An inverse problems for the matrix Sturm-Liouville equation on a finite interval*, Inverse Problems **22** (2006), 1139–1149.
27. D. Chelkak and E. Korotyaev, *Weyl-Titchmarsh functions of vector-valued Sturm-Liouville operators on the unit interval*, J. Funct. Anal. **257** (2009), no. 5, 1546–1588.
28. N. P. Bondarenko, *Necessary and sufficient conditions for the solvability of the inverse problem for the matrix Sturm-Liouville operator*, Funct. Anal. Appl. **257** (2011) (to appear).
29. B. M. Levitan and I. S. Sargsyan, *Introduction to Spectral Theory*, Amer. Math. Soc., Transl. Math. Monographs, Vol. 39, Providence, RI, 1975. (Russian edition: Nauka, Moscow, 1970)
30. Yu. M. Berezanskii, *The uniqueness theorem in the inverse spectral problem for the Schrödinger equation*, Trudy Moskov. Mat. Obshch. **7** (1958), 3–62. (Russian)
31. A. L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2000.
32. S. I. Kabanikhin, A. D. Satybaev, and M. A. Shishlenin, *Direct Methods of Solving Multidimensional Inverse Hyperbolic Problems*, Inverse and Ill-posed Problems Series. VSP, Utrecht, 2005.
33. L. P. Nizhnik, *Inverse Scattering Problems for Hyperbolic Equations*, Naukova Dumka, Kiev, 1991. (Russian)
34. A. I. Prilepko, D. G. Orlovsky, and I. A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker, New York, 2000.
35. V. G. Romanov, *Inverse Problems in Mathematical Physics*, Nauka, Moscow, 1984. (Russian); English transl. VNU Sci. Press, Utrecht, 1987.
36. M. A. Naimark, *Linear Differential Operators,* I, II, GITTL, Moscow, 1954. (Russian); English transl. of 1st ed., Parts I, II, Ungar, New York, 1967, 1968.
37. G. Freiling and V. A. Yurko, *Inverse problems for differential operators on graphs with general matching conditions*, Applicable Analysis **86** (2007), no. 6, 653–667.

Department of Mathematics, Saratov University, 83 Astrakhanskaya, Saratov, 410012, Russia

E-mail address: `yurkova@info.sgu.ru`

Received 12/04/2011