A Tight MIP Formulation of the Unit Commitment Problem with Start-up and Shut-down Constraints

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Abstract

This paper provides the convex hull description for the following basic operating constraints of a single power generation unit in Unit Commitment (UC) problems: 1) generation limits, 2) startup and shutdown capabilities, and 3) minimum up and down times. Although the model does not consider some crucial constraints, such as ramping, the proposed constraints can be used as the core of any UC formulation, thus tightening the final UC model. We provide evidence that dramatic improvements in computational time are obtained by solving a self-UC problem for different case studies.

Keywords: Unit Commitment (UC), Mixed-Integer Programming (MIP), Facet/Convex hull description.

1. Introduction

The short-term Unit Commitment problem requires to optimally operate a set of power generation units over a time horizon ranging from a day to a week. Despite significant improvements in Mixed-Integer Programming (MIP) solvers, the time required to solve Unit Commitment (UC) problems continues to be a critical limitation that restricts its size and scope. Nevertheless, improving the UC formulation can dramatically reduce its computational burden and so allow the implementation of more advanced and computationally demanding problems.

Ideally, an MIP problem can be reformulated so that the feasible region of the corresponding Linear Programming (LP) model becomes the convex hull of the feasible points. If this is possible, we could solve an MIP as an LP since each vertex is a point satisfying the integrality constraints and hence it always exists an optimal solution of the LP that

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is optimal for the corresponding MIP [14]. Unfortunately, in many practical problems there is an enormous number of inequalities needed to describe the convex hull, and the effort required to obtain them outweighs the computation needed to solve the original formulation of the MIP problem [14, 13]. For the UC case, however, it is possible to tighten the feasible region of the relaxed LP problem, consequently obtaining dramatic improvements in computation [14, 13, 11, 8, 7].

In particular, an UC formulation can be considerably tightened by providing the convex hull (or tight) description of some set of constraints. Even though other constraints in the problem might add some fractional vertices, this solution should be nearer to the optimal solution than would be the original model [14, 13]. Some efforts in tightening specific set of constraints have been done, such as: the convex hull of the minimum up and down times [5, 6, 12], cuts to tighten ramping limits [11], tighter approximation for quadratic generation costs [4], and simultaneously tight and compact description of thermal units operation [8, 7].

This paper further improves the work in Morales-Espana et al. [7] by providing the convex hull description for the following set of constraints: generation limits, startup and shutdown capabilities, and minimum up and down times. In addition, different case studies for a self-UC were solved as LP obtaining feasible MIP solutions; if compared with three other MIP formulations, the same optimal results were obtained but significantly faster.

The remainder of this paper is organized as follows. Section 2 introduces the main notation used to describe the proposed formulation. Section 3 details the basic operating constraints of a single generating unit. Section 4 contains the facet inducing and convex hull proofs for the proposed linear description of the self-UC subproblem. Section 5 provides and discusses results from several case studies, where a comparison with other three UC formulations is made. Finally, some relevant conclusions are drawn in Section 6.

2. Notation

Here we introduce the main notation used in this paper. Lowercase letters are used for denoting variables and indexes. Uppercase letters denote parameters.

2.1. Indexes

t Time periods, running from 1 to T hours.

2.2. Unit's Technical Parameters

$\overline{P}$ Maximum power output [MW].

$\underline{P}$ Minimum power output [MW].

SD Shutdown capability [MW].

SU Startup capability [MW].

TD Minimum down time [h].

TU Minimum up time [h].

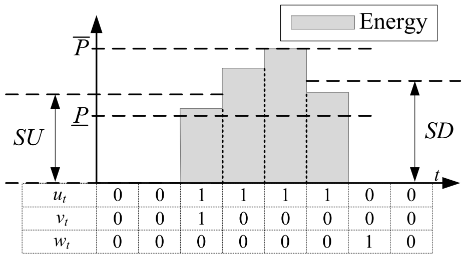

Figure 1: Unit's operation including its startup and shutdown capabilities

2.3. Continuous Decision Variables

p_t Power output of the unit for period t, production above the unit's minimum output $\underline{P}$ [MW].

2.4. Binary Decision Variables

u_t Commitment status of the unit for period t, which is equal to 1 if the unit is online and 0 offline

v_t Startup status of the unit, which takes the value of 1 if the unit starts up in period t and 0 otherwise.

w_t Shutdown status of the unit, which takes the value of 1 if the unit shuts down in period t and 0 otherwise.

3. Modelling the Unit's Operation

This section describes the mathematical formulation of the basic operation of a single generating unit in Unit Commitment (UC) problems. The following set of constraints are modelled: generation limits, minimum up and down times, and startup and shutdown capabilities. As shown in Figure 1, the startup capability SU is the maximum energy that a generating unit can produce when it starts up. Similarly, the unit should be producing bellow its shutdown capability SD when it shuts down. All these constraints are inherent to units' operation and they are included in recent Unit Commitment literature, see [1, 4, 11, 7, 9] and references therein for further details.

The unit's generation limits taking into account its maximum $\overline{P}$ and minimum $\underline{P}$ production, as well as its startup SU and shutdown SD capabilities are set as follows:

$$p_1 \leq \left(\overline{P} - \underline{P}\right) u_1 - \left(\overline{P} - SD\right) w_2 \quad (1)$$

$$p_t \leq \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - SU\right) v_t - \left(\overline{P} - SD\right) w_{t+1} \quad t \in [2, T-1] \quad (2)$$

$$p_T \leq \left(\overline{P} - \underline{P}\right) u_T - \left(\overline{P} - SU\right) v_T \quad (3)$$

It is important to highlight that the continuous decision variable p_t is the generation over $\underline{P}$. The total generation output can be obtained as $u_t\underline{P} + p_t$.

Be aware that (2) may be infeasible in the event that the unit is online for just one period. That is, $v_t = w_{t+1} = 1$ and the right side of (2) can be negative. Consequently, (2) is only valid for units with uptime $TU \geq 2$. Therefore, the correct formulation for units with $TU = 1$ is given by:

$$\begin{aligned} p_t \leq & \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - SD\right) w_{t+1} \\ & - \max\left(SD{-}SU, 0\right) v_t \quad \forall t \in [2, T-1] \qquad (4) \\ p_t \leq & \left(\overline{P} - \underline{P}\right) u_t - \left(\overline{P} - SU\right) v_t \\ & - \max\left(SU{-}SD, 0\right) w_{t+1} \quad \forall t \in [2, T-1]\,. \qquad (5) \end{aligned}$$

Note that if $SU{=}SD$ then (4)-(5) would be equivalent to the power limit constraints proposed in [7].

The logical relationship between the decision variables u_t, v_t and w_t; and the minimum uptime TU and downtime TD limits are ensured with

$$u_t - u_{t-1} = v_t - w_t \quad \forall t \in [2, T] \qquad (6)$$

$$\sum_{i=t-TU+1}^{t} v_i \leq u_t \quad \forall t \in [TU+1, T] \qquad (7)$$

$$\sum_{i=t-TD+1}^{t} w_i \leq 1 - u_t \quad \forall t \in [TD+1, T] \qquad (8)$$

where (6)-(8) are the constraints proposed in [12] to describe the convex hull formulation of the minimum-up and -down time constraints. Finally, the variable bounds are given by

$$0 \leq u_t \leq 1 \quad \forall t \qquad (9)$$

$$v_t \geq 0, \quad w_t \geq 0 \quad \forall t \in [2, T] \qquad (10)$$

$$p_t \geq 0 \quad \forall t. \qquad (11)$$

In summary, inequalities (1)-(3) together with (6)-(11) describe the operation for units with $TU \geq 2$; and (1) together with (3)-(11) for the cases in which $TU = 1$.

4. Strength of the Proposed Inequalities

In this section, we prove that inequalities (1)-(5) and (11) are facet defining.

Note that constraints (6) uniquely define the value of the variables w as a function of variables u and v. Unless differently specified, in the following, we will consider only the space defined by the variables u, v, and p. Moreover, we suppose that all constraints (1)-(5) and (7)-(11) are rewritten by substituting the w variables accordingly.

Definition 1. Let $\overline{C}_T\left(TU, TD, \overline{P}, \underline{P}, SU, SD\right)$ be the convex hull of the feasible integer solution for the problem. That is, for $TU \geq 2$, we write

$$\begin{aligned} \overline{C}_T \;\; & (TU \geq 2, TD, \overline{P}, \underline{P}, SU, SD) = \\ & conv\{(u, v, p) \in \{0, 1\}^{2T-1} \times R_+^T |\, (u, v, p) \text{ satisfy (1)-(3) and (7)-(11)}\}; \end{aligned}$$

	u_1	u_2	$\cdots$	u_{t-1}	u_t	u_{t+1}	$\cdots$	u_{T-1}	u_T	p_1	p_2	$\cdots$	p_{t-1}	p_t	p_{t+1}	$\cdots$	p_{T-1}	p_T	v_2	$\cdots$	v_{t-1}	v_t	v_{t+1}	$\cdots$	v_{T-1}	v_T	w_2	$\cdots$	w_{t-1}	w_t	w_{t+1}	$\cdots$	w_{T-1}	w_T
$x^{(1)}$	(1	0	⋯	0	0	0	⋯	0	0	g_t^x	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	1	⋯	0	0	0	⋯	0	0)
$x^{(2)}$	(1	1	⋯	0	0	0	⋯	0	0	g_t^x	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
$x^{(t-1)}$	(1	1	⋯	1	0	0	⋯	0	0	g_t^x	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	1	0	⋯	0	0)
$x^{(t)}$	(1	1	⋯	1	1	0	⋯	0	0	g_t^x	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	1	⋯	0	0)
$x^{(t+1)}$	(1	1	⋯	1	1	1	⋯	0	0	g_t^x	0	⋯	0	g_t	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
$x^{(T-1)}$	(1	1	⋯	1	1	1	⋯	1	0	g_t^x	0	⋯	0	g_t	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	1)
$x^{(T)}$	(1	1	⋯	1	1	1	⋯	1	1	g_t^x	0	⋯	0	g_t	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
$y^{(1)}$	(1	0	⋯	0	0	0	⋯	0	0	D	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	1	⋯	0	0	0	⋯	0	0)
$y^{(2)}$	(1	1	⋯	0	0	0	⋯	0	0	P	D	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
$y^{(t-1)}$	(1	1	⋯	1	0	0	⋯	0	0	P	P	⋯	D	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	1	0	⋯	0	0)
$y^{(t)}$	(1	1	⋯	1	1	0	⋯	0	0	P	P	⋯	P	D	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	1	⋯	0	0)
$y^{(t+1)}$	(1	1	⋯	1	1	1	⋯	0	0	P	P	⋯	P	g_t^y	D	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
$y^{(T-1)}$	(1	1	⋯	1	1	1	⋯	1	0	P	P	⋯	P	g_t^y	P	⋯	D	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	1)
$y^{(T)}$	(1	1	⋯	1	1	1	⋯	1	1	P	P	⋯	P	g_t^y	P	⋯	P	P	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
$z^{(1)}$	(0	1	⋯	1	1	1	⋯	1	1	0	U	⋯	0	g_t	0	⋯	0	0	1	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
$z^{(t-2)}$	(0	0	⋯	1	1	1	⋯	1	1	0	0	⋯	U	g_t	0	⋯	0	0	0	⋯	1	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
$z^{(t-1)}$	(0	0	⋯	0	1	1	⋯	1	1	0	0	⋯	0	g_t^z	0	⋯	0	0	0	⋯	0	1	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
$z^{(t)}$	(0	0	⋯	0	0	1	⋯	1	1	0	0	⋯	0	0	U	⋯	0	0	0	⋯	0	0	1	⋯	0	0	0	⋯	0	0	0	⋯	0	0)
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮
$z^{(T-2)}$	(0	0	⋯	0	0	0	⋯	1	1	0	0	⋯	0	0	0	⋯	U	0	0	⋯	0	0	0	⋯	1	0	0	⋯	0	0	0	⋯	0	0)
$z^{(T-1)}$	(0	0	⋯	0	0	0	…	0	1	0	0	⋯	0	0	0	…	0	U	0	⋯	0	0	0	…	0	1	0	…	0	0	0	…	0	0)
$z^{(T)}$	(0	0	⋯	0	0	0	…	0	0	0	0	⋯	0	0	0	…	0	0	0	⋯	0	0	0	…	0	0	0	…	0	0	0	…	0	0)
$q^{(t)}$	(0	0	⋯	0	1	0	…	0	0	0	0	⋯	0	g_t^q	0	…	0	0	0	⋯	0	1	0	…	0	0	0	…	0	0	1	…	0	0)
$y^{(T+1)}$	(1	1	⋯	1	1	1	⋯	1	1	P	P	⋯	P	g_t^y	P	⋯	P	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0	0	⋯	0	0)

Figure 2: $3T$ Affinely independent points for $g_t^x, g_t = 0$ and $g_t^z = U$, where $U = SU - \underline{P}$, $D = SD - \underline{P}$ and $P = \overline{P} - \underline{P}$.

for $TU = 1$, we write

$$\begin{aligned}\overline{C}_T \ &\big(TU = 1, TD, \overline{P}, \underline{P}, SU, SD\big) = \\ &conv\left\{(u, v, p) \in \{0,1\}^{2T-1} \times R_+^T \middle| \, (u, v, p) \ \text{ satisfy (1), (3)-(5), and (7)-(11)}\right\}.\end{aligned}$$

For short we write $\overline{C}_T$ for $\overline{C}_T\big(TU, TD, \overline{P}, \underline{P}, SU, SD\big)$, $\overline{C}_T(TU \geq 2)$ for $\overline{C}_T(TU \geq 2, TD, \overline{P}, \underline{P}, SU, SD)$, and $\overline{C}_T(TU = 1)$ for $\overline{C}_T(TU = 1, TD, \overline{P}, \underline{P}, SU, SD)$.

In order to simplify the proofs, we introduce the points $x^i, y^i, z^i \in \overline{C}_T$, as shown in Figure 2. For short, we introduce the parameters U, D, and P which are equivalent to $U = SU - \underline{P}$, $D = SD - \underline{P}$, and $P = \overline{P} - \underline{P}$, respectively.

Proposition 2. *$\overline{C}_T\big(TU, TD, \overline{P}, \underline{P}, SU, SD\big)$ is full-dimensional in terms of u, v and p.*

Proof. From Figure 2, it can be easily shown that the $3T$ points x^i, y^i, and z^i for $i \in [1, T]$ are affinely independent when $g_t^x = g_t = 0$, $g_t^y = P$, and $g_t^z = U$. Note that in case $D = 0$ the point $y^{(1)}$ must be replaced by $y^{(T+1)}$, thus keeping the $3T$ affinely independent points. This applies for all the following proofs; but for the sake of brevity, we assume in the following that $D \neq 0$. □

Proposition 3. *The inequalities (2) describe facets of the polytope $\overline{C}_T\big(TU \geq 2\big)$.*

Proof. We show that (2) describe facets of $\overline{C}_T\big(TU \geq 2\big)$ by the direct method [14]. We do so by presenting $3T-1$ affinely independent points in $\overline{C}_T\big(TU \geq 2\big)$ that are tight (i.e., that satisfy as an equality) for inequality (2). Note in Figure 2 that the point z^T (the origin) satisfies (1)-(5) and (11) as equality. Therefore, to get $3T-1$ affinely independent points, we need $3T-2$ other linearly independent points.

The following $3T-2$ points are linearly independent and tight for (2) when $g_t^x = 0$, $g_t = g_t^y = P$ and $g_t^z = U$: $T-1$ points x^i for $i \in [1, t-1] \cup [t+1, T]$, T points y^i for $i \in [1, T]$, and $T-1$ points z^i for $i \in [1, T-1]$. □

Proposition 4. *The inequalities* (4) *and* (5) *describe facets of the polytope* $\overline{C}_T\big(TU = 1\big)$.

Proof. As z^T (the origin) satisfies both (4) and (5) as equality, it suffices to show $3T-2$ linearly independent points that are tight for (4) and the same for (5). The following $3T-2$ points are linearly independent and tight for (4) when $g_t^x = 0$, $g_t = g_t^y = P$: $T-1$ points x^i for $i \in [1, t-1] \cup [t+1, T]$, T points y^i for $i \in [1, T]$, $T-2$ points z^i for $i \in [1, t-2] \cup [t, T-1]$, and one point $q^{(t)}$ where $g_t^q = D$ if $SD \leq SU$ and $g_t^q = U$ if $SD \geq SU$.

The following $3T-2$ points are linearly independent and tight for (5) when $g_t^x = 0$, $g_t = g_t^y = P$, and $g_t^z = U$: $T-1$ points x^i for $i \in [1, t-1] \cup [t+1, T]$, $T-1$ points y^i for $i \in [1, t-1] \cup [t+1, T]$, $T-1$ points z^i for $i \in [1, T-1]$, and one point $q^{(t)}$ where $g_t^q = D$ if $SD \leq SU$ and $g_t^q = U$ if $SD \geq SU$. □

Proposition 5. *The inequalities* (1) *and* (3) *describe facets of the polytope* $\overline{C}_T$.

Proof. As z^T (the origin) satisfies both (1) and (3) as equality, it suffices to provide a set of $3T-2$ linearly independent points that are tight for each of the above inequalities. The following $3T-2$ points are linearly independent and tight for (1) when $g_t = 0$, $g_t^x = g_t^y = P$ and $g_t^z = U$: $T-1$ points x^i for $i \in, [2, T]$, T points y^i for $i \in [1, T]$, and $T-1$ points z^i for $i \in [1, T-1]$.

The following $3T-2$ points are linearly independent and tight for (3) when $g_t^x = 0$, $g_t = g_t^y = P$, and $g_t^z = U$: $T-1$ points x^i for $i \in [1, T]$, T points y^i for $i \in [1, T]$, and $T-1$ points z^i for $i \in [1, T-1]$. □

Proposition 6. *The inequality* (11) *describes a facet of the polytope* $\overline{C}_T$.

Proof. The point z^T satisfies the inequality (11) as equality. So, as above discussed, it suffices to show $3T-2$ linearly independent solutions that are tight for (11). The following $3T-2$ points are linearly independent and tight for (11) when $g_t = g_t^x = g_t^y = g_t^z = 0$: T points x^i for $i \in [1, T]$, $T-1$ points y^i for $i \in [1, t-1] \cup [t+1, T]$, and $T-1$ points z^i for $i \in [1, T-1]$. □

Summing up (1)-(5) and (11) describe facets of $\overline{C}_T$. Finally, we prove that the inequalities (1)-(11) are sufficient to describe the convex hull of the feasible solutions.

We need a preliminary lemma.

Lemma 7. *Let* $P = \{x \in \mathbb{R}^n | Ax \leq b\}$ *be an integral polyhedron, i.e,* $P = conv(P \cap \mathbb{Z}^n)$. *Define* $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | x \in P, 0 \leq y_i \leq c_i x, i = 1, \ldots, k,\ y_i = d_i x, i = k+1, \ldots, m\}$, *where* $1 \leq k \leq m$, $c_i, d_i \in \mathbb{R}^n$, *and* $c_i x \geq 0$, $d_i x \geq 0$ *for* $i = 1, \ldots, m$ *and for each* $x \in P$. *Then every vertex* $(\tilde{x}, \tilde{y})$ *of* Q *has the property that* $\tilde{x}$ *is integral.*

Proof. Suppose by contradiction that there exists a vertex $(\tilde{x}, \tilde{y})$ of Q such that $\tilde{x}$ is not integral. Then $\tilde{x}$ is not a vertex of P and therefore there exist $\bar{x}^1, \bar{x}^2 \in P$ such that $\tilde{x} = \frac{1}{2}\bar{x}^1 + \frac{1}{2}\bar{x}^2$. Moreover, $\tilde{y}_i = c_i\tilde{x}$ for $i = 1, \ldots, k$, indeed if there exists r, $1 \leq r \leq k$, such that $0 \leq \tilde{y}_r < c_r\tilde{x}$, then $(\tilde{x}, \tilde{y})$ is a convex combination of the point $(\tilde{x}, \hat{y})$ and the point $(\tilde{x}, \check{y})$, where $\hat{y}_r = c_r\tilde{x}$, $\check{y}_r = 0$, and $\hat{y}_i = \check{y}_i = \tilde{y}_i$ for $1 \leq i \leq m$, $i \neq r$.

For $j = 1, 2$, let $\bar{y}_i^j = c_i\bar{x}^j$ for $i = 1, \ldots, k$ and $\bar{y}_i^j = d_i\bar{x}^j$ for $i = k+1, \ldots, m$. Then $(\tilde{x}, \tilde{y}) = \frac{1}{2}(\bar{x}^1, \bar{y}^1) + \frac{1}{2}(\bar{x}^2, \bar{y}^2)$, i.e., $(\tilde{x}, \tilde{y})$ is a convex combination of $(\bar{x}^1, \bar{y}^1)$ and $(\bar{x}^2, \bar{y}^2)$. Contradiction. □

Theorem 8. *Let* $\overline{D}_T\big(TU, TD, \overline{P}, \underline{P}, SU, SD\big)$ *be a polyhedron defined as follows:*

- *for* $TU \geq 2$

$$\overline{D}_T\ \big(TU \geq 2, TD, \overline{P}, \underline{P}, SU, SD\big) = \big\{(u, v, p) \in [0,1]^{2T-1} \times R_+^T \,|\, (u, v, p) \text{ satisfy } (1)\text{-}(3) \text{ and } (7)\text{-}(11)\ \big\};$$

- *for* $TU = 1$

$$\overline{D}_T\ \big(TU = 1, TD, \overline{P}, \underline{P}, SU, SD\big) = \big\{(u, v, p) \in [0,1]^{2T-1} \times R_+^T \,|\, (u, v, p) \text{ satisfy } (1),\ (3)\text{-}(5),\text{ and } (7)\text{-}(11)\ \big\}.$$

Then $\overline{C}_T\big(TU, TD, \overline{P}, \underline{P}, SU, SD\big) = \overline{D}_T\big(TU, TD, \overline{P}, \underline{P}, SU, SD\big)$.

Proof. As for $\overline{C}_T$, we use short notations $\overline{D}_T$, $\overline{D}_T\big(TU \geq 2\big)$, and $\overline{D}_T\big(TU = 1\big)$. The proof for $TU \geq 2$ easily follows from Lemma 7. Indeed, $\overline{D}_T\big(TU \geq 2\big)$ is described by the inequalities (6)-(10), that describe an integral polyhedron in u and v as proved in [12], together with inequalities (1)-(3) and (11) satisfying the hypothesis of Lemma 7.

For $TU = 1$ let us suppose that $SU \geq SD$. We follow Approach 8 in [14] (see Section 9.2.3, Problem 2, Approach 8). We first introduce an extended formulation of the problem, then we prove that the extended formulation is integral, and finally we prove that the projection of the new polyhedron correspond to $\overline{D}_T\big(TU = 1\big)$. To accomplish to this task we need to prove some preliminary claims. We define the following new binary variables for $t = 2, \ldots, T-1$:

- $x_t = 1$ if and only if $v_t = 1$ and $w_{t+1} = 1$,
- $\tilde{v}_t = 1$ if and only if $v_t = 1$ and $w_{t+1} = 0$,
- $\tilde{w}_{t+1} = 1$ if and only if $v_t = 0$ and $w_{t+1} = 1$,
- $\tilde{u}_t = 1$ if and only if $u_t = 1$, $v_t = 0$, and $w_{t+1} = 0$.

Moreover, $\tilde{u}_T = 1$ if and only if $u_T = 1$ and $v_T = 0$.

Claim 1. The polyhedron P defined by the points $(u, v, w, \tilde{u}, \tilde{v}, \tilde{w}, x)$ satisfying the following inequalities is integral:

$$v_t \leq u_t \quad t = 2, \ldots, T \tag{12}$$

$$\sum_{i=t-TD+1}^{t} w_i \leq 1 - u_t \quad t \in [TD+1, T] \tag{13}$$

$$u_t - u_{t-1} = v_t - w_t \quad t \in [2, T] \tag{14}$$

$$w_{t+1} = \tilde{w}_{t+1} + x_t \quad t \in [2, T-1] \tag{15}$$

$$v_t = \tilde{v}_t + x_t \quad t \in [2, T-1] \tag{16}$$

$$u_t = \tilde{v}_t + \tilde{w}_{t+1} + x_t + \tilde{u}_t \quad t \in [2, T-1] \tag{17}$$

$$u_T = v_T + \tilde{u}_T \tag{18}$$

$$0 \leq u_t \leq 1 \quad t \in [1, T] \tag{19}$$

$$v_t \geq 0 \quad t \in [2, T] \tag{20}$$

$$w_t \geq 0 \quad t \in [2, T] \tag{21}$$

$$\tilde{v}_t, x_t \geq 0 \quad t \in [2, T-1] \tag{22}$$

$$\tilde{w}_t \geq 0 \quad t \in [3, T] \tag{23}$$

$$\tilde{u}_t \geq 0 \quad t \in [2, T] \tag{24}$$

Proof of Claim 1. The proof is carried on by showing that the coefficient matrix associated with the above linear system is totally unimodular.

We exploit this well-known property (proved by Ghouila-Houri, see [10], Chapter III.1, Theorem 2.7): let A be a $\{0, 1, -1\}$-matrix, if each subset J of columns of A can be partitioned into J_1 and J_2 such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1 \tag{25}$$

for each row i, then A is totally unimodular. This part of the proof has been inspired by the proof of Malkin [6] for the polyhedron defined by minimum-up and down-time constraints.

First we assign the variables $w_i \in J$ alternatively to J_1 and to J_2 in lexicographic order. Then the variables $u_t \in J$ are assigned either to J_1 if $w_k \in J_2$, where $k = \max\{i | 1 \leq i \leq t, w_i \in J\}$, or to J_2 if $w_k \in J_1$, or to the same set with respect to u_{t-1} if $\{i | 1 \leq i \leq t, w_i \in J\}$ is empty. Thus condition (25) is satisfied for constraints (13).

Variables $v_t \in J$ are assigned either to J_1 if $u_t \in J_1$, or to J_2 if $u_t \in J_2$, or to the opposite set with respect to u_{t-1} if $u_t \notin J$, or to the same set as w_t if both $u_{t-1}, u_t \notin J$. This ensures that condition (25) is satisfied for constraints (12) and (14).

If $v_t, w_{t+1} \in J$, then assign $\tilde{v}_t \in J$ to the same subset as v_t, $x_t \in J$ to the opposite set with respect to $\tilde{v}_t$, and $\tilde{w}_t \in J$ to the same subset as w_t. These assignments guarantee that condition (25) is satisfied for constraints (15) and (16) both in the case that v_t and w_{t+1} are in the same set or in different sets. Moreover, the assignment for $\tilde{u}_t$ can be chosen to satisfy condition (25) for constraints (17). If one between v_t and w_{t+1} does not belong to J then proceed as follows: suppose w.l.o.g. that $v_t \notin J$, then assign w_{t+1}, $\tilde{w}_{t+1}$, and $\tilde{v}_t$ to the same set and x_t to the other set, then $\tilde{u}_t$ can be chosen to satisfy condition (25) for constraints (17). Similar choices can be done if some of the variables $\tilde{v}_t$, $\tilde{w}_{t+1}$, x_t, $\tilde{u}_t$ do not belong to J and the claim follows. **End of Claim 1.**

Then we define the polyhedron $\tilde{Q}$ by adding to the linear system defining P the following inequalities:

$$p^v_t \leq (SU - \underline{P})\tilde{v}_t \quad t \in [2, T-1] \tag{26}$$
$$p^x_t \leq (SD - \underline{P})x_t \quad t \in [2, T-1] \tag{27}$$
$$p^w_t \leq (SD - \underline{P})\tilde{w}_{t+1} \quad t \in [2, T-1] \tag{28}$$
$$p^u_t \leq (\overline{P} - \underline{P})\tilde{u}_t \quad t \in [2, T] \tag{29}$$
$$p^v_T \leq (SU - \underline{P})v_T \tag{30}$$
$$p_1 \leq (\overline{P} - \underline{P})u_1 - (\overline{P} - SD)w_2 \tag{31}$$

where p^v, p^x, p^w, p^u and p_1 are non-negative variables.

Claim 2. The polyhedron $\tilde{Q}$ is integral with respect to variables u, v, w, x, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$. **End of Claim 2.**

The proof of Claim 2 is a direct application of Lemma 7 to the polyhedron P of Claim 1.

Then we define the polyhedron Q by adding to the linear system defining $\tilde{Q}$ the following inequalities

$$p_t = p^v_t + p^x_t + p^w_t + p^u_t \qquad t \in [2, \ldots, T-1] \tag{32}$$
$$p_T = p^v_T + p^u_T \tag{33}$$

where p_t for $t \in [2 \ldots T]$ are non-negative variables.

Claim 3. The polyhedron Q is integral with respect to variables u, v, w, x, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$. **End of Claim 3.**

Claim 3 follows from Claim 2 and by the straightforward extension of Lemma 7, where the role of P is played by the integral polyhedron $\tilde{Q}$.

Finally we prove that

Claim 4. The projection of Q onto the space of the variables u, v, p is equivalent to $\overline{D}_T$.

Proof of Claim 4. We start by eliminating the variables p^v_t, p^x_t, p^w_t, and p^u_t by simply substituting constraints (32)-(33) with the following:

$$\begin{aligned} p_t \leq& (SU - \underline{P})\tilde{v}_t + (SD - \underline{P})x_t + \\ &+ (SD - \underline{P})\tilde{w}_{t+1} + (\overline{P} - \underline{P})\tilde{u}_t \qquad t \in [2, T-1] \end{aligned} \tag{34}$$
$$p_T \leq (SU - \underline{P})v_T + (\overline{P} - \underline{P})\tilde{u}_T, \tag{35}$$

which are obtained by using constraints (26)-(30).

Now, we replace $\tilde{u}_T$ from (18) in (35) to obtain:

$$p_T \leq \left(\overline{P} - \underline{P}\right) u_T - \left(\overline{P} - SU\right) v_T \tag{36}$$

then we eliminate variables in (34) according to the following order

- $\tilde{u}_t$ by using the equation (17);

- $\tilde{w}_{t+1}$ by using the equation (15);
- $\tilde{v}_t$ by using the equation (16).

It is not difficult to see that for $t \in [2, T-1]$ we obtain the following constraints:

$$p_t \leq (\overline{P} - \underline{P})u_t - (\overline{P} - SU)v_t$$
$$- (\overline{P} - SD)w_{t+1} + (\overline{P} - SU)x_t \quad (37)$$
$$x_t \geq 0 \quad (38)$$
$$x_t \geq v_t + w_{t+1} - u_t \quad (39)$$
$$x_t \leq v_t \quad (40)$$
$$x_t \leq w_{t+1}. \quad (41)$$

Now we can apply Fourier-Motzkin elimination to variables x_t by considering the following pairs of constraints:

- by constraints (40) and (37) we obtain
$$p_t \leq (\overline{P} - \underline{P})u_t - (\overline{P} - SD)w_{t+1}; \quad (42)$$
- by constraints (40) and (38) we obtain $v_t \geq 0$;
- by constraints (40) and (39) we obtain
$$w_{t+1} \leq u_t; \quad (43)$$
- by constraints (41) and (37) we obtain
$$p_t \leq (\overline{P} - \underline{P})u_t - (\overline{P} - SU)v_t - (SU - SD)w_{t+1}; \quad (44)$$
- by constraints (41) and (38) we obtain $w_{t+1} \geq 0$;
- by constraints (41) and (39) we obtain $u_t \geq v_t$.

By using equation (14), $w_{t+1} \leq u_t$ is equivalent to $v_{t+1} \leq u_{t+1}$, which is one of the inequalities (12). We can simply see that the new constraints (42) and (44) coincide with constraints (4) and (5) for the case $SU \geq SD$, respectively; and constraints (31) and (36) coincide with constraints (1) and (3), respectively. **End of Claim 4.**

From Claim 4 it follows that $\overline{D}_T$ is integral with respect to the variables u and v. The proof for $SD \geq SU$ can be performed in a symmetric way.

□

5. Numerical Results

To illustrate the computational performance of the Tight and Compact formulation proposed in this paper, the self-UC problem for a price-taker producer is solved for different time spans. The goal of a price-taker producer is to maximize his profit during the planning horizon (which is the difference between the revenue and the total operating

Table 1: Generator Data

	Technical Information							Cost Coefficients†		
Gen	$\overline{P}$ [MW]	$\underline{P}$ [MW]	TU/TD [h]	SU [MW]	SD [MW]	p_0* [MW/h]	Ste$_0\star$ [h]	C^{NL} [$/h]	C^{LV} [$/MWh]	C^{SU} [$]
1	455	150	8	252	303	150	8	1000	16.19	9000
2	455	150	8	252	303	150	8	970	17.26	10000
3	130	20	5	57	75	20	5	700	16.60	1100
4	130	20	5	57	75	20	5	680	16.50	1120
5	162	25	6	71	94	25	6	450	19.70	1800
6	80	20	3	40	50	20	3	370	22.26	340
7	85	25	3	45	55	25	3	480	27.74	520
8	55	10	1	25	33	10	1	660	25.92	60
9	55	10	1	25	33	10	1	665	27.27	60
10	55	10	1	25	33	10	1	670	27.79	60

* p_0 is the unit's initial production prior to the first period of the time span.

$\star$Ste$_0$ is the number of hours that the unit has been online prior to the first period of the time span.

†C^{NL}, C^{LV} and C^{SU} stand for non-load, linear-variable and startup costs, respectively.

Table 2: Energy Price ($/MWh)

$t = 1 \ldots 12 \rightarrow$	13.0	7.2	4.6	3.3	3.9	5.9	9.8	15.0	22.1	31.3	33.2	24.8
$t = 13 \ldots 24 \rightarrow$	19.5	16.3	14.3	13.7	15.0	17.6	20.2	29.3	49.5	53.4	30.0	20.2

cost [8]). The self-UC is also associated with the scheduling problem of a single generation unit [2], which arises when solving UC with decomposition methods such as Lagrangian Relaxation [3]. The 10-unit system data is presented in Table 1 and the energy prices are shown in Table 2. The power system data are based on information presented in [1, 7]. All tests were carried out using CPLEX 12.5 on an Intel-i7 3.4-GHz personal computer with 8 GB of RAM memory. The problems are solved until they hit the time limit of 10000 seconds or until they reach optimality (more precisely to 10^{-6} of relative optimality tolerance).

The formulation presented in this paper, labelled as *TC*, is compared with the previous Tight and Compact formulation presented in [7], labelled as *TC0,* and with those in [1] and [11], labelled as *1bin* and *3bin*, respectively.

Table 3 shows the computational performance for four cases with different time spans.

Table 3: Computational Performance Comparison

Case	Optimum	IntGap (%)				LP time (s)				MIP time (s)*				Nodes			
(days)	(M$)	*TC*	*TC0*	*3bin*	*1bin*	*TC*	*TC0*	*3bin*	*1bin*	*TC*	*TC0*	*3bin*	*1bin*	*TC*	*TC0*	*3bin*	*1bin*
64	7.259361	0	0.09	0.88	2.57	0.57	0.47	0.80	0.95	0.57	1.92	12.01	13.79	0	0	496	487
128	14.517096	0	0.09	0.87	2.57	1.17	1.20	2.06	2.60	1.17	4.81	45.54	(3.33E-4)	0	0	528	603915
256	29.032567	0	0.09	0.87	2.57	3.16	3.29	5.38	6.88	3.16	7.75	199.18	(5.21E-4)	0	0	533	229035
512	58.063509	0	0.09	0.87	2.57	8.08	8.39	14.29	18.83	8.08	17.29	734.03	(5.35E-4)	0	0	488	136128

* If the time limit is reached then the final optimality tolerance is shown between parentheses

Table 4: Problem Size Comparison

Case	# constraints			# nonzero elements				# real var		# binary var	
(days)	*TC**	*3bin*	*1bin*	*TC*	*TC0*	*3bin*	*1bin*	*TC*†	*1bin*	*TC*†	*1bin*
64	65997	107459	138225	338994	334389	417313	469719	15360	46080	46080	15360
128	132045	214979	276465	678450	669237	835105	939735	30720	92160	92160	30720
256	264141	430019	552945	1357362	1338933	1670689	1879767	61440	184320	184320	61440
512	528333	860099	1105905	2715186	2678325	3341857	3759831	122880	368640	368640	122880

* *TC* is equal to *TC0* for these cases

† *TC, TC0* and *3bin* are equal for these cases

All formulations achieve the same MIP optimum since all of them model the same MIP problem. However, they present different LP optimums, the relative distance between their MIP and LP optimums is measured with the Integrality Gap [13, 7]. Note that the MIP optimums of *TC* were achieved by just solving the LP over (1)-(11), IntGap$=0$, hence solving the problems in LP time. On the other hand, as usual, the branch-and-cut method was needed to solve the MIP for *TC0*, *3bin* and *1bin*. Table 3 also shows the MIP time and nodes explored that were required by the different formulations to reach optimality. It is interesting to note that although *TC0* reached optimality exploring zero nodes, *TC0* needed to make use of the solver's cutting planes strategy because the relaxed LP solution did not achieve the integer one, IntGap$\neq 0$ (the solver used 227 and 1224 cuts for the smallest and largest case, respectively). This tightening process took more time than the time required to solve the initial LP relaxation, that is why the MIP time for *TC0* is more than twice its LP relaxation time.

Table 4 shows the dimensions for all of the formulations for four selected instances. Note that *TC* and *TC0* are more compact, in terms of quantity of constraints and nonzero elements, than *3bin* and *1bin*. The formulation *1bin* presents a third of binary variables in comparison with the other formulations, but 3 times more continuous variables. This is because the work in [1] reformulated the units' operation model to avoid the startup and shutdown binary variables, claiming that this would reduce the node enumeration in the branch-and-bound process. Note however that this reformulation considerable damaged the strength of *1bin*, hence it presented the worst computational performance, similar results are obtained in [11, 7]. The formulation *1bin* presents more continuous variables than the other formulations because it requires the introduction of new continuous variables to model the startup and shutdown costs of generating units.

In conclusion, *TC* presents a dramatic improvement in computation in comparison with *3bin* and *1bin* due to its tightness (speedups above 90x and 8500x, respectively); and it also presents a lower LP burden due to its compactness, see Table 4. Compared with *TC0*, the formulation *TC* is tighter; consequently, *TC* requires less time to solve the MIP problem (speedup above 4.1x).

6. Conclusion

This paper presented the convex hull description of the basic constraints of generating units for unit commitment (UC) problems. These constraints are: generation limits, startup and shutdown capabilities, and minimum up and down times. The model does not include some crucial constraints, such as ramping, but the proposed constraints can be used as the core of any UC formulation and they can help to tighten the final UC model. Finally, different case studies for a self-UC were solved as LP obtaining MIP solutions; if compared with three other formulations, the same optimal results were obtained but significantly faster.

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