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Quasilinear and singular elliptic systems

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Abstract. In this paper, we investigate the following quasilinear elliptic and singular system $(\mathbf{P})$:

$$-\Delta_p u = f_1(x,u,v) \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega,$$

$$-\Delta_q v = f_2(x,u,v) \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega,$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $1 < p,q < \infty$ and $f_1, f_2 \in \mathscr{C}^1(\Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$ two positive functions. Under suitable conditions on f_1 and f_2, we first give a general result on the existence of positive weak solutions pairs $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ to $(\mathbf{P})$. Next, we give some applications to Biology.

Keywords. Quasilinear singular elliptic systems, weak comparison principle, sub and super solutions, cone condition, Schauder fixed point Theorem..

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1 Introduction

In this paper we are interested in the following quasilinear elliptic and singular system,

$$(\mathbf{P}) \begin{cases} -\Delta_p u = f_1(x,u,v) & \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ -\Delta_q v = f_2(x,u,v) & \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega. \end{cases}$$

Here, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ with $\mathscr{C}^2$ boundary $\partial\Omega$, $\Delta_r u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{r-2}\nabla u)$ denotes the r-Laplace operator and $1 < p,q < \infty$. In the right-hand sides, f_1 and f_2 are two Carathéodory functions in $\Omega \times \left(\mathbb{R}_+^* \times \mathbb{R}_+^*\right)$ possibly singular. More precisely, for every $(t_1,t_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and for almost every $x \in \Omega$, we assume that

$(\boldsymbol{H_1})$ $f_1(\cdot,t_1,t_2)$ and $f_2(\cdot,t_1,t_2)$ are Lebesgue measurable in Ω,

$(\boldsymbol{H_2})$ $f_1(x,\cdot,\cdot)$ and $f_2(x,\cdot,\cdot)$ are in $\mathscr{C}^1(\mathbb{R}_+^* \times \mathbb{R}_+^*)$.

We aim to establish the existence of a positive weak solutions pair to problem $(\mathbf{P})$ using the Schauder Fixed Point Theorem. Namely, if we can compose two order-reversing mappings,

$$(u,v) \mapsto T_1(u,v) \stackrel{\text{def}}{=} \tilde{u} \quad \text{and} \quad (u,v) \mapsto T_2(u,v) \stackrel{\text{def}}{=} \tilde{v}, \tag{1.1}$$

where $\tilde{u} \in \mathrm{W}_0^{1,p}(\Omega)$ and $\tilde{v} \in \mathrm{W}_0^{1,q}(\Omega)$ are defined to be the (unique) positive weak solution to the Dirichlet problems

$$-\Delta_p \tilde{u} + h_1(x,\tilde{u}) = f_1(x,u,v) + h_1(x,u) \text{ in } \Omega; \quad \tilde{u}|_{\partial\Omega=0}, \quad \tilde{u} > 0 \quad \text{in } \Omega, \tag{1.2}$$

$$-\Delta_q \tilde{v} + h_2(x,\tilde{v}) = f_2(x,u,v) + h_2(x,v) \text{ in } \Omega; \quad \tilde{v}|_{\partial\Omega=0}, \quad \tilde{v} > 0 \quad \text{in } \Omega, \tag{1.3}$$

respectively, in suitable conical shells of positive cones in $\mathrm{W}_0^{1,p}(\Omega)$ and $\mathrm{W}_0^{1,q}(\Omega)$, with appropriate functions h_1 and h_2; then any fixed point of the mapping

$$(u,v) \mapsto T(u,v) \stackrel{\text{def}}{=} (T_1(u,v), T_2(u,v)) \tag{1.4}$$

is a positive weak solution pair to $(\mathbf{P})$ and conversely. To prove that T is well defined and invariant in some conical shell, we use monotonicity methods together with the existence of sub- and supersolutions which prescribe the behaviour of the right-hand side singular non-linearities, namely f_1 and f_2, near the boundary $\partial\Omega$. The continuity and the compactness in $\mathscr{C}^{0,\alpha}(\overline{\Omega}) \times \mathscr{C}^{0,\alpha}(\overline{\Omega})$ for some suitable $0 < \alpha < 1$ follow from the regularity result Theorem 1.1 in [11] we recall in the appendix (see Theorem A.1). We derive further uniqueness results in case where the system $(\mathbf{P})$ is competitive or cooperative (see Theorem 3.8). To establish the uniqueness of a positive pair of solutions to $(\mathbf{P})$, it is essential that the mapping T is *subhomogeneous*. In the cooperative and "strong" singular case, we also prove the existence of very weak solutions in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ (see Theorem 2.3).

Quasilinear elliptic systems have been quite intensely investigated in the literature with various methods. In [23], the authors take advantage of the variational structure of the problem to apply variational methods. In [3], a blow up argument combined with a Liouville theorem yields universal a priori bounds. Then, the existence of solutions is obtained by a topological degree argument (see also the review article [5]). In [4], the key ingredients to prove existence of solutions are the Strong Comparison Principle and Kreǐn-Rutman theorem for homogeneous non-linear mapping. While dealing with subhomogeneous systems, one usually appeals the method of sub and supersolutions.

Related problems for *singular* quasilinear systems have been also studied in [16] and [12]. Accordingly, we study in our paper a more general situation that handle more singular cases. We point out additionally that in the present work non-linearities f_1 and f_2 are not necessary non-negative.

The case of singular semi-linear systems ($p = q = 2$) has been studied even more frequently in [1], [2], [21], [14], [15] and [8]. We refer to [14] for additional references on the subject.

Throughout this paper, we will use the following notations and definitions:

(i) To $r \in (1, +\infty)$ we associate $r' \stackrel{\text{def}}{=} \frac{r}{r-1} > 1$ and we denote by $\mathrm{W}^{-1,r'}(\Omega)$ the dual space of $\mathrm{W}_0^{1,r}(\Omega)$ with respect to the standard inner product in $\mathrm{L}^2(\Omega)$.

(ii) We denote by $d(x) \stackrel{\text{def}}{=} \inf_{y\in\partial\Omega} d(x,y)$, the distance from $x \in \Omega$ to $\partial\Omega$.

(iii) We denote by $D \stackrel{\text{def}}{=} \sup_{x,y\in\Omega} d(x,y)$, the diameter of the domain Ω.

(iv) Let $f, g : \Omega \longrightarrow [0, +\infty]$ be two functions of $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$. Then, we write

$$f(x) \sim g(x) \qquad \text{in } \Omega$$

if there exist two positive constants C_1 and C_2 such that for almost every $x \in \Omega$,

$$C_1 g(x) \le f(x) \le C_2 g(x).$$

(v) The function $\varphi_{1,r} \in \mathrm{W}_0^{1,r}(\Omega)$ denotes the positive and L^r-renormalized eigenfunction corresponding to the first eigenvalue of $-\Delta_r$,

$$\lambda_{1,r} \stackrel{\text{def}}{=} \inf\left\{ \int_\Omega |\nabla v|^r dx \in \mathbb{R}, \quad v \in \mathrm{W}_0^{1,r}(\Omega) \quad \text{and} \quad \int_\Omega |v|^r dx = 1 \right\}.$$

It is a weak solution of the following eigenvalue problem:

$$-\Delta_r w = \lambda_{1,r} w^{r-1} \text{ in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega.$$

Using Moser iterations, $\varphi_{1,r} \in \mathrm{L}^\infty(\Omega)$ and using the Hölder regularity result in LIEBERMAN [19], $\varphi_{1,r} \in \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$ for some $0 < \alpha < 1$. Moreover the strong maximum and boundary principles from VÁSQUEZ [24], guarantee that $\varphi_{1,r}$ satisfies

$$\varphi_{1,r}(x) \sim d(x) \quad \text{in } \Omega. \tag{1.5}$$

(vi) We say that a Lebesgue measurable function $f : \Omega \to \mathbb{R}$ is *locally uniformly positive* if $\operatorname{essinf}_K f > 0$ holds over every compact set $K \subset \Omega$.

(vii) In this paper, we primarily look for *positive weak solution pairs* (*positive solutions*, for short) of problem $(\mathbf{P})$, that is, pairs of functions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ with both u and v locally uniformly positive and each satisfying the respective equation in problem $(\mathbf{P})$ in the weak sense. More precisely, given $1 < r < \infty$ and $f \in \mathrm{W}^{-1,r'}(\Omega)$, we say that a function $u \in \mathrm{W}_0^{1,r}(\Omega)$ satisfies the equation

$$-\Delta_r u = f \quad \text{in } \Omega \tag{1.6}$$

in the weak sense if u is locally uniformly positive and satisfies

$$\forall w \in \mathrm{W}_0^{1,r}(\Omega), \quad \int_\Omega |\nabla u|^{r-2}\nabla u.\nabla w\, dx = \langle f, w\rangle_{\mathrm{W}^{-1,r'}(\Omega)\times \mathrm{W}_0^{1,r}(\Omega)}.$$

In the case where the existence of positive solutions of $(\mathbf{P})$ cannot be established, we discuss the existence of weaker solutions. Then, we say that $(u,v) \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega) \times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ is a *positive very weak solution pair* of $(\mathbf{P})$ if both u and v are locally uniformly positive and satisfy the respective equation in problem $(\mathbf{P})$ in the sense of distributions.

In the three last points, for $1 < r < +\infty$, $\mathscr{A}_r(\Omega)$ represents the space $\mathrm{W}_0^{1,r}(\Omega)$ or the space $\mathrm{W}^{1,r}_{\mathrm{loc}}(\Omega)$.

(viii) Let $\underline{w}, \overline{w} \in \mathscr{A}_r(\Omega)$, two locally uniformly positive functions such that $\underline{w} \leq \overline{w}$ a.e. in Ω. We define the convex set

$$[\underline{w}, \overline{w}] \stackrel{\text{def}}{=} \left\{ w \in \mathscr{A}_r(\Omega) \cap \mathscr{C}\left(\overline{\Omega}\right), \quad \underline{w} \leq w \leq \overline{w} \quad \text{a.e. in } \Omega \right\}.$$

(ix) Let $\underline{u}, \overline{u} \in \mathscr{A}_p(\Omega)$ and $\underline{v}, \overline{v} \in \mathscr{A}_q(\Omega)$ four locally uniformly positive functions such that $\underline{u} \leq \overline{u}$ a.e. in Ω and $\underline{v} \leq \overline{v}$ a.e. in Ω. The couples $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are said to be *sub and supersolutions* pairs to $(\mathbf{P})$ if the following inequalities are satisfied in the distribution sense

$$-\Delta_p\, \underline{u} \leq f_1\left(x, \underline{u}, v\right) \quad \text{in } \Omega, \quad \text{for any } v \in [\underline{v}, \overline{v}], \tag{1.7}$$

$$-\Delta_q\, \underline{v} \leq f_2\left(x, u, \underline{v}\right) \quad \text{in } \Omega, \quad \text{for any } u \in [\underline{u}, \overline{u}], \tag{1.8}$$

$$-\Delta_p\, \overline{u} \geq f_1\left(x, \overline{u}, v\right) \quad \text{in } \Omega, \quad \text{for any } v \in [\underline{v}, \overline{v}], \tag{1.9}$$

$$-\Delta_q\, \overline{v} \geq f_2\left(x, u, \overline{v}\right) \quad \text{in } \Omega, \quad \text{for any } u \in [\underline{u}, \overline{u}]. \tag{1.10}$$

(x) Let$(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in \mathscr{A}_p(\Omega) \times \mathscr{A}_q(\Omega)$ be respectively sub and supersolutions pairs to $(\mathbf{P})$. Then, the conical shell $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ is denoted by $\mathcal{C}$.

The paper is organised as follows. The next section (Section 2) contains the statements and the proofs of our main results (Theorems 2.1 and Theorems 2.3). Different applications of Theorems 2.1 and 2.3 arising in population dynamics models are given in Section 3. The appendix contains the regularity result (Theorem A.1) used to prove Hölder continuity of solutions. Theorem A.1 is proved in [11].

2 General results

Theorem 2.1. *Let* $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *be sub and supersolutions pairs to* $(\mathbf{P})$ *and assume in addition that the following conditions hold:*

(i) *there exist constants* $k_1, k_2 > 0$ *and* $\delta_1, \delta_2 \in \mathbb{R}$ *such that*

$$|f_1(x,u,v)| \le k_1 d(x)^{\delta_1} \quad \textit{and} \quad |f_2(x,u,v)| \le k_2 d(x)^{\delta_2} \quad \textit{in } \Omega \times \mathcal{C}, \tag{2.1}$$

(ii) *there exist constants* $C_1, C_2 > 0$ *and* $b_1, b_2 > 0$ *such that*

$$\overline{u} \le C_1 d(x)^{b_1} \quad \textit{and} \quad \overline{v} \le C_2 d(x)^{b_2} \quad \textit{in } \Omega, \tag{2.2}$$

(iii) *and there exist* $\kappa_1, \kappa_2 > 0$ *and* $\alpha_1, \alpha_2 > 0$ *such that*

$$\left| \frac{\partial f_1}{\partial u}(x,u,v) \right| \le \kappa_1 d(x)^{\delta_1 - \alpha_1} \quad \textit{in } \Omega \times \mathcal{C}, \tag{2.3}$$

$$\left| \frac{\partial f_2}{\partial v}(x,u,v) \right| \le \kappa_2 d(x)^{\delta_2 - \alpha_2} \quad \textit{in } \Omega \times \mathcal{C}, \tag{2.4}$$

with the following conditions on the coefficients

$$\delta_1 > -2 + \frac{1}{p} + (\alpha_1 - b_1)^+, \qquad \delta_2 > -2 + \frac{1}{q} + (\alpha_2 - b_2)^+. \tag{2.5}$$

Then, there exists a positive weak solutions pair $(u,v) \in \mathcal{C}$.

Remark 2.2. Instead of conditions (2.3) and (2.4), as in [12], we can rather suppose that there exist $\kappa_1, \kappa_2 > 0$ and $\alpha_1, \alpha_2 > 0$ such that for all $(u,v) \in \mathcal{C}$,

$$w \mapsto f_1(x,w,v) + \kappa_1 d(x)^{\delta_1 - \alpha_1} w^{p-1} \text{ is non decreasing on } [\underline{u}, \overline{u}],$$

$$w \mapsto f_2(x,u,w) + \kappa_2 d(x)^{\delta_2 - \alpha_2} w^{q-1} \text{ is non decreasing on } [\underline{v}, \overline{v}].$$

Replacing condition (2.5) by

$$\delta_1 > -2 - \frac{1}{p} + (\alpha_1 - (p-1)b_1)^+, \qquad \delta_2 > -2 + \frac{1}{q} + (\alpha_2 - (q-1)\, b_2)^+ ,$$

we get the same result and the condition is sharper if $p, q > 2$. For that, it suffices to replace the first equation of the problem $(\mathbf{Q})$, given below, by

$$-\Delta_p w + \tilde{g}_1(x,w) = f_1(x,u,v) + \kappa_1 d(x)^{\delta_1 - \alpha_1} u^{p-1} \text{ in } \Omega,$$

with $\tilde{g}_1 : \Omega \times \mathbb{R} \to \mathbb{R}_+^*$ the cut-off function defined as follows:

$$\tilde{g}_1(x,z) \stackrel{\text{def}}{=} \begin{cases} \kappa_1 d(x)^{\delta_1-\alpha_1}\overline{u}^{p-1} & \text{if } z \geq \overline{u}(x), \\ \kappa_1 d(x)^{\delta_1-\alpha_1} z^{p-1} & \text{if } z \in [0,\overline{u}(x)]\,, \\ 0 & \text{if } z \leq 0 \end{cases} \tag{2.6}$$

and proceed similarly for the second equation of $(\mathbf{P})$.

Proof. Let $(u,v) \in \mathcal{C}$. We first prove the existence of $T_1(u,v) \in \mathrm{W}_0^{1,p}(\Omega)$, where $T_1(u,v)$ is defined in (1.2) with $h_1(x,u) \stackrel{\text{def}}{=} \kappa_1 d(x)^{\delta_1-\alpha_1}u$ in $\Omega \times [\underline{u},\overline{u}]$. For that, let us introduce the following problem :

$$(\mathbf{Q}) \begin{cases} -\Delta_p w + g_1(x,w) = f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u \ \text{ in } \Omega, \\ w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega, \end{cases}$$

with $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}_+^*$ the cut-off function defined as follows:

$$g_1(x,z) \stackrel{\text{def}}{=} \begin{cases} \kappa_1 d(x)^{\delta_1-\alpha_1}\overline{u} & \text{if } z \geq \overline{u}(x), \\ \kappa_1 d(x)^{\delta_1-\alpha_1} z & \text{if } z \in [0,\overline{u}(x)]\,, \\ 0 & \text{if } z \leq 0. \end{cases} \tag{2.7}$$

Then, g_1 is a Carathéodory function on $\Omega \times \mathbb{R}$. Thus, for $(x,s) \in \Omega \times \mathbb{R}$, setting $G_1(x,s) \stackrel{\text{def}}{=} \int_0^s g_1(x,z)dz$, we consider the following functional: $\forall w \in \mathrm{W}_0^{1,p}(\Omega)$,

$$E(w) \stackrel{\text{def}}{=} \frac{1}{p}\int_\Omega |\nabla w|^p\,dx + \int_\Omega G_1(x,w)\,dx - \int_\Omega \left(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\right) w\,dx.$$

By assumption (2.5) and Hardy's inequality, E is well defined in $\mathrm{W}_0^{1,p}(\Omega)$ and for all $w \in \mathrm{W}_0^{1,p}(\Omega)$,

$$E(w) \geq \frac{1}{p}\|w\|^p_{\mathrm{W}_0^{1,p}(\Omega)} - C\left\|\left(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\right)d(x)\right\|_{\mathrm{L}^{p'}(\Omega)}\|w\|_{\mathrm{W}_0^{1,p}(\Omega)}. \tag{2.8}$$

So, let us define

$$I \stackrel{\text{def}}{=} \inf_{w\in \mathrm{W}_0^{1,p}(\Omega)} E(w) \tag{2.9}$$

and let $(w_n)_{n\in\mathbb{N}} \subset \mathrm{W}_0^{1,p}(\Omega)$ be a minimizing sequence of E, *i.e.* $\lim_{n\to\infty} E(w_n) = I$. Using (2.8), $(w_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{W}_0^{1,p}(\Omega)$, therefore there exists a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ and $\tilde{u} \in \mathrm{W}_0^{1,p}(\Omega)$ such that $w_{n_k} \underset{k\to\infty}{\longrightarrow} \tilde{u}$, weakly in $\mathrm{W}_0^{1,p}(\Omega)$ and a.e. in Ω. Therefore,

$$\liminf_{k\to\infty} \|w_{n_k}\|_{\mathrm{W}_0^{1,p}(\Omega)} \geq \|\tilde{u}\|_{\mathrm{W}_0^{1,p}(\Omega)}$$

and using Fatou's lemma,

$$\liminf_{k\to\infty}\int_\Omega G_1(x,w_{n_k})dx \geq \int_\Omega \liminf_{k\to\infty} G_1(x,w_{n_k})\,dx = \int_\Omega G_1(x,\tilde{u})dx.$$

Hence, $E(\tilde{u}) = I$ and $\tilde{u}$ is a solution to the Euler-Lagrange equation associated to E_0, that is:

$$\int_\Omega |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}.\nabla w\,dx + \int_\Omega g_1(x,\tilde{u})w\,dx$$
$$= \int_\Omega \Big(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\Big)\,w\,dx, \quad (2.10)$$

for any $w \in \mathrm{W}_0^{1,p}(\Omega)$. Now let us prove that $\tilde{u} \in [\underline{u},\overline{u}]$. Combining (1.7) and (2.10), we get for all $w \in \mathrm{W}_0^{1,p}(\Omega)^+ \overset{\text{def}}{=} \{w \in \mathrm{W}_0^{1,p}(\Omega),\ w \geq 0 \text{ a.e in } \Omega\}$,

$$\int_\Omega \left(|\nabla\tilde{u}|^{p-2}\nabla\tilde{u} - |\nabla\underline{u}|^{p-2}\nabla\underline{u}\right).\nabla w\,dx + \int_\Omega \left(g_1(x,\tilde{u}) - g_1(x,\underline{u})\right)w\,dx$$
$$\geq \int_\Omega \left[\left(f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u\right) - \left(f_1(x,\underline{u},v) + \kappa_1 d(x)^{\delta_1-\alpha_1}\underline{u}\right)\right]w\,dx. \quad (2.11)$$

By assumption (2.3), applying this inequality with $w = (\tilde{u}-\underline{u})^- \in \mathrm{W}_0^{1,p}(\Omega)^+$, we get $\tilde{u} \geq \underline{u}$ a.e. in Ω. Similarly, combining (1.9) and (2.10) we also get $\tilde{u} \leq \overline{u}$ a.e. in Ω. Then, $\tilde{u}$ satisfies the equation

$$-\Delta_p\tilde{u} + \kappa_1 d(x)^{\delta_1-\alpha_1}\tilde{u} = f_1(x,u,v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u \quad \text{in } \Omega, \quad (2.12)$$

in the weak sense. Moreover, using a classical local regularity result in [22], $\tilde{u} \in \mathscr{C}^{1,\gamma}(K)$ for some $\gamma > 0$ in any compact subset K of Ω. So using inequality (2.2), $\tilde{u} \in \mathscr{C}\left(\overline{\Omega}\right)$, which gives us that $\tilde{u} \in [\underline{u},\overline{u}]$. Finally, by the weak maximum principle, $\tilde{u}$ is the unique function in the conical shell $[\underline{u},\overline{u}]$ satisfying (2.12). Then, the mapping $T_1 : (u,v) \mapsto \tilde{u}$ is well-defined from $\mathcal{C}$ to $[\underline{u},\overline{u}]$. In the same spirit, we get the existence of the mapping $T_2 : (u,v) \mapsto \tilde{v}$ defined from $\mathcal{C}$ to $[\underline{v},\overline{v}]$, where $\tilde{v}$ is the unique weak solution in $[\underline{v},\overline{v}]$ of

$$-\Delta_p\tilde{v} + \kappa_2 d(x)^{\delta_2-\alpha_2}\tilde{v} = f_2(x,u,v) + \kappa_2 d(x)^{\delta_2-\alpha_2}v \quad \text{in } \Omega. \quad (2.13)$$

This proves that the operator T defined in (1.4) is well-defined and makes invariant the conical shell $\mathcal{C}$.

Now, the continuity and the compactness of T follow from a regularity result in

[11] we recall in appendix A. Indeed, let $(u_n, v_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ and $(u, v) \in \mathcal{C}$ such that $(u_n, v_n) \to (u, v)$ in $\mathscr{C}\left(\overline{\Omega}\right) \times \mathscr{C}\left(\overline{\Omega}\right)$ as $n \to +\infty$. Then, from Theorem A.1 and assumptions (2.1), $(T_1(u_n, v_n) = \tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{C}^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. By Ascoli-Arzelà theorem, there exists a subsequence $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ and $\tilde{u} \in [\underline{u}, \overline{u}]$ such that $\tilde{u}_{n_k} \to \tilde{u}$ uniformly in $\overline{\Omega}$ as $k \to \infty$. Moreover, using the local regularity result in [22], $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ is bounded in $\mathscr{C}^{1,\gamma}(K)$ for some $\gamma > 0$ and for any compact subset K of Ω which entails that up to a subsequence denoted again $(\tilde{u}_{n_k})_{k\in\mathbb{N}}$ such that $\nabla\tilde{u}_{n_k} \to \nabla\tilde{u}$ uniformly in K as $k \to +\infty$. Then, $\tilde{u}$ satisfies

$$-\Delta_p\tilde{u} + \kappa_1 d(x)^{\delta_1-\alpha_1}\tilde{u} = f_1(x, u, v) + \kappa_1 d(x)^{\delta_1-\alpha_1}u \quad \text{in } \Omega \tag{2.14}$$

in the sense of distributions. Moreover, since $\tilde{u} \le \overline{u}$ a.e in Ω, $f_1(x, u, v) + \kappa_1 d(x)^{\delta_1-\alpha_1}(u - \tilde{u}) \in \mathrm{W}^{-1,p'}(\Omega)$, which implies that $\tilde{u} \in \mathrm{W}_0^{1,p}(\Omega)$. Hence $\tilde{u} \in [\underline{u}, \overline{u}]$ and is a weak solution of (2.14). By uniqueness of a such solution in $[\underline{u}, \overline{u}]$, it follows that $\tilde{u} = T_1(u, v)$ and all the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ converges to $\tilde{u}$ in $\mathscr{C}(\overline{\Omega})$. The same arguments hold to prove that $T_2(u_n, v_n) \to T_2(u, v)$ uniformly in $\overline{\Omega}$ as $n \to +\infty$. Then, $T : \mathcal{C} \to \mathcal{C}$ is continuous. Finally, it easy from the compact embedding of $\mathscr{C}^{0,\alpha}(\overline{\Omega})$ in $\mathscr{C}(\overline{\Omega})$ to get the compactness of T. Applying the Schauder Fixed Point Theorem to T in $\mathcal{C}$, the proof of Theorem 2.1 is now complete. □

We now give a more general result which guarantees the existence of a "very weak" positive solutions pair, in the cooperative case, when the inequalities (2.5) may not be satisfied.

Theorem 2.3. *Assume that* $(\mathbf{P})$ *is a* ***cooperative system****, i.e.*

$$\frac{\partial f_1}{\partial v}(x, u, v) > 0 \quad \textit{and} \quad \frac{\partial f_2}{\partial u}(x, u, v) > 0 \quad \textit{in } \Omega \times \mathbb{R}_+^* \times \mathbb{R}_+^*. \tag{2.15}$$

Let $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in \left[\mathscr{C}(\overline{\Omega}) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\right] \times \left[\mathscr{C}(\overline{\Omega}) \cap \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)\right]$ *be sub and super-solutions pairs to* $(\mathbf{P})$. *Assume in addition that the following conditions hold:*

(i) *there exist constants* $C_1, C_2 > 0$ *and* $b_1, b_2 > 0$ *such that*

$$\overline{u} \le C_1 d(x)^{b_1} \quad \textit{and} \quad \overline{v} \le C_2 d(x)^{b_2} \quad \textit{in } \Omega, \tag{2.16}$$

(ii) *there exist* $\kappa_1, \kappa_2 > 0$ *and* $\delta_1, \delta_2 \in \mathbb{R}$ *such that*

$$\left|\frac{\partial f_1}{\partial u}(x, u, v)\right| \le \kappa_1 d(x)^{\delta_1} \;\textit{and}\; \left|\frac{\partial f_2}{\partial v}(x, u, v)\right| \le \kappa_2 d(x)^{\delta_2} \;\textit{in } \Omega\times\mathcal{C}. \tag{2.17}$$

Then, there exists a positive very weak solution pair $(u, v) \in \left[\mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\right] \times \left[\mathrm{L}^\infty(\Omega) \cap \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)\right]$ *to* $(\mathbf{P})$ *such that* $(u, v) \in \mathcal{C}$.

Remark 2.4. Since f_1 and f_2 are continuous with respect to the two last variables in $\mathbb{R}_+^* \times \mathbb{R}_+^*$, assumptions (2.16) and (2.17) imply that for any $K \subset\subset \Omega$, there exist $C_K, C'_K > 0$ such that

$$|f_1(x,u,v)| \leq C_K \quad \text{and} \quad |f_2(x,u,v)| \leq C'_K \quad \text{in } K \times \mathcal{C}. \tag{2.18}$$

Proof. Since Ω is a smooth domain, we can introduce $(\Omega_n)_{n\in\mathbb{N}^*} \subset \Omega$ an increasing sequence of smooth subdomains of Ω such that $\Omega_n \underset{n\to\infty}{\longrightarrow} \Omega$ in the Hausdorff topology with

$$\forall n \in \mathbb{N}^*, \quad \frac{1}{n+1} < \operatorname{dist}(\partial\Omega, \partial\Omega_n) < \frac{1}{n}.$$

Then, for all $n \in \mathbb{N}^*$ we consider the following iterative scheme:

$$(\mathbf{P_n}) \begin{cases} -\Delta_p u_n + \kappa_1 d(x)^{\delta_1} u_n = f_1(x, \tilde{u}_{n-1}, \tilde{v}_{n-1}) + \kappa_1 d(x)^{\delta_1} \tilde{u}_{n-1} & \text{in } \Omega_n, \\ -\Delta_q v_n + \kappa_2 d(x)^{\delta_2} v_n = f_2(x, \tilde{u}_{n-1}, \tilde{v}_{n-1}) + \kappa_2 d(x)^{\delta_2} \tilde{v}_{n-1} & \text{in } \Omega_n, \\ u_n|_{\partial\Omega_n} = \underline{u}, \quad v_n|_{\partial\Omega_n} = \underline{v} \quad \text{and} \quad u_n > 0, \quad v_n > 0 & \text{in } \Omega_n, \end{cases}$$

with initial data $u_0 = \underline{u}$ and $v_0 = \underline{v}$ in Ω_0 and for all $n \in \mathbb{N}$,

$$\tilde{u}_n \overset{\text{def}}{=} \mathbb{1}_{\Omega_n}.u_n + \mathbb{1}_{\Omega\setminus\Omega_n}.\underline{u} \quad \text{and} \quad \tilde{v}_n \overset{\text{def}}{=} \mathbb{1}_{\Omega_n}.v_n + \mathbb{1}_{\Omega\setminus\Omega_n}.\underline{v} \quad \text{in} \quad \Omega.$$

By induction on $n \in \mathbb{N}^*$, $(\mathbf{P_n})$ has a solution $(u_n, v_n) \in \mathrm{W}^{1,p}(\Omega_n) \times \mathrm{W}^{1,q}(\Omega_n)$ satisfying for all $n \in \mathbb{N}^*$,

$$\underline{u} \leq \tilde{u}_n \leq \tilde{u}_{n+1} \leq \overline{u} \quad \text{and} \quad \underline{v} \leq \tilde{v}_n \leq \tilde{v}_{n+1} \leq \overline{v} \quad \text{a.e. in } \Omega. \tag{2.19}$$

Indeed, using estimates (2.16) and (2.18),

$$f_1(x, \underline{u}, \underline{v}) + \kappa_1 d(x)^{\delta_1} \underline{u} \in \mathrm{L}^\infty(\Omega_1) \hookrightarrow \mathrm{W}^{-1,p'}(\Omega_1)$$

and since $\underline{u} \in \mathrm{W}^{1,p}(\Omega_1) \hookrightarrow \mathrm{W}^{1/p',p}(\partial\Omega_1)$ in the sense of the traces, we get $u_1 \in \mathrm{W}^{1,p}(\Omega_1)$ as a minimum of the functional E_1 defined for $w \in \mathrm{W}^{1,p}(\Omega_1)$ by

$$E_1(w) \overset{\text{def}}{=} \frac{1}{p} \int_{\Omega_1} |\nabla(w + \underline{u})|^p \, dx + \frac{\kappa_1}{2} \int_{\Omega_1} d(x)^{\delta_1} (w + \underline{u})^2 \, dx - \int_{\Omega_1} (f_1(x, \underline{u}, \underline{v}) + \kappa_1 d(x)^{\delta_1} \underline{u}) w \, dx. \tag{2.20}$$

Since the operator $u \mapsto -\Delta_p u + \kappa_1 d(x)^{\delta_1} u$ is monotone in $\mathrm{W}^{1,p}(\Omega_1)$, applying the weak comparison principle we get

$$\underline{u} \leq u_1 \leq \overline{u} \quad \text{a.e. in } \Omega_1.$$

Using the same arguments as above, we prove the existence of $v_1 \in \mathrm{W}^{1,q}(\Omega_1)$ satisfying $\underline{v} \leq v_1 \leq \overline{v}$ a.e. in Ω_1. Now, let us fix $n \in \mathbb{N}^*$ and suppose that for all $k \leq n$, $(\mathbf{P}_{\boldsymbol{k}})$ has a solution $(u_k, v_k) \in \mathrm{W}^{1,p}(\Omega_k) \times \mathrm{W}^{1,q}(\Omega_k)$ satisfying (2.19). The existence of positive solutions of $(\mathbf{P}_{\boldsymbol{n+1}})$, $(u_{n+1}, v_{n+1}) \in \mathrm{W}^{1,p}(\Omega_{n+1}) \times \mathrm{W}^{1,q}(\Omega_{n+1})$ satisfying

$$\underline{u} \leq u_{n+1} \leq \overline{u} \quad \text{and} \quad \underline{v} \leq v_{n+1} \leq \overline{v} \quad \text{a.e. in } \Omega_{n+1},$$

can be established using similar techniques as above. To prove the monotonicity of the sequences $(\tilde{u}_m)_{m\in\mathbb{N}^*}$ and $(\tilde{v}_m)_{m\in\mathbb{N}^*}$, we remark that $\tilde{u}_n \in \mathrm{W}^{1,p}(\Omega_{n+1})$ and satisfies

$$-\Delta_p \tilde{u}_n + \kappa_1 d(x)^{\delta_1} \tilde{u}_n \leq f_1(x, \tilde{u}_{n-1}, \tilde{v}_{n-1}) + \kappa_1 d(x)^{\delta_1} \tilde{u}_{n-1} \quad \text{in } \Omega_{n+1}, \quad (2.21)$$

in the weak sense. Then, using (2.21) together with (2.18), we deduce from the previous inequality that,

$$-\Delta_p \tilde{u}_n + \kappa_1 d(x)^{\delta_1} \tilde{u}_n \leq f_1(x, \tilde{u}_{n-1}, \tilde{v}_n) + \kappa_1 d(x)^{\delta_1} \tilde{u}_{n-1} \quad \text{in } \Omega_{n+1},$$

in the weak sense. Hence, by estimate (2.17) and from the weak comparison principle applied in $\mathrm{W}^{1,p}(\Omega_{n+1})$, we obtain

$$\tilde{u}_n \leq u_{n+1} \quad \text{a.e. in } \Omega_{n+1}.$$

Similarly, we get the existence and the behaviour of v_{n+1}. Then, for almost every $x \in \Omega$, we define

$$u(x) = \lim_{n\to\infty} \tilde{u}_n(x) \quad \text{and} \quad v(x) = \lim_{n\to\infty} \tilde{v}_n(x).$$

Moreover, using a classical local regularity result of SERRIN[22], $\tilde{u}_n, \tilde{v}_n \in \mathscr{C}^{1,\gamma}_{\text{loc}}(\Omega_n)$ for some $0 < \gamma < 1$ and $\nabla \tilde{u}_n \underset{n\to\infty}{\longrightarrow} \nabla u$ and $\nabla \tilde{v}_n \underset{n\to\infty}{\longrightarrow} \nabla v$, uniformly in any compact set K of Ω. Thus, $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ and passing to the limit in $(\mathbf{P}_{\boldsymbol{n}})$, (u, v) is a solution of $(\mathbf{P})$ in the sense of distributions. □

3 Applications

3.1 Example 1

In this section we focus on the following quasilinear elliptic and singular system,

$$\textbf{(P)}\begin{cases} -\Delta_p u = K_1(x)u^{a_1}v^{b_1} \quad \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ -\Delta_q v = K_2(x)v^{a_2}u^{b_2} \quad \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega. \end{cases}$$

In this problem,

(i) The exponents $a_1 < p-1$, $a_2 < q-1$ and $b_1, b_2 \neq 0$ satisfy the subhomogeneous condition

$$(p-1-a_1)(q-1-a_2) - |b_1 b_2| > 0, \tag{3.1}$$

which is equivalent to the existence of a positive constant $\sigma > 0$ such that

$$(p-1-a_1) - \sigma|b_1| > 0 \quad \text{and} \quad \sigma(q-1-a_2) - |b_2| > 0. \tag{3.2}$$

(ii) K_1, K_2 are two positive functions in Ω satisfying

$$K_1(x) = d(x)^{-k_1}L_1(d(x)) \text{ and } K_2(x) = d(x)^{-k_2}L_2(d(x)) \text{ in } \Omega, \tag{3.3}$$

with $0 \leq k_1 < p$, $0 \leq k_2 < q$ and for $i = 1, 2$, L_i a lower perturbation in $\mathscr{C}^2((0, D])$ (D the diameter of the domain Ω), of the form:

$$\forall t \in (0, D],\ L_i(t) = \exp\left(\int_t^{2D} \frac{z_i(s)}{s} ds\right), \tag{3.4}$$

with $z_i \in \mathscr{C}([0, D]) \cap \mathscr{C}^1((0, D])$ and $z_i(0) = 0$.

Remark 3.1. a. Let us notice that (3.4) implies that

$$\forall \varepsilon > 0, \qquad \lim_{t\to 0^+} t^{-\varepsilon} L_i(t) = +\infty \quad \text{and} \quad \lim_{t\to 0^+} t^{\varepsilon} L_i(t) = 0. \tag{3.5}$$

b. Definition (3.4) also implies that

$$\lim_{t\to 0^+} \frac{tL_i'(t)}{L_i(t)} = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{tL_i''(t)}{L_i'(t)} = -1.$$

c. If L_1, L_2 are two functions satisfying (3.4), then for any $\alpha, \beta \in \mathbb{R}$, the function ${L_1}^{\alpha}.{L_2}^{\beta}$ also satisfies (3.4).

d. Such functions L_1, L_2 defined as above belong to the Karamata Class [17].

Example 3.2. Let $m \in \mathbb{N}^*$ and $A >> D$ large enough. Let us define

$$\forall t \in (0, D], \qquad L_i(t) = \prod_{n=1}^{m} \left(\log_n \left(\frac{A}{t} \right) \right)^{\mu_n},$$

where, $\log_n \overset{\text{def}}{=} \log \circ \cdots \circ \log$ (n times) and $\mu_n > 0$. Then L_i satisfies (3.4).

In our study, $b_1 \neq 0$ and $b_2 \neq 0$. In the case where $b_1 > 0$ and $b_2 > 0$, the expression of the right-hand sides of the two coupled equations in system **(P)** define a ***cooperative*** interaction between the two components (species) u and v:

$$\frac{\partial}{\partial v} \left(K_1(x) u^{a_1} v^{b_1} \right) = b_1 K_1(x) u^{a_1} v^{b_1 - 1} > 0, \tag{3.6}$$

$$\frac{\partial}{\partial u} \left(K_2(x) v^{a_2} u^{b_2} \right) = b_2 K_2(x) v^{a_2} u^{b_2 - 1} > 0. \tag{3.7}$$

In the case where $b_1 < 0$ and $b_2 < 0$, both partial derivative in (3.6) and (3.7) are negative and the expression of the right-hand sides of the two coupled equations of **(P)** defines a ***competitive*** interaction between u and v.

First, we discuss the existence of positive weak solutions pairs to problem $(\mathbf{P})$. For that, regarding Theorem 2.1, we take

$$f_1(x, u, v) = K_1(x) u^{a_1} v^{b_1}, \quad f_2(x, u, v) = K_2(x) v^{a_2} u^{b_2}$$

and construct suitable sub and supersolutions pairs of **(P)** in $\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$.

Then, in the cases where $(\mathbf{P})$ is either competitive or cooperative, we investigate the uniqueness of such positive weak solutions pairs. For that, it is essential that the mappings $T_1 \circ T_2$ and $T_2 \circ T_1$ (where T_1 and T_2 are defined in (1.1)) is ***subhomogeneous***, which is equivalent to condition (3.1).

Preliminary results

Let $1 < r < \infty$, $\delta < r - 1$ and $K : x \longmapsto d(x)^{-k} L(d(x))$, with $0 \leq k < r$ and L a perturbation function satisfying (3.4). In view of constructing suitable pairs of sub and supersolutions to $(\mathbf{P})$, we first introduce the following problem:

$$-\Delta_r w = K(x) w^{\delta} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega. \tag{3.8}$$

Theorem 3.3. *Under the above hypothesis, we have:*

(i) *If* $k-1<\delta<r-1$, *problem* (3.8) *has a unique positive weak solution* $\psi \in \mathrm{W}_0^{1,r}(\Omega)$ *that satisfies the following estimate:*

$$\psi(x) \sim d(x) \qquad \textit{in } \Omega. \tag{3.9}$$

In addition, we have $\psi \in \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(ii) *If* $\delta = k-1$, *problem* (3.8) *has a unique positive weak solution* $\psi \in \mathrm{W}_0^{1,r}(\Omega)$ *that satisfies the following estimate:*

$$\psi(x) \sim d(x)\left(\int_{d(x)}^{2D} L(t)t^{-1}\,dt\right)^{\frac{1}{r-k}} \qquad \textit{in } \Omega. \tag{3.10}$$

In addition, we have $\psi \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(iii) *If* $k-2+\frac{k-1}{r-1}<\delta<k-1$, *problem* (3.8) *has a unique positive weak solution* $\psi \in \mathrm{W}_0^{1,r}(\Omega)$ *that satisfies the following estimate:*

$$\psi(x) \sim d(x)^{\frac{r-k}{r-1-\delta}} L(d(x))^{\frac{1}{r-1-\delta}} \qquad \textit{in } \Omega. \tag{3.11}$$

In addition, we have $\psi \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(iv) *If* $\delta \leq k-2+\frac{k-1}{r-1}$, *problem* (3.8) *has at least one positive weak solution* $\psi \in \mathrm{W}_{\mathrm{loc}}^{1,r}(\Omega)\cap\mathscr{C}_0\left(\overline{\Omega}\right)$ *that satisfies the following estimate:*

$$\psi(x) \sim d(x)^{\frac{r-k}{r-1-\delta}} L(d(x))^{\frac{1}{r-1-\delta}} \qquad \textit{in } \Omega. \tag{3.12}$$

Proof. See Lemma 3.3 in GIACOMONI, MÂAGLI, SAUVY[9]. □

Remark 3.4. In (iv) above, it can be proved that $\forall\,\gamma>\frac{(r-1)(r-1-\delta)}{r(r-k)}$, $\psi^\gamma \in \mathrm{W}_0^{1,r}(\Omega)$.

We give now a weak comparison principle used to establish the uniqueness of a positive weak solutions pair of **(P)**.

Theorem 3.5. *Let* $K:\Omega\to\mathbb{R}_+$ *be a* $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ *function and* $\delta<r-1$. *Assume* $u,v\in \mathrm{W}_0^{1,r}(\Omega)\cap \mathrm{L}^\infty(\Omega)$ *are two locally uniformly positive functions satisfying the sub and supersolution inequalities:*

$$-\Delta_r u \leq K(x)u^\delta \quad \textit{and} \quad -\Delta_r v \geq K(x)v^\delta \qquad \textit{in } \Omega, \tag{3.13}$$

in the sense of distributions (i.e. Radon measures) in $\mathrm{W}^{-1,r'}(\Omega)$. *Then*

(i) *If* $\delta < 0$*, inequality* $u \leq v$ *holds a.e. in* Ω*.*

(ii) *If* $\delta > 0$ *and if we suppose in addition that there exist* $C_1, C_2 > 0$ *and a locally uniformly positive function* $w_0 \in \mathrm{L}^\infty(\Omega)$ *such that* $C_1 w_0 \leq u, v \leq C_2 w_0$ *a.e. in* Ω *and*

$$\int_\Omega K(x) {w_0}^{\delta+1}\, dx < +\infty, \tag{3.14}$$

inequality $u \leq v$ *holds a.e. in* Ω*.*

To prove this theorem, we use the well-known inequality in Lemma 3.6 and the Díaz-Saa inequality (see DÍAZ-SAA[6]).

Lemma 3.6. *There exists a constant* $C_r > 0$ *such that, for all* $x, y \in \mathbb{R}^N$*,*

$$|x|^r - |y|^r - r|x|^{r-2}x.(y-x) \geq \begin{cases} C_r|x-y|^r & if \quad r \geq 2, \\ C_r \dfrac{|x-y|^2}{(|x|+|y|)^{2-r}} & if \quad 1 < r < 2. \end{cases}$$

Proof. See Lemma 4.2 in LINDQVIST [20]. □

Proof. (OF THEOREM 3.5)

(i) If $\delta < 0$, we wish to prove that the function $w = (u-v)^+$ satisfies $w = 0$ a.e. in Ω. First notice that $0 \leq w \in \mathrm{W}_0^{1,r}(\Omega)$. Applying the duality between $\mathrm{W}_0^{1,r}(\Omega)$ and $\mathrm{W}^{-1,r'}(\Omega)$, respectively, to w and the the difference

$$-\Delta_r u + \Delta_r v \leq K(x)\left(u^\delta - v^\delta\right)$$

which is ≤ 0 on the set $\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega, \quad w(x) > 0\}$, we obtain

$$\int_{\Omega_+} \left(|\nabla u|^{r-2}\nabla u - |\nabla v|^{r-2}\nabla v\right).(\nabla u - \nabla v)\, dx$$
$$= \int_\Omega \left(|\nabla u|^{r-2}\nabla u - |\nabla v|^{r-2}\nabla v\right).\nabla w\, dx \leq 0.$$

This forces $\nabla w = 0$ a.e. in Ω_+ and, consequently, also in Ω. Since $w \in \mathrm{W}_0^{1,r}(\Omega)$, we conclude that $w = 0$ a.e. in Ω as claimed, that is, $u \leq v$ a.e. in Ω.

(ii) If $0 < \delta < r-1$, following some ideas in LINDQUIST[20] (see also DRÁBEK-HERNÁNDEZ[7]), we use the Díaz-Saa inequality.

More precisely, for $\varepsilon > 0$, we set $u_\varepsilon \stackrel{\text{def}}{=} u + \varepsilon$ and $v_\varepsilon \stackrel{\text{def}}{=} v + \varepsilon$ in Ω and we define

$$\phi \stackrel{\text{def}}{=} \frac{{u_\varepsilon}^r - {v_\varepsilon}^r}{{u_\varepsilon}^{r-1}} \quad \text{and} \quad \psi \stackrel{\text{def}}{=} \frac{{v_\varepsilon}^r - {u_\varepsilon}^r}{{v_\varepsilon}^{r-1}} \quad \text{in } \Omega.$$

Then, $\frac{u_\varepsilon}{v_\varepsilon}, \frac{v_\varepsilon}{u_\varepsilon} \in \mathrm{L}^\infty(\Omega)$ and $\phi, \psi \in \mathrm{W}_0^{1,r}(\Omega)$ with

$$\nabla\phi = \left[1 + (r-1)\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^r\right]\nabla u - r\left(\frac{v_\varepsilon}{u_\varepsilon}\right)^{r-1}\nabla v \quad \text{in } \Omega, \tag{3.15}$$

$$\nabla\psi = \left[1 + (r-1)\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^r\right]\nabla v - r\left(\frac{u_\varepsilon}{v_\varepsilon}\right)^{r-1}\nabla u \quad \text{in } \Omega. \tag{3.16}$$

Setting $\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega, \quad u(x) > v(x)\}$, we have that $\phi > 0$ and $\psi < 0$ in Ω_+ and

$$\int_{\Omega_+} |\nabla u|^{r-2}\nabla u.\nabla\phi\, dx \leq \int_{\Omega_+} K(x)u^\delta\phi\, dx < +\infty,$$

$$\int_{\Omega_+} |\nabla v|^{r-2}\nabla v.\nabla\psi\, dx \leq \int_{\Omega_+} K(x)v^\delta\psi\, dx < +\infty.$$

Using equalities (3.15) and (3.16) and the fact that

$$|\nabla \ln u_\varepsilon| = \frac{|\nabla u|}{u_\varepsilon} \quad \text{and} \quad |\nabla \ln v_\varepsilon| = \frac{|\nabla v|}{v_\varepsilon} \quad \text{in } \Omega, \tag{3.17}$$

we get

$$\begin{aligned}\int_{\Omega_+} |\nabla u|^{r-2}\nabla u.\nabla\phi\, dx &+ \int_{\Omega_+} |\nabla v|^{r-2}\nabla v.\nabla\psi\, dx \\ &= \int_{\Omega_+} ({u_\varepsilon}^r - {v_\varepsilon}^r)(|\nabla \ln u_\varepsilon|^r - |\nabla \ln v_\varepsilon|^r)\, dx \\ &- \int_{\Omega_+} r{v_\varepsilon}^r|\nabla \ln u_\varepsilon|^{r-2}(\nabla \ln u_\varepsilon).\left(\nabla \ln v_\varepsilon - \nabla \ln u_\varepsilon\right)\, dx \\ &- \int_{\Omega_+} r{u_\varepsilon}^r|\nabla \ln v_\varepsilon|^{r-2}(\nabla \ln v_\varepsilon).\left(\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon\right)\, dx.\end{aligned}$$

a. If $r \geq 2$, from Lemma 3.6, it follows that

$$\begin{aligned}
&\int_{\Omega_+} |\nabla u|^{r-2}\nabla u.\nabla\phi\,dx + \int_{\Omega_+} |\nabla v|^{r-2}\nabla v.\nabla\psi\,dx \\
&\quad \geq \int_{\Omega_+} ({u_\varepsilon}^r - {v_\varepsilon}^r)(|\nabla \ln u_\varepsilon|^r - |\nabla \ln v_\varepsilon|^r)\,dx \\
&\quad + \int_{\Omega_+} {v_\varepsilon}^r \left(|\nabla \ln u_\varepsilon|^r - |\nabla \ln v_\varepsilon|^r + C_r|\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon|^r\right)\,dx \\
&\quad + \int_{\Omega_+} {u_\varepsilon}^r \left(|\nabla \ln v_\varepsilon|^r - |\nabla \ln u_\varepsilon|^r + C_r|\nabla \ln u_\varepsilon - \nabla \ln v_\varepsilon|^r\right)\,dx \\
&\quad = C_r \int_{\Omega_+} |u_\varepsilon \nabla v_\varepsilon - v_\varepsilon \nabla u_\varepsilon|^r \left(\frac{1}{{u_\varepsilon}^r} + \frac{1}{v_\varepsilon^r}\right)\,dx.
\end{aligned}$$

b. If $1 < r < 2$, Lemma 3.6 entails

$$\begin{aligned}
&\int_{\Omega_+} |\nabla u|^{r-2}\nabla u.\nabla\phi\,dx + \int_{\Omega_+} |\nabla v|^{r-2}\nabla v.\nabla\psi\,dx \\
&\qquad \geq C_r \int_{\Omega_+} \frac{|u_\varepsilon \nabla v_\varepsilon - v_\varepsilon \nabla u_\varepsilon|^2}{(u_\varepsilon|\nabla v_\varepsilon| + v_\varepsilon|\nabla u_\varepsilon|)^{2-r}} \left(\frac{1}{{u_\varepsilon}^r} + \frac{1}{{v_\varepsilon}^r}\right)\,dx.
\end{aligned}$$

In the right-hand side, we get

$$\begin{aligned}
&\int_{\Omega_+} K(x)\left(u^\delta\phi + v^\delta\psi\right)\,dx = \\
&\qquad \int_{\Omega_+} K(x)\left[\frac{u^\delta}{u^{r-1}}\left(\frac{u}{u_\varepsilon}\right)^{r-1} - \frac{v^\delta}{v^{r-1}}\left(\frac{v}{v_\varepsilon}\right)^{r-1}\right]({u_\varepsilon}^r - {v_\varepsilon}^r)\,dx.
\end{aligned}$$

Then, since $\frac{u}{u_\varepsilon} \to 1$, $\frac{v}{v_\varepsilon} \to 1$ as $\varepsilon \to 0^+$ a.e. in Ω, we get from (3.14) and Lebesgue's Theorem that

$$\lim_{\varepsilon\to 0^+} \int_{\Omega_+} K(x)\left(u^\delta\phi + v^\delta\psi\right)\,dx \leq 0.$$

By Fatou's Lemma and using the above estimates, we obtain in the both cases that $|u\nabla v - v\nabla u| = 0$ a.e. in Ω_+, from which we get that on each connected component set ω of Ω_+, there exists $k > 0$ such that $u = kv$ a.e. in ω. From sub an supersolution inequalities (3.13) we have,

$$\begin{aligned}
k^r \int_\omega K(x)v^{\delta+1}\,dx \leq k^r \int_\omega |\nabla v|^r\,dx = \int_\omega |\nabla u|^r\,dx \\
\leq \int_\omega K(x)u^{\delta+1}\,dx = k^{\delta+1}\int_\omega K(x)v^{\delta+1}\,dx. \quad (3.18)
\end{aligned}$$

Hence, $k \leq 1$ which implies that $u \leq v$ a.e. in Ω_+ and from the definition of Ω_+, $u \leq v$ a.e. in Ω.

□

Main results

Theorem 3.7. *Assume that the exponents $a_1 < p-1$, $a_2 < q-1$ and $b_1, b_2 \neq 0$ in problem $(\mathbf{P})$ satisfy the hypothesis* (3.1)*.*

(i) *Set*

$$\alpha_1 = \frac{q-1-a_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \alpha_2 = \frac{p-1-a_1}{(p-1-a_1)(q-1-a_2)-b_1b_2},$$

$$\beta_1 = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \beta_2 = \frac{b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2},$$

$$\gamma_1 = \frac{(p-k_1)(q-1-a_2)+(q-k_2)b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \gamma_2 = \frac{(q-k_2)(p-1-a_1)+(p-k_1)b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}$$

and assume that

$$1-\frac{1}{p} < \gamma_1 < 1 \quad \text{and} \quad 1-\frac{1}{q} < \gamma_2 < 1. \tag{3.19}$$

Then, problem $(\mathbf{P})$ possesses positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ that satisfy the following estimates:

$$u(x) \sim d(x)^{\gamma_1} L_1(d(x))^{\alpha_1} L_2(d(x))^{\beta_1} \quad \text{in } \Omega, \tag{3.20}$$

$$v(x) \sim d(x)^{\gamma_2} L_2(d(x))^{\alpha_2} L_1(d(x))^{\beta_2} \quad \text{in } \Omega. \tag{3.21}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$.

(ii) *Now assume that*

$$k_1 - 1 < a_1 + b_1 < p-1 \quad \text{and} \quad k_2 - 1 < a_2 + b_2 < q-1. \tag{3.22}$$

Then, problem $(\boldsymbol{P})$ possesses positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ that satisfy the following estimates:

$$u(x) \sim d(x) \quad \text{and} \quad v(x) \sim d(x) \quad \text{in } \Omega. \tag{3.23}$$

In addition, we have $(u,v) \in \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$.

(iii) *Set*

$$\gamma = \frac{p-k_1+b_1}{p-1-a_1}$$

and assume that

$$1-\frac{1}{p}<\gamma<1 \quad \textit{and} \quad k_2-1<a_2+b_2\gamma<q-1. \tag{3.24}$$

Then, problem $(\boldsymbol{P})$ *possesses positive solutions* $(u,v)\in \mathrm{W}_0^{1,p}(\Omega)\times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates:*

$$u(x)\ \sim\ d(x)^{\gamma}L_1(d(x))^{\frac{1}{p-1-a_1}} \quad \textit{and} \quad v(x)\ \sim\ d(x) \quad \textit{in}\ \Omega. \tag{3.25}$$

In addition, we have $(u,v)\in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)\times \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(iv) *Symmetrically to part (iii) above, set*

$$\gamma = \frac{q-k_2+b_2}{q-1-a_2}$$

and assume that

$$k_1-1<a_1+b_1\gamma<p-1 \quad \textit{and} \quad 1-\frac{1}{q}<\gamma<1. \tag{3.26}$$

Then, problem $(\boldsymbol{P})$ *possesses positive solutions* $(u,v)\in \mathrm{W}_0^{1,p}(\Omega)\times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates:*

$$u(x)\ \sim\ d(x) \quad \textit{and} \quad v(x)\ \sim\ d(x)^{\gamma}L_2(d(x))^{\frac{1}{q-1-a_2}} \quad \textit{in}\ \Omega. \tag{3.27}$$

In addition, we have $(u,v)\in \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)\times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

Theorem 3.8. *Let* $a_1<p-1$, $a_2<q-1$ *and* $b_1,b_2\neq 0$ *satisfying the subhomogeneity hypothesis* (3.1). *Assume that* $(\boldsymbol{P})$ *is either a competitive or a cooperative system, i.e.* $\boldsymbol{b_1b_2>0}$. *Then, each solution provided by Theorem* 3.7 *is unique.*

The cooperative case is further analysed in the following result:

Theorem 3.9. *Let us suppose that the exponents* $a_1<p-1$, $a_2<q-1$ *and* $b_1,b_2\neq 0$ *satisfy the subhomogeneity hypothesis* (3.1). *Moreover, assume that* $(\boldsymbol{P})$ *is a cooperative system, i.e.,* $\boldsymbol{b_1>0}$ *and* $\boldsymbol{b_2>0}$.

(i) *Set*

$$\gamma_1 = \frac{(p-k_1)(q-1-a_2)+(q-k_2)b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.28}$$

$$\gamma_2 = \frac{(q-k_2)(p-1-a_1)+(p-k_1)b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2} \tag{3.29}$$

and assume that one of the three following conditions are satisfied:

$$0<\gamma_1\leq 1-\frac{1}{p} \quad \textit{and} \quad 0<\gamma_2\leq 1-\frac{1}{q}, \tag{3.30}$$

$$1-\frac{1}{p}<\gamma_1<1 \quad \textit{and} \quad 0<\gamma_2\leq 1-\frac{1}{q}, \tag{3.31}$$

$$0<\gamma_1\leq 1-\frac{1}{p} \quad \textit{and} \quad 1-\frac{1}{q}<\gamma_2<1. \tag{3.32}$$

Then, problem $(\mathbf{P})$ *admits positive solutions* $(u,v)\in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ *in the sense of distributions satisfying the estimates* (3.20) *and* (3.21).

(ii) *Set*

$$\gamma = \frac{p-k_1+b_1}{p-1-a_1} \tag{3.33}$$

and assume that

$$0<\gamma\leq 1-\frac{1}{p} \quad \textit{and} \quad k_2-1<a_2+b_2\gamma<q-1. \tag{3.34}$$

Then, problem $(\mathbf{P})$ *nevertheless admits positive solutions* $(u,v)\in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)\times \mathrm{W}^{1,q}_{0}(\Omega)$ *in the sense of distributions satisfying the estimates* (3.20) *and* (3.21).

(iii) *Symmetrically to part (ii) above, set*

$$\gamma = \frac{q-k_2+b_2}{q-1-a_2} \tag{3.35}$$

and assume that

$$k_1-1<a_1+b_1\gamma<p-1 \quad \textit{and} \quad 0<\gamma\leq 1-\frac{1}{q}. \tag{3.36}$$

Then, problem $(\boldsymbol{P})$ *possesses positive solutions* $(u,v)\in \mathrm{W}^{1,p}_{0}(\Omega)\times \mathrm{W}^{1,q}_{\mathrm{loc}}(\Omega)$ *in the sense of distributions that satisfies the estimates given in* (3.27).

The next result deals with some limiting cases:

Theorem 3.10. *Assume that the exponents* $a_1 < p-1, a_2 < q-1$ *and* $b_1, b_2 \neq 0$ *satisfy the subhomogeneity hypothesis* (3.1).

(i) *Assume that*

$$a_1 + b_1 = k_1 - 1 \quad \textit{and} \quad k_2 - 1 \leq a_2 + b_2 < q - 1. \tag{3.37}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C_1', C_2' > 0$ *such that problem* $(\boldsymbol{P})$ *possesses positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates:*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \textit{ and } C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \textit{ in } \Omega, \tag{3.38}$$

where $\sigma > 0$ *is given in* (3.2).*In addition, we have* $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$*, for some* $0 < \alpha < 1$.

(ii) *Symmetrically, assume that*

$$a_2 + b_2 = k_2 - 1 \quad \textit{and} \quad k_1 - 1 \leq a_1 + b_1 < q - 1. \tag{3.39}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C_1', C_2' > 0$ *such that problem* $(\boldsymbol{P})$ *possesses positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates:*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \textit{ and } C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \textit{ in } \Omega. \tag{3.40}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$*, for some* $0 < \alpha < 1$.

(iii) *Let us abbreviate*

$$\gamma = \frac{p - k_1 + b_1}{p - 1 - a_1}$$

and assume that

$$1 - \frac{1}{p} < \gamma < 1 \quad \textit{and} \quad a_2 + b_2\gamma = k_2 - 1. \tag{3.41}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C_1', C_2' > 0$ *such that problem* $(\boldsymbol{P})$ *possesses positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates in* Ω*:*

$$C_1 d(x)^{\gamma+\varepsilon} \leq u \leq C_2 d(x)^{\gamma-\varepsilon} \textit{ and } C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma}. \tag{3.42}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$*, for some* $0 < \alpha < 1$.

(iv) *Symmetrically, let us abbreviate*

$$\gamma = \frac{q - k_2 + b_2}{q - 1 - a_2}$$

and assume that

$$a_1 + b_1\gamma = k_2 - 1 \quad \text{and} \quad 1 - \frac{1}{q} < \gamma < 1. \tag{3.43}$$

Then, for all $\varepsilon > 0$ *small enough, there exist* $C_1, C_2 > 0$ *and* $C_1', C_2' > 0$ *such that problem* $(\boldsymbol{P})$ *possesses positive solutions* $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *that satisfy the following estimates in* Ω*:*

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \text{ and } C_1' d(x)^{\gamma+\varepsilon\sigma} \leq v \leq C_2' d(x)^{\gamma-\varepsilon\sigma}. \tag{3.44}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$*, for some* $0 < \alpha < 1$.

Proof of Theorem 3.7

Thanks to Theorem 3.3, we apply Theorem 2.1 with a suitable choice of sub and supersolutions pairs $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ in the following form:

$$\underline{u} \equiv m\psi_1 \quad \text{and} \quad \overline{u} \equiv m^{-1}\psi_1 \quad \text{in } \Omega,$$

$$\underline{v} \equiv m^{\sigma}\psi_2 \quad \text{and} \quad \overline{v} \equiv m^{-\sigma}\psi_2 \quad \text{in } \Omega,$$

where $\sigma > 0$ is given in (3.2), $0 < m < 1$ is an appropriate constant small enough and $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ are given by Theorem 3.3 as the respective unique solutions of problems

$$-\Delta_p w = d(x)^{-k_1}\mathcal{L}_1(d(x))w^{\delta_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega, \tag{3.45}$$

$$-\Delta_q w = d(x)^{-k_2}\mathcal{L}_2(d(x))w^{\delta_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega, \tag{3.46}$$

satisfying some cone conditions we specify below. In the following alternatives, we choose suitable perturbations $\mathcal{L}_1, \mathcal{L}_2$ as in (3.4) and suitable values of exponents $k_1 - 2 + \frac{k_1-1}{p-1} < \delta_1 < p - 1$ and $k_2 - 2 + \frac{k_2-1}{q-1} < \delta_2 < q - 1$ in order to satisfy

$$-\Delta_p\psi_1 \sim K_1(x){\psi_1}^{a_1}{\psi_2}^{b_1} \quad \text{and} \quad -\Delta_q\psi_2 \sim K_2(x){\psi_2}^{a_2}{\psi_1}^{b_2} \quad \text{in } \Omega, \tag{3.47}$$

which provide us the inequalities (1.7) to (1.10) in order to apply Theorem 2.1.

Alternative 1: We look for positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ to $(\mathbf{P})$ by making the "Ansatz" that

$$u(x) \sim d(x)^{\gamma_1} L_1(d(x))^{\alpha_1} L_2(d(x))^{\beta_1} \quad \text{in } \Omega,$$

$$v(x) \sim d(x)^{\gamma_2} L_2(d(x))^{\alpha_2} L_1(d(x))^{\beta_2} \quad \text{in } \Omega,$$

for some $\gamma_1 \in (1-\frac{1}{p}, 1)$, $\gamma_2 \in (1-\frac{1}{q}, 1)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. For that, we take in (3.45) and (3.46)

$$k_1 - 2 + \frac{k_1-1}{p-1} < \delta_1 < k_1 - 1 \quad \text{and} \quad \mathrm{k}_2 - 2 + \frac{k_2-1}{q-1} < \delta_2 < k_2 - 1, \tag{3.48}$$

$$\mathcal{L}_1 = {L_1}^{\lambda_1}.{L_2}^{\mu_1} \quad \text{and} \quad \mathcal{L}_2 = {L_2}^{\lambda_2}.{L_1}^{\mu_2} \quad \text{in } \Omega,$$

where $\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2 \in \mathbb{R}$ are suitable exponents we fix later. By Theorem 3.3, $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ and satisfy

$$\psi_1(x) \ \sim\ d(x)^{\frac{p-k_1}{p-1-\delta_1}} L_1(d(x))^{\frac{\lambda_1}{p-1-\delta_1}} L_2(d(x))^{\frac{\mu_1}{p-1-\delta_1}} \quad \text{in } \Omega, \tag{3.49}$$

$$\psi_2(x) \ \sim\ d(x)^{\frac{q-k_2}{q-1-\delta_2}} L_2(d(x))^{\frac{\lambda_2}{q-1-\delta_2}} L_1(d(x))^{\frac{\mu_2}{q-1-\delta_2}} \quad \text{in } \Omega. \tag{3.50}$$

In view of satisfying estimates given in (3.47), the comparison of the term $-\Delta_p\psi_1$ with $K_1(x){\psi_1}^{a_1}{\psi_2}^{b_1}$ on one side, and the term $-\Delta_q\psi_2$ with $K_2(x){\psi_2}^{a_2}{\psi_1}^{b_2}$ on the other side, imposes the exponents $\lambda_1, \lambda_2, \mu_1, \mu_2$ and δ_1, δ_2 to satisfy the following system:

$$\begin{cases} \delta_1 \frac{p-k_1}{p-1-\delta_1} = a_1 \frac{p-k_1}{p-1-\delta_1} + b_1 \frac{q-k_2}{q-1-\delta_2}, \\ \delta_2 \frac{q-k_2}{q-1-\delta_2} = a_2 \frac{q-k_2}{q-1-\delta_2} + b_2 \frac{p-k_1}{p-1-\delta_1}, \\ \lambda_1 \frac{p-1}{p-1-\delta_1} = 1 + a_1 \frac{\lambda_1}{p-1-\delta_1} + b_1 \frac{\mu_2}{q-1-\delta_2}, \\ \lambda_2 \frac{q-1}{q-1-\delta_2} = 1 + b_2 \frac{\mu_1}{p-1-\delta_1} + a_2 \frac{\lambda_2}{q-1-\delta_2}, \\ \mu_1 \frac{p-1}{p-1-\delta_1} = a_1 \frac{\mu_1}{p-1-\delta_1} + b_1 \frac{\lambda_2}{q-1-\delta_2}, \\ \mu_2 \frac{q-1}{q-1-\delta_2} = b_2 \frac{\lambda_1}{p-1-\delta_1} + a_2 \frac{\mu_2}{q-1-\delta_2}. \end{cases}$$

Then, we get

$$\gamma_1 = \frac{p-k_1}{p-1-\delta_1} = \frac{(p-k_1)(q-1-a_2)+(q-k_2)b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.51}$$

$$\gamma_2 = \frac{q-k_2}{q-1-\delta_2} = \frac{(q-k_2)(p-1-a_1)+(p-k_1)b_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.52}$$

$$\alpha_1 = \frac{\lambda_1}{p-1-\delta_1} = \frac{q-1-a_2}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.53}$$

$$\alpha_2 = \frac{\lambda_2}{q-1-\delta_2} = \frac{p-1-a_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.54}$$

$$\beta_1 = \frac{\mu_1}{p-1-\delta_1} = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.55}$$

$$\beta_2 = \frac{\mu_2}{q-1-\delta_2} = \frac{b_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \tag{3.56}$$

which imply estimate (3.47). Moreover, inequalities (3.19) are then equivalent to inequalities (3.48). Let $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. On one hand, we have

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 L_1(d(x))^{\lambda_1+\delta_1\gamma_1} L_2(d(x))^{\mu_1+\delta_1\beta_1} d(x)^{\delta_1\gamma_1-k_1} \text{ in } \Omega,$$

$$-\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' L_2(d(x))^{\lambda_2+\delta_2\alpha_2} L_1(d(x))^{\mu_2+\delta_2\beta_2} d(x)^{\delta_2\gamma_2-k_2} \text{ in } \Omega.$$

On the other hand,

$$K_1(x)\underline{u}^{a_1} v^{b_1} \geq C_2 m^{a_1+\sigma|b_1|} \Lambda_1(d(x)) d(x)^{a_1\gamma_1+b_1\gamma_2-k_1} \text{ in } \Omega$$

with $\Lambda_1 = {L_1}^{1+a_1\alpha_1+b_1\beta_2} . {L_2}^{a_1\beta_1+b_1\alpha_2}$. Similarly,

$$K_2(x)\underline{v}^{a_2} u^{b_2} \geq C_2' m^{\sigma a_2+|b_2|} \Lambda_2(d(x)) d(x)^{a_2\gamma_2+b_2\gamma_1-k_2} \text{ in } \Omega,$$

with $\Lambda_2 = {L_2}^{1+a_2\alpha_2+b_2\beta_1} . {L_1}^{a_2\beta_2+b_2\alpha_1}$. Then, under condition (3.2) and thanks to (3.51) to (3.56), $(\underline{u},\underline{v})$ is a subsolutions pair of problem $(\mathbf{P})$, for m small enough. Next,

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 L_1(d(x))^{\lambda_1+\delta_1\gamma_1} L_2(d(x))^{\mu_1+\delta_1\beta_1} d(x)^{\delta_1\gamma_1-k_1} \text{ in } \Omega,$$

$$-\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' L_2(d(x))^{\lambda_2+\delta_2\alpha_2} L_1(d(x))^{\mu_2+\delta_2\beta_2} d(x)^{\delta_2\gamma_2-k_2} \text{ in } \Omega.$$

Furthermore,

$$K_1(x)\overline{u}^{a_1} v^{b_1} \leq C_4 m^{-a_1-\sigma|b_1|} \Lambda_1(d(x)) d(x)^{a_1\gamma_1+b_1\gamma_2-k_1} \text{ in } \Omega.$$

Similarly,

$$K_2(x)\overline{v}^{a_2} u^{b_2} \leq C_4' m^{-\sigma a_2-|b_2|} \Lambda_2(d(x)) d(x)^{a_2\gamma_2+b_2\gamma_1-k_2} \text{ in } \Omega.$$

Then under (3.2) and thanks to (3.51) to (3.56), $(\overline{u},\overline{v})$ is a supersolutions pair of problem $(\mathbf{P})$, for m small enough. Therefore estimates (1.7) to (1.10) hold. Let us check that conditions (2.1) to (2.5) of Theorem 2.1 are satisfied. By estimates (3.49) and (3.50) and using the properties of the perturbations L_1 and L_2 given in point (a) and (c) of Remark 3.1, for all $\varepsilon > 0$ there exist positive constants C_1, C_2 and C_1', C_2' such that

$$C_1 d(x)^{\gamma_1} \leq \underline{u}, \overline{u} \leq C_2 d(x)^{\gamma_1-\varepsilon} \quad \text{and} \quad C_1' d(x)^{\gamma_2} \leq \underline{v}, \overline{v} \leq C_2' d(x)^{\gamma_2-\varepsilon} \quad \text{in } \Omega.$$

In addition, using (3.51) to (3.56), there exist positive constants κ_1, κ_2 such that

$$|f_1(x,u,v)| = K_1(x) u^{a_1} v^{b_1} \leq \kappa_1 d(x)^{\delta_1\gamma_1 - k_1 - \varepsilon} \quad \text{in } \Omega \times \mathcal{C},$$

$$|f_2(x,u,v)| = K_2(x) v^{a_2} u^{b_2} \leq \kappa_2 d(x)^{\delta_2\gamma_2 - k_2 - \varepsilon} \quad \text{in } \Omega \times \mathcal{C}$$

and

$$\left|\frac{\partial f_1}{\partial u}(x,u,v)\right| = |a_1| K_1(x) u^{a_1-1} v^{b_1} \leq \kappa_1 d(x)^{(\delta_1\gamma_1 - k_1 - \varepsilon)-\gamma_1} \quad \text{in } \Omega \times \mathcal{C},$$

$$\left|\frac{\partial f_2}{\partial v}(x,u,v)\right| = |a_2| K_2(x) v^{a_2-1} u^{b_2} \leq \kappa_2 d(x)^{(\delta_2\gamma_2 - k_2 - \varepsilon)-\gamma_2} \quad \text{in } \Omega \times \mathcal{C}.$$

Since $\gamma_1 \in (1-\frac{1}{p}, 1)$ and $\gamma_2 \in (1-\frac{1}{q}, 1)$, inequalities (2.5) hold for ε small enough. Then, applying Theorem 2.1 we conclude about the existence of positive solutions to $(\mathbf{P})$ in $\mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying the estimates (3.20) and (3.21).

Finally, using Theorem A.1, we get that any positive weak solutions pair to $(\mathbf{P})$ in the conical shell $\mathcal{C}$ belongs to $\mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$. This proves (i) of Theorem 3.7.

Alternative 2: In this part, we look for positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ by making the "Ansatz" that both function u and v behave like the distance function $d(x)$ for $x \in \Omega$ near the boundary $\partial\Omega$. For that, similarly as in *Alternative 1*, we take in (3.45) and (3.46)

$$k_1 - 1 < \delta_1 < p-1 \quad \text{and} \quad \mathrm{k}_2 - 1 < \delta_2 < q-1, \tag{3.57}$$

$$\mathcal{L}_1 = L_1 \quad \text{and} \quad \mathcal{L}_2 = L_2 \quad \text{in } \Omega.$$

By Theorem 3.3, $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ and satisfy

$$\psi_1(x) \sim d(x) \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

In view of satisfying estimates given in (3.47), we fix δ_1 and δ_2 as follows:

$$\delta_1 = a_1 + b_1 \quad \text{and} \quad \delta_2 = a_2 + b_2. \tag{3.58}$$

Then, (3.47) holds and inequalities given in (3.57) entail (3.22). The rest of the proof is as in *Alternative 1*. This proves (ii) of Theorem 3.7.

Alternative 3: Now we combine our methods from *Alternative 1* and *Alternative 2*. We search for positive solutions $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ to problem **(P)** by again making the "Ansatz" that

$$u(x) \sim d(x)^{\gamma} L_1(d(x))^{\alpha} L_2(d(x))^{\beta} \text{ in } \Omega,$$

for some $\gamma \in (1-\frac{1}{p}, 1)$ and $\alpha, \beta \in \mathbb{R}$, and v behave like the distance function in Ω. For that, we take in (3.45) and (3.46)

$$k_1 - 2 + \frac{k_1 - 1}{p-1} < \delta_1 < k_1 - 1 \quad \text{and} \quad \mathrm{k}_2 - 1 < \delta_2 < q - 1, \tag{3.59}$$

$$\mathcal{L}_1 = {L_1}^{\lambda_1}.{L_2}^{\mu_1} \quad \text{and} \quad \mathcal{L}_2 = {L_2}^{\lambda_2}.{L_1}^{\mu_2} \quad \text{in } \Omega,$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$ are suitable exponents to be fixed. By Theorem 3.3, $\psi_1 \in W_0^{1,p}(\Omega)$, $\psi_2 \in W_0^{1,q}(\Omega)$ and satisfy

$$\psi_1(x) \sim d(x)^{\frac{p-k_1}{p-1-\delta_1}} L_1(d(x))^{\frac{\lambda_1}{p-1-\delta_1}} L_2(d(x))^{\frac{\mu_1}{p-1-\delta_1}} \text{ and } \psi_2(x) \sim d(x) \text{ in } \Omega.$$

In view of (3.47), the exponents have to satisfy

$$\begin{cases} \delta_1 \frac{p-k_1}{p-1-\delta_1} = a_1 \frac{p-k_1}{p-1-\delta_1} + b_1, & \delta_2 = b_2 \frac{p-k_1}{p-1-\delta_1} + a_2, \\ \lambda_1 \frac{p-1}{p-1-\delta_1} = a_1 \frac{\lambda_1}{p-1-\delta_1} + 1, & \lambda_2 = b_2 \frac{\mu_1}{p-1-\delta_1} + 1, \\ \mu_1 \frac{p-1}{p-1-\delta_1} = a_1 \frac{\mu_1}{p-1-\delta_1}, & \mu_2 = b_2 \frac{\lambda_1}{p-1-\delta_1}. \end{cases}$$

Hence we obtain

$$\gamma = \frac{p-k_1}{p-1-\delta_1} = \frac{p-k_1+b_1}{p-1-a_1} \quad \text{and} \quad \delta_2 = a_2 + b_2 \frac{p-k_1+b_1}{p-1-a_1},$$

$$\alpha = \frac{\lambda_1}{p-1-\delta_1} = \frac{1}{p-1-\delta_1} \quad \text{and} \quad \beta = \frac{\mu_1}{p-1-\delta_1} = 0.$$

The rest of the proof is as in *Alternative 1*. This proves (iii) of Theorem 3.7 and (iv) is the corresponding symmetric case of (iii). □

Proof of Theorem 3.8

To prove uniqueness of solutions, we apply a classical argument of KRANSNOSELSKII [18]. Let $(u,v), (\tilde{u}, \tilde{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, be two distinct positive weak solutions pairs to problem $(\mathbf{P})$ in the conical shell $\mathcal{C} = [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, where $(\underline{u}, \underline{v})$,

$(\overline{u},\overline{v})$ are given in the proof of Theorem 3.7. This means that $T(u,v) = (u,v)$ and $T(\tilde{u},\tilde{v}) = (\tilde{u},\tilde{v})$, which implies that, $T_1 \circ T_2(u) = u$, $T_2 \circ T_1(v) = v$ and $T_1 \circ T_2(\tilde{u}) = \tilde{u}$, $T_1 \circ T_2(\tilde{v}) = \tilde{v}$, respectively. Let us define

$$C_{\max} \stackrel{\text{def}}{=} \sup\{C \in \mathbb{R}_+, \quad C\tilde{u} \leq u \quad \text{and} \quad C\tilde{v} \leq v \quad \text{a.e. in } \Omega\}. \tag{3.60}$$

$$T_1 \circ T_2(C_{\max}\tilde{u}) = (C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}} T_1 \circ T_2(\tilde{u}) = (C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}} \tilde{u},$$

$$T_2 \circ T_1(C_{\max}\tilde{v}) = (C_{\max})^{\frac{b_2}{q-1-a_2}\cdot\frac{b_1}{p-1-a_1}} T_2 \circ T_1(\tilde{v}) = (C_{\max})^{\frac{b_2}{q-1-a_2}\cdot\frac{b_1}{p-1-a_2}} \tilde{v}.$$

Therefore, by Theorem 3.5, both mappings $T_1 \circ T_2$ and $T_2 \circ T_1$ being (pointwise) order-preserving, we arrive at

$$u = T_1 \circ T_2(u) \geq T_1 \circ T_2(C_{\max}\tilde{u}) = (C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}} \tilde{u}, \tag{3.61}$$

$$v = T_2 \circ T_1(v) \geq T_2 \circ T_1(C_{\max}\tilde{v}) = (C_{\max})^{\frac{b_2}{q-1-a_2}\cdot\frac{b_1}{p-1-a_1}} \tilde{v}. \tag{3.62}$$

From $0 < C_{\max} < 1$ combined with the subhomogeneity condition (3.1) we deduce that

$$C'_{\max} \stackrel{\text{def}}{=} (C_{\max})^{\frac{b_1}{p-1-a_1}\cdot\frac{b_2}{q-1-a_2}} > C_{\max},$$

which contradicts the maximality of the constant $C_{\max}$ in (3.60), by inequalities (3.61) and (3.62). Then, $C_{\max} \geq 1$ which entails $\tilde{u} \leq u$ and $\tilde{v} \leq v$ a.e. in Ω. Interchanging the roles of (u,v) and $(\tilde{u},\tilde{v})$, we finally get $(u,v) = (\tilde{u},\tilde{v})$ a.e. in Ω. □

Proof of Theorem 3.9 The proof is very similar to the proof of Theorem 3.7. So we omit it. □

Proof of Theorem 3.10

Alternative 1: Assume that $a_1 + b_1 = k_1 - 1$ and $k_2 - 1 \leq a_2 + b_2 < q - 1$. We look for positive sub and supersolutions pairs $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ in the form:

$$\underline{u} = m\psi_1 \quad \text{and} \quad \overline{u} = m^{-1}(\varphi_{1,p})^{1-\varepsilon} \quad \text{in } \Omega,$$

$$\underline{v} = m^{\sigma}\psi_2 \quad \text{and} \quad \overline{v} = m^{-\sigma}(\varphi_{1,q})^{1-\sigma\varepsilon} \quad \text{in } \Omega,$$

where $\sigma > 0$ is given by (3.2), $\varepsilon < 1$ and $m < 1$ are appropriate positive constants small enough and $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$ and $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ are the respective solutions to

$$-\Delta_p w = K_1(x)w^{\delta_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

$$-\Delta_q w = K_2(x)w^{\delta_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

with $k_1 - 1 < \delta_1 < p - 1$ and $a_2 + b_2 < \delta_2 < q - 1$. By Theorem 3.3, both ψ_1 and ψ_2 behave like the distance function in Ω. Let us remark that by estimate (1.5), $\underline{u} \leq \overline{u}$ and $\underline{v} \leq \overline{v}$ in Ω, for m small enough. Now, let $1 < r < \infty$ and $\gamma \in (0, 1)$, then we have

$$\begin{aligned} -\Delta_r\left[(\varphi_{1,r})^\gamma\right] &= \gamma^{r-1}\left[\lambda_{1,r}(\varphi_{1,r})^{\gamma(r-1)} - (\gamma-1)(r-1)(\varphi_{1,r})^{(\gamma-1)(r-1)-1}|\nabla\varphi_{1,r}|^r\right] \\ &= \gamma^{r-1}(\varphi_{1,r})^{-(1-\gamma)(r-1)-1}\left[\lambda_{1,r}(\varphi_{1,r})^r + (1-\gamma)(r-1)|\nabla\varphi_{1,r}|^r\right] \end{aligned}$$

in Ω. By estimate (1.5), we conclude that

$$-\Delta_r\left[(\varphi_{1,r}(x))^\gamma\right] \sim d(x)^{-(1-\gamma)(r-1)-1} \quad \text{in } \Omega. \tag{3.63}$$

So, let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. On one hand, we have

$$-\Delta_p \underline{u} \leq m^{p-1}C_1K_1(x)d(x)^{\delta_1} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{q-1}C_1'K_2(x)d(x)^{\delta_2} \quad \text{in } \Omega.$$

On the other hand, we also have

$$\begin{aligned} K_1(x)\underline{u}^{a_1}v^{b_1} &\geq \begin{cases} m^{a_1+\sigma b_1}K_1(x){\psi_1}^{a_1}{\psi_2}^{b_1} & \text{if } b_1 > 0, \\ m^{a_1-\sigma b_1}K_1(x){\psi_1}^{a_1}(\varphi_{1,q})^{b_1(1-\varepsilon\sigma)} & \text{if } b_1 < 0, \end{cases} \\ &\geq m^{a_1+\sigma|b_1|}C_2K_1(x)d(x)^{k_1-1+\varepsilon\sigma b_1^-} \quad \text{in } \Omega, \end{aligned}$$

in Ω. Similarly, we get

$$K_2(x)\underline{v}^{a_2}u^{b_2} \geq m^{\sigma a_2+|b_2|}C_2'K_2(x)d(x)^{a_2+b_2+\varepsilon b_2^-} \quad \text{in } \Omega.$$

Then, for m and ε small enough, $(\underline{u}, \underline{v})$ is a subsolutions pair of problem $(\mathbf{P})$. Similarly, using estimate (3.63), we obtain

$$-\Delta_p\overline{u} \geq m^{1-p}C_3d(x)^{-1-\varepsilon(p-1)} \quad \text{and} \quad -\Delta_q\overline{v} \geq m^{\sigma(1-q)}C_3'd(x)^{-1-\varepsilon\sigma(q-1)} \quad \text{in } \Omega.$$

Furthermore, by (3.5), for any $\varepsilon' > 0$, there exists $C_4 = C_4(\varepsilon') > 0$ such that

$$\begin{aligned} K_1(x)\overline{u}^{a_1}v^{b_1} &\leq \begin{cases} m^{-(a_1+\sigma b_1)}K_1(x)(\varphi_{1,p})^{a_1(1-\varepsilon)}(\varphi_{1,q})^{b_1(1-\varepsilon\sigma)} & \text{if } b_1 > 0, \\ m^{-(a_1-\sigma b_1)}K_1(x)(\varphi_{1,p})^{a_1(1-\varepsilon)}{\psi_2}^{b_1} & \text{if } b_1 < 0, \end{cases} \\ &\leq m^{-(a_1+\sigma|b_1|)}C_4d(x)^{-1-\varepsilon(a_1+\sigma b_1^+)-\varepsilon'} \quad \text{in } \Omega, \end{aligned}$$

Similarly, we have

$$K_2(x)\underline{v}^{a_2}u^{b_2} \leq m^{-(\sigma a_2+|b_2|)}C_4'd(x)^{-k_1+a_2+b_2-\varepsilon(\sigma a_2+b_2^+)-\varepsilon'} \quad \text{in } \Omega,$$

with $C_4' = C_4'(\varepsilon')$. Then, for m, ε and ε' small enough, $(\overline{u}, \overline{v})$ is a supersolutions pair of problem $(\mathbf{P})$. Applying Theorem 2.1, we get the existence of positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ of $(\mathbf{P})$ satisfying (3.69). This proves (i) of Theorem 3.10.

Alternative 2: When $k_1 - 1 \leq a_1 + b_1 < q - 1$ and $a_2 + b_2 = k_2 - 1$, interchanging the role of u and v, the proof of (ii) is the same as above.

Alternative 3: Assume that (3.41) is satisfied. To prove *(3)*, we follow the proof in *Alternative 1*. We construct positive sub and supersolutions pairs $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ in the form

$$\underline{u} = m(\varphi_{1,p})^{\gamma+\varepsilon},\ \overline{u} = m^{-1}(\varphi_{1,p})^{\gamma-\varepsilon} \text{ and } \underline{v} = m^{\sigma}\psi,\ \overline{v} = m^{-\sigma}(\varphi_{1,q})^{1-\sigma\varepsilon} \text{ in } \Omega,$$

where $\sigma > 0$ is given by (3.2), and ε, m are appropriate positive constants small enough and $\psi \in \mathrm{W}_0^{1,q}(\Omega)$ is the solution (see Theorem 3.3) of

$$-\Delta_q w = K_2(x)w^{\delta} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

with $a_2 + \gamma b_2 < \delta < q - 1$. (iv) is the symmetric case of *(3)* by interchanging the role of u and v. Finally, from Theorem A.1, we get the Hölder regularity of (u, v). $\square$

3.2 Example 2

We consider now the following singular system

$$(\mathbf{P}) \begin{cases} -\Delta_p u = u^{a_1}v^{b_1} - u^{\alpha_1}v^{\beta_1} & \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ -\Delta_q v = v^{a_2}u^{b_2} - v^{\alpha_2}u^{\beta_2} & \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

where the above exponents satisfy

$$(p - 1 - a_1) - \sigma|b_1| > 0 \quad \text{and} \quad (\alpha_1 - a_1) - \sigma(|\beta_1| - |b_1|) > 0, \tag{3.64}$$

$$\sigma(q - 1 - a_2) - |b_2| > 0 \quad \text{and} \quad \sigma(\alpha_2 - a_2) - (|\beta_2| - |b_2|) > 0, \tag{3.65}$$

for some constant $\sigma > 0$. Then, we have the following result:

Theorem 3.11.

(i) *Let*

$$\gamma_1 = \frac{p(q-1-a_2)+qb_1}{(p-1-a_1)(q-1-a_2)-b_1b_2}, \quad \gamma_2 = \frac{q(p-1-a_1)+pb_2}{(p-1-a_1)(q-1-a_2)-b_1b_2} \tag{3.66}$$

and assume that

$$1-\frac{1}{p} < \gamma_1 < 1 \quad \textit{and} \quad (\alpha_1 - a_1)\gamma_1 + (\beta_1 - b_1)\gamma_2 > 0, \tag{3.67}$$

$$1-\frac{1}{q} < \gamma_2 < 1 \quad \textit{and} \quad (\alpha_2 - a_2)\gamma_2 + (\beta_2 - b_2)\gamma_1 > 0. \tag{3.68}$$

Then, problem $(\mathbf{P})$ *has a positive solution* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x) \sim d(x)^{\gamma_1} \quad \textit{and} \quad v(x) \sim d(x)^{\gamma_2} \quad \textit{in } \Omega. \tag{3.69}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(ii) *Assume that*

$$-1 < a_1 + b_1 < p-1 \quad \textit{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1) > 0, \tag{3.70}$$

$$-1 < a_2 + b_2 < q-1 \quad \textit{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2) > 0. \tag{3.71}$$

Then, $(\mathbf{P})$ *has a positive solution* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x) \sim d(x) \quad \textit{and} \quad v(x) \sim d(x) \quad \textit{in } \Omega. \tag{3.72}$$

In addition, we have $(u,v) \in \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(iii) *Let*

$$\gamma = \frac{p+b_1}{p-1-a_1} \tag{3.73}$$

and assume that

$$1-\frac{1}{p} < \gamma < 1 \quad \textit{and} \quad (\alpha_1 - a_1)\gamma + (\beta_1 - b_1) > 0, \tag{3.74}$$

$$-1 < a_2 + b_2\gamma < p-1 \quad \textit{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2)\gamma > 0. \tag{3.75}$$

Then, $(\mathbf{P})$ *has a positive solution* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x) \sim d(x)^{\gamma} \quad \textit{and} \quad v(x) \sim d(x) \quad \textit{in } \Omega. \tag{3.76}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$, *for some* $0<\alpha<1$.

(iv) *Symmetrically, set*

$$\gamma = \frac{q+b_2}{q-1-a_2} \tag{3.77}$$

and assume that

$$-1 < a_1 + b_1\gamma < p-1 \quad \textit{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1)\gamma > 0, \tag{3.78}$$

$$1 - \frac{1}{q} < \gamma < 1 \quad \textit{and} \quad (\alpha_2 - a_2)\gamma + (\beta_2 - b_2) > 0. \tag{3.79}$$

Then, $(\mathbf{P})$ *has a positive solution* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying*

$$u(x) \sim d(x) \quad \textit{and} \quad v(x) \sim d(x)^{\gamma} \quad \textit{in } \Omega. \tag{3.80}$$

In addition, we have $(u,v) \in \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$ *, for some* $0 < \alpha < 1$.

Proof. We apply Theorem 2.1 with

$$\underline{u} \equiv m\psi_1, \quad \overline{u} \equiv m^{-1}\psi_1 \quad \text{and} \quad \underline{v} \equiv m^{\sigma}\psi_2, \quad \overline{v} \equiv m^{-\sigma}\psi_2 \quad \text{in } \Omega,$$

where $\sigma > 0$ is the constant given in (3.64) and (3.65), $m < 1$ is a positive constant small enough and $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ are given by Theorem 3.3 as the respective unique solutions of problems

$$-\Delta_p w = w^{\delta_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

$$-\Delta_q w = w^{\delta_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

satisfying some cone conditions we precise below. In the following *Alternatives*, we choose $-2 - \frac{1}{p-1} < \delta_1 < p-1$ and $-2 - \frac{1}{q-1} < \delta_2 < q-1$ such that

$$-\Delta_p \psi_1 \sim {\psi_1}^{a_1}{\psi_2}^{b_1} \quad \text{and} \quad -\Delta_q \psi_2 \sim {\psi_2}^{a_2}{\psi_1}^{b_2} \quad \text{in } \Omega. \tag{3.81}$$

Alternative 1: Assume that conditions (3.67) and (3.68) hold. Then, arguing as in *Alternative 1* in the proof of Theorem 3.7, we choose $-2 - \frac{1}{p-1} < \delta_1 < -1$ and $-2 - \frac{1}{q-1} < \delta_2 < -1$ unique solutions pair of the following system:

$$\frac{\delta_1 p}{p-1-\delta_1} = \frac{a_1 p}{p-1-\delta_1} + \frac{b_1 q}{q-1-\delta_2} \quad \text{and} \quad \frac{\delta_2 q}{q-1-\delta_2} = \frac{a_2 q}{q-1-\delta_2} + \frac{b_2 p}{p-1-\delta_2}.$$

Since

$$\psi_1(x) \sim d(x)^{\gamma_1} \quad \text{and} \quad \psi_2(x) \sim d(x)^{\gamma_2} \quad \text{in } \Omega,$$

where $\gamma_1 = \frac{p}{p-1-\delta_1}$ and $\gamma_2 = \frac{q}{q-1-\delta_2}$ are given by (3.66), estimates (3.81) follows. Let $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. First, we have

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{\delta_1\gamma_1} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{\delta_2\gamma_2} \quad \text{in } \Omega. \tag{3.82}$$

On the other hand, by (3.64) and (3.67),

$$\begin{aligned} \underline{u}^{a_1} v^{b_1} - \underline{u}^{\alpha_1} v^{\beta_1} &\geq m^{a_1+\sigma|b_1|} {\psi_1}^{a_1} {\psi_2}^{b_1} \left[1 - m^{\alpha_1 - a_1 - \sigma(|\beta_1|-|b_1|)} {\psi_1}^{\alpha_1-a_1} {\psi_2}^{\beta_1-b_1}\right] \\ &\geq m^{a_1+\sigma|b_1|} C_2 d(x)^{a_1\gamma_1+b_1\gamma_2} \quad \text{in } \Omega. \end{aligned} \tag{3.83}$$

for m small enough. By (3.65) and (3.68), we also have

$$\underline{v}^{a_2} u^{b_2} - \underline{v}^{\alpha_2} u^{\beta_2} \geq m^{\sigma a_2 + |b_2|} C_2' d(x)^{a_2\gamma_2+b_2\gamma_1} \quad \text{in } \Omega, \tag{3.84}$$

for m small enough. Then, under conditions (3.64), (3.65), (3.67) and (3.68) and for m small enough, $(\underline{u}, \underline{v})$ is a subsolutions pair of problem $(\mathbf{P})$.

Similarly, we have

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{\delta_1\gamma_1} \quad \text{and} \quad -\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{\delta_2\gamma_2} \quad \text{in } \Omega. \tag{3.85}$$

In addition,

$$\overline{u}^{a_1} v^{b_1} - \overline{u}^{\alpha_1} v^{\beta_1} \leq m^{-a_1-\sigma|b_1|} {\psi_1}^{a_1} {\psi_2}^{b_1} \leq m^{-a_1-\sigma|b_1|} C_4 d(x)^{a_1\gamma_1+b_1\gamma_2} \tag{3.86}$$

in Ω. We obtain further

$$\overline{v}^{a_2} u^{b_2} - \overline{v}^{\alpha_2} u^{\beta_2} \leq m^{-\sigma a_2 - |b_2|} C_4' d(x)^{a_2\gamma_2+b_2\gamma_1} \quad \text{in } \Omega. \tag{3.87}$$

Then, under conditions (3.64), (3.65) and for m small enough, $(\underline{u}, \underline{v})$ is a supersolutions pair of problem $(\mathbf{P})$.

Applying Theorem 2.1, we get the existence of positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ of $(\mathbf{P})$ satisfying (3.69). Again from Theorem A.1, (u,v) are Hölder continuous. This proves the assertion (i).

Alternative 2: Now, assume that conditions (3.70) and (3.71) are satisfied. Then, we choose $\delta_1 = a_1 + b_1$ and $\delta_2 = a_2 + b_2$. By Theorem 3.3, since

$$\psi_1(x) \sim d(x) \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega,$$

estimates (3.81) hold. Instead of inequalities (3.82), we have in this case

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{a_1+b_1} \text{ and } -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{a_2+b_2} \quad \text{in } \Omega.$$

From (3.64), (3.65), (3.70) and (3.71), we get for any $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$:

$$\underline{u}^{a_1}v^{b_1}-\underline{u}^{\alpha_1}v^{\beta_1}\geq m^{a_1+\sigma|b_1|}C_2d(x)^{a_1+b_1}\quad \text{in } \Omega,$$
$$\underline{v}^{a_2}u^{b_2}-\underline{v}^{\alpha_2}u^{\beta_2}\geq m^{\sigma a_2+|b_2|}C_2'd(x)^{a_2+b_2}\quad \text{in } \Omega,$$

for m small enough. Then, under conditions (3.64), (3.65), (3.70), (3.71) and for m small enough, $(\underline{u},\underline{v})$ is a subsolution pair of problem $(\mathbf{P})$. Instead of inequalities (3.85), we have in this case in Ω,

$$-\Delta_p\overline{u}\geq m^{1-p}C_3d(x)^{a_1+b_1}\quad \text{and}\quad -\Delta_q\overline{v}\geq m^{\sigma(1-q)}C_3'd(x)^{a_2+b_2}.$$

In addition, instead of inequalities (3.86) and (3.87), we get

$$\overline{u}^{a_1}v^{b_1}-\overline{u}^{\alpha_1}v^{\beta_1}\leq m^{-a_1-\sigma|b_1|}C_4d(x)^{a_1+b_1},$$
$$\overline{v}^{a_2}u^{b_2}-\overline{v}^{\alpha_2}u^{\beta_2}\leq m^{-\sigma a_2-|b_2|}C_4'd(x)^{a_2+b_2},$$

in Ω. Then, under conditions (3.64), (3.65) and for m small enough, $(\overline{u},\overline{v})$ is a supersolution pair of problem $(\mathbf{P})$. Then, we conclude as in the *Alternative 1* and (ii) is proved.

Alternative 3: Now, assume conditions (3.74) and (3.75) hold. Then, arguing as in the proof of Theorem 3.7, we choose $-2-\frac{1}{p}<\delta_1<-1$ and $-1<\delta_2<q-1$ unique solutions pair of the following system:

$$\frac{\delta_1 p}{p-1-\delta_1}=\frac{a_1p}{p-1-\delta_1}+b_1\quad \text{and}\quad \delta_2=a_2+\frac{b_2p}{p-1-\delta_2}.$$

Estimates in (3.81) hold since

$$\psi_1(x)\sim d(x)^\gamma\quad \text{and}\quad \psi_2(x)\sim d(x)\quad \text{in } \Omega,$$

with γ given by (3.73). Instead of inequalities (3.82), we have in this case

$$-\Delta_p\underline{u}\leq m^{p-1}C_1d(x)^{\delta_1\gamma}\quad \text{and}\quad -\Delta_q\underline{v}\leq m^{\sigma(q-1)}C_1'd(x)^{\delta_2}\quad \text{in } \Omega.$$

From (3.64), (3.65), (3.74) and (3.75), we obtain now

$$\underline{u}^{a_1}v^{b_1}-\underline{u}^{\alpha_1}v^{\beta_1}\geq m^{a_1+\sigma|b_2|}C_2d(x)^{a_1\gamma+b_1}\quad \text{in } \Omega,$$
$$\underline{v}^{a_2}u^{b_2}-\underline{v}^{\alpha_2}u^{\beta_2}\geq m^{\sigma a_2+|b_2|}C_2'd(x)^{a_2+b_2\gamma}\quad \text{in } \Omega,$$

for m small enough. Then, under conditions (3.64), (3.65), (3.74), (3.75) and for m small enough, $(\underline{u},\underline{v})$ is a subsolution pair of problem $(\mathbf{P})$. Instead of (3.85), we have

$$-\Delta_p\overline{u}\geq m^{1-p}C_3d(x)^{\delta_1\gamma}\quad \text{and}\quad -\Delta_p\overline{v}\geq m^{\sigma(1-q)}C_3'd(x)^{\delta_2}\quad \text{in } \Omega.$$

And inequalities (3.86) are replaced by

$$\overline{u}^{a_1}v^{b_1} - \overline{u}^{\alpha_1}v^{\beta_1} \leq m^{-a_1-\sigma|b_1|}C_4 d(x)^{a_1\gamma+b_1} \quad \text{in } \Omega,$$

$$\overline{v}^{a_2}u^{b_2} - \overline{v}^{\alpha_2}u^{\beta_2} \leq m^{-\sigma a_2-|b_2|}C_4' d(x)^{a_2+b_2\gamma} \quad \text{in } \Omega.$$

Then, under conditions (3.64), (3.65) and for m small enough, $(\overline{u}, \overline{v})$ is a supersolution pair of problem $(\mathbf{P})$. We conclude as in the *Alternative 1*. Thus, (iii) is proved. Note that (iv) is the symmetric case of (iii) by interchanging u and v. □

We can further prove similarly (we omit the proof):

Theorem 3.12. *Assume that conditions* (3.64) *and* (3.65) *are satisfied.*

(i) *Assume that*

$$a_1 + b_1 = -1 \quad \text{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1) > 0, \tag{3.88}$$

$$-1 \leq a_2 + b_2 < q - 1 \quad \text{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2) > 0. \tag{3.89}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that $(\boldsymbol{P})$ admits positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying:

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \text{ and } C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \text{ in } \Omega, \tag{3.90}$$

with $\sigma > 0$ is given in (3.2)*. In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$.*

(ii) *Symmetrically, assume that*

$$-1 \leq a_1 + b_1 < q - 1 \quad \text{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1) > 0, \tag{3.91}$$

$$a_2 + b_2 = -1 \quad \text{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2) > 0. \tag{3.92}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that $(\boldsymbol{P})$ admits positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying:

$$C_1 d(x) \leq u \leq C_2 d(x)^{1-\varepsilon} \text{ and } C_1' d(x) \leq v \leq C_2' d(x)^{1-\varepsilon\sigma} \text{ in } \Omega. \tag{3.93}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$.

(iii) *Let*

$$\gamma = \frac{p + b_1}{p - 1 - a_1}$$

and assume that

$$1 - \frac{1}{p} < \gamma < 1 \quad \text{and} \quad (\alpha_1 - a_1)\gamma + (\beta_1 - b_1) > 0, \tag{3.94}$$

$$a_2 + b_2\gamma = -1 \quad \text{and} \quad (\alpha_2 - a_2) + (\beta_2 - b_2)\gamma > 0. \tag{3.95}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that $(\boldsymbol{P})$ admits positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying:

$$C_1 d(x)^{\gamma+\varepsilon} \le u \le C_2 d(x)^{\gamma-\varepsilon} \text{ and } C_1' d(x) \le v \le C_2' d(x)^{1-\varepsilon\sigma} \text{ in } \Omega. \tag{3.96}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$.

(iv) *Symmetrically, let*

$$\gamma = \frac{q + b_2}{q - 1 - a_2}$$

and assume that

$$a_1 + b_1\gamma = -1 \quad \text{and} \quad (\alpha_1 - a_1) + (\beta_1 - b_1)\gamma > 0, \tag{3.97}$$

$$1 - \frac{1}{q} < \gamma < 1 \quad \text{and} \quad (\alpha_2 - a_2)\gamma + (\beta_2 - b_2) > 0. \tag{3.98}$$

Then, for all $\varepsilon > 0$ small enough, there exist $C_1, C_2 > 0$ and $C_1', C_2' > 0$ such that $(\boldsymbol{P})$ admits positive solutions $(u, v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ satisfying:

$$C_1 d(x) \le u \le C_2 d(x)^{1-\varepsilon} \text{ and } C_1' d(x)^{\gamma+\varepsilon\sigma} \le v \le C_2' d(x)^{\gamma-\varepsilon\sigma} \text{ in } \Omega. \tag{3.99}$$

In addition, we have $(u, v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, for some $0 < \alpha < 1$.

3.3 Example 3

In this section, we consider the following singular competition system

$$(\mathbf{P}) \begin{cases} -\Delta_p u = \lambda_1 u^{\alpha_1} - u^{\beta_1} - \mu_1 u^{a_1} v^{b_1} & \text{in } \Omega\,; \quad u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \\ -\Delta_q v = \lambda_2 v^{\alpha_2} - v^{\beta_2} - \mu_2 v^{a_2} u^{b_2} & \text{in } \Omega\,; \quad v|_{\partial\Omega} = 0, \quad v > 0 \quad \text{in } \Omega, \end{cases}$$

where λ_1, λ_2 and μ_1, μ_2 are positive and $\alpha_1, \alpha_2, \beta_1, \beta_2, a_1, a_2, b_1, b_2$ satisfy

$$-2 - \frac{1}{p-1} < \alpha_1 < p - 1, \quad \alpha_1 < \beta_1 \quad \text{and} \quad a_1 - \alpha_1 - \sigma|b_1| > 0, \tag{3.100}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < q - 1, \quad \alpha_2 < \beta_2 \quad \text{and} \quad \sigma(a_2 - \alpha_2) - |b_2| > 0, \tag{3.101}$$

for some constant $\sigma > 0$. Then, we have

Theorem 3.13. (i) *Assume that*

$$-2-\frac{1}{p-1}<\alpha_1<-1 \quad \text{and} \quad \frac{(a_1-\alpha_1)p}{p-1-\alpha_1}+\frac{b_1q}{q-1-\alpha_2}>0, \tag{3.102}$$

$$-2-\frac{1}{q-1}<\alpha_2<-1 \quad \text{and} \quad \frac{(a_2-\alpha_2)q}{q-1-\alpha_2}+\frac{b_2p}{p-1-\alpha_1}>0. \tag{3.103}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ *satisfying:*

$$u(x)\sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \text{and} \quad v(x)\sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \text{in } \Omega. \tag{3.104}$$

In addition, we have $(u,v)\in\mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)\times\mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$ *, for some* $0<\alpha<1$.

(ii) *Assume that*

$$-1<\alpha_1<p-1 \quad \text{and} \quad a_1-\alpha_1+b_1>0, \tag{3.105}$$

$$-1<\alpha_2<q-1 \quad \text{and} \quad a_2-\alpha_2+b_2>0. \tag{3.106}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ *satisfying:*

$$u(x)\sim d(x) \quad \text{and} \quad v(x)\sim d(x) \quad \text{in } \Omega. \tag{3.107}$$

In addition, we have $(u,v)\in\mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)\times\mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$ *, for some* $0<\alpha<1$.

(iii) *Assume that*

$$-2-\frac{1}{p-1}<\alpha_1<-1 \quad \text{and} \quad (a_1-\alpha_1+b_1)p-b_1(\alpha_1+1)>0, \tag{3.108}$$

$$-1<\alpha_2<q-1 \quad \text{and} \quad (a_2-\alpha_2+b_2)p-(a_2-\alpha_2)(\alpha_1+1)>0. \tag{3.109}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ *satisfying:*

$$u(x)\sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \text{and} \quad v(x)\sim d(x) \quad \text{in } \Omega. \tag{3.110}$$

In addition, we have $(u,v)\in\mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)\times\mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$ *, for some* $0<\alpha<1$.

(iv) *Symmetrically, assume that*

$$-1<\alpha_1<p-1 \quad \text{and} \quad (a_1-\alpha_1+b_1)q-(a_1-\alpha_1)(\alpha_2+1)>0, \tag{3.111}$$

$$-2-\frac{1}{q-1}<\alpha_2<-1 \quad \text{and} \quad (a_2-\alpha_2+b_2)q-b_2(\alpha_2+1)>0. \tag{3.112}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ *satisfying:*

$$u(x)\sim d(x) \quad \text{and} \quad v(x)\sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \text{in } \Omega. \tag{3.113}$$

In addition, we have $(u,v)\in\mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)\times\mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$ *, for some* $0<\alpha<1$.

Proof. We apply Theorem 2.1 with

$$\underline{u} \equiv m\psi_1, \quad \overline{u} \equiv m^{-1}\psi_1 \quad \text{and} \quad \underline{v} \equiv m^{\sigma}\psi_2, \quad \overline{v} \equiv m^{-\sigma}\psi_2 \quad \text{in } \Omega, \tag{3.114}$$

where $\sigma > 0$ is the constant given in (3.100) and (3.101), $m < 1$ is a suitable small positive constant and $\psi_1 \in \mathrm{W}_0^{1,p}(\Omega)$, $\psi_2 \in \mathrm{W}_0^{1,q}(\Omega)$ are (given by Theorem 3.3) the respective unique solutions of problems

$$-\Delta_p w = w^{\alpha_1} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega,$$

$$-\Delta_q w = w^{\alpha_2} \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega.$$

Alternative 1: Assume conditions (3.102) and (3.103) are satisfied. Then, from Theorem 3.3, we get

$$\psi_1(x) \sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \text{and} \quad \psi_2(x) \sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \text{in } \Omega.$$

Let us prove that, for m small enough, $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are respectively sub and supersolutions pairs of $(\mathbf{P})$. Let $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. We have in Ω,

$$-\Delta_p \underline{u} \leq m^{p-1} C_1 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \ \text{and} \ -\Delta_q \underline{v} \leq m^{\sigma(q-1)} C_1' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}}. \tag{3.115}$$

From (3.100) and (3.102), we obtain:

$$\begin{aligned} &\lambda_1 \underline{u}^{\alpha_1} - \underline{u}^{\beta_1} - \mu_1 \underline{u}^{a_1} v^{b_1} \\ &\geq \lambda_1 (m\psi_1)^{\alpha_1} \left[1 - \tfrac{1}{\lambda_1}(m\psi_1)^{\beta_1-\alpha_1} - \tfrac{\mu_1}{\lambda_1}(m\psi_1)^{a_1-\alpha_1} \left(m^{-\sigma \operatorname{sign}(b_1)}\psi_2\right)^{b_1}\right] \\ &\geq \tfrac{\lambda_1}{2} m^{\alpha_1} C_2 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}}, \end{aligned} \tag{3.116}$$

for m small enough. In addition, from (3.101) and (3.103), we get:

$$\lambda_2 \underline{v}^{\alpha_2} - \underline{v}^{\beta_2} - \mu_2 \underline{v}^{a_2} u^{b_2} \geq \frac{\lambda_1}{2} m^{\sigma\alpha_2} C_2' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \quad \text{in } \Omega, \tag{3.117}$$

for m small enough. Then, under conditions (3.102), (3.103) and for m small enough, $(\underline{u}, \underline{v})$ is a subsolutions pair of problem $(\mathbf{P})$. We also get

$$-\Delta_p \overline{u} \geq m^{1-p} C_3 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \text{ and } -\Delta_q \overline{v} \geq m^{\sigma(1-q)} C_3' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \text{ in } \Omega. \tag{3.118}$$

Similarly, one has

$$\lambda_1 \overline{u}^{\alpha_1} - \overline{u}^{\beta_1} - \mu_1 \overline{u}^{a_1} v^{b_1} \leq \lambda_1 m^{-\alpha_1} C_4 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega, \tag{3.119}$$

$$\lambda_2\overline{v}^{\alpha_2} - \overline{v}^{\beta_2} - \mu_2\overline{v}^{a_2}u^{b_2} \leq \lambda_2 m^{-\sigma\alpha_2}C_4' d(x)^{\frac{\alpha_2 q}{q-1-\alpha_2}} \quad \text{in } \Omega. \tag{3.120}$$

Then, for m small enough, $(\overline{u},\overline{v})$ is a supersolutions pair of problem $(\mathbf{P})$.

Applying Theorem 2.1, we get the existence of positive solutions $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ of $(\mathbf{P})$ satisfying (3.104). From Theorem A.1, we get the Hölder regularity of u and v. This proves (i).

Alternative 2: Now, let conditions (3.105) and (3.106) be satisfied. Then,

$$\psi_1(x) \sim d(x) \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

Let $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. Instead of (3.115), we now get

$$-\Delta_p \underline{u} \leq m^{p-1}C_1 d(x)^{\alpha_1} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{\sigma(q-1)}C_1' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

From (3.100), (3.101), (3.105) and (3.106), instead of (3.116) and (3.117), we have

$$\lambda_1\underline{u}^{\alpha_1} - \underline{u}^{\beta_1} - \mu_1\underline{u}^{a_1}v^{b_1} \geq \frac{\lambda_1}{2}m^{\alpha_1}C_2 d(x)^{\alpha_1} \quad \text{in } \Omega,$$

$$\lambda_2\underline{v}^{\alpha_2} - \underline{v}^{\beta_2} - \mu_2\underline{v}^{a_2}u^{b_2} \geq \frac{\lambda_2}{2}m^{\sigma\alpha_2}C_2' d(x)^{\alpha_2} \quad \text{in } \Omega,$$

for m small enough. Then, under conditions (3.105), (3.106) and for m small enough, $(\underline{u},\underline{v})$ is a subsolutions pair of problem $(\mathbf{P})$. Instead of (3.118), we have

$$-\Delta_p \overline{u} \geq m^{1-p}C_3 d(x)^{\alpha_1} \quad \text{and} \quad -\Delta_q \overline{v} \geq m^{\sigma(1-q)}C_3' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

Furthermore, the following inequalities

$$\lambda_1\overline{u}^{\alpha_1} - \overline{u}^{\beta_1} - \mu_1\overline{u}^{a_1}v^{b_1} \leq \lambda_1 m^{-\alpha_1}C_4 d(x)^{\alpha_1} \ \text{ in } \Omega,$$

$$\lambda_2\overline{v}^{\alpha_2} - \overline{v}^{\beta_2} - \mu_2\overline{v}^{a_2}u^{b_2} \leq \lambda_1 m^{-\sigma\alpha_2}C_4' d(x)^{\alpha_2} \ \text{ in } \Omega$$

replace (3.119) and (3.120). Then, for m small enough, $(\overline{u},\overline{v})$ is a supersolutions pair of problem $(\mathbf{P})$. We conclude as in the Alternative 1 and (ii) is proved.

Alternative 3: Now, assume that conditions (3.108) and (3.109) are satisfied. Then,

$$\psi_1(x) \sim d(x)^{\frac{p}{p-1-\alpha_1}} \quad \text{and} \quad \psi_2(x) \sim d(x) \quad \text{in } \Omega.$$

Let $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. Instead of (3.115), we have

$$-\Delta_p \underline{u} \leq m^{p-1}C_1 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{and} \quad -\Delta_q \underline{v} \leq m^{\sigma(q-1)}C_1' d(x)^{\alpha_2} \quad \text{in } \Omega.$$

From (3.100), (3.101), (3.108) and (3.109), instead of (3.116) and (3.117), we get

$$\lambda_1\underline{u}^{\alpha_1} - \underline{u}^{\beta_1} - \mu_1\underline{u}^{a_1}v^{b_1} \geq \frac{\lambda_1}{2}m^{\alpha_1}C_2 d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega,$$

$$\lambda_2\underline{v}^{\alpha_2} - \underline{v}^{\beta_2} - \mu_2\underline{v}^{a_2}u^{b_2} \geq \frac{\lambda_2}{2}m^{\sigma\alpha_2}C_2'd(x)^{\alpha_2} \quad \text{in } \Omega,$$

for m small enough. Then, under conditions (3.108), (3.109) and for m small enough, $(\underline{u},\underline{v})$ is a subsolutions pair of problem $(\mathbf{P})$. Finally, Instead of (3.118), we have

$$-\Delta_p\overline{u} \geq m^{1-p}C_3d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{and} \quad -\Delta_q\overline{v} \geq m^{\sigma(1-q)}C_3'd(x)^{\alpha_2} \quad \text{in } \Omega.$$

Instead of (3.119) and (3.120), we obtain

$$\lambda_1\overline{u}^{\alpha_1} - \overline{u}^{\beta_1} - \mu_1\overline{u}^{a_1}v^{b_1} \leq \lambda_1 m^{-\alpha_1}C_4d(x)^{\frac{\alpha_1 p}{p-1-\alpha_1}} \quad \text{in } \Omega,$$

$$\lambda_2\overline{v}^{\alpha_2} - \overline{v}^{\beta_2} - \mu_2\overline{v}^{a_2}u^{b_2} \leq \lambda_1 m^{-\sigma\alpha_2}C_4'd(x)^{\alpha_2} \quad \text{in } \Omega.$$

Then, for m small enough, $(\overline{u},\overline{v})$ is a supersolutions pair of problem $(\mathbf{P})$. Then, we conclude as in the *Alternative 1*. Thus, (iii) and by symmetry (iv) are proved. □

Concerning the above theorem, we analyse further some limiting cases. The proof of the next result follows the proof of Theorem 3.10. So we omit it.

Theorem 3.14. (i) *Let*

$$\alpha_1 = -1 \quad \textit{and} \quad (a_1 - \alpha_1 + b_1)q - (a_1 - \alpha_1)(\alpha_2 + 1) > 0, \tag{3.121}$$

$$-2 - \frac{1}{q-1} < \alpha_2 < -1 \quad \textit{and} \quad (a_2 - \alpha_2 - b_2)q - b_2(\alpha_2 + 1) > 0. \tag{3.122}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying:*

$$u(x) \sim d(x)|\ln(d(x))|^{\frac{1}{p}} \quad \textit{and} \quad v(x) \sim d(x)^{\frac{q}{q-1-\alpha_2}} \quad \textit{in } \Omega. \tag{3.123}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, *for some* $0 < \alpha < 1$.

(ii) *Let*

$$\alpha_1 = -1 \quad \textit{and} \quad a_1 - \alpha_1 + b_1 > 0, \tag{3.124}$$

$$\alpha_2 = -1 \quad \textit{and} \quad a_2 - \alpha_2 + b_2 > 0. \tag{3.125}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying:*

$$u(x) \sim d(x)|\ln(d(x))|^{\frac{1}{p}} \quad \textit{and} \quad v(x) \sim d(x)|\ln(d(x))|^{\frac{1}{q}} \quad \textit{in } \Omega. \tag{3.126}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right)$, *for some* $0 < \alpha < 1$.

(iii) *Let*

$$\alpha_1 = -1 \quad \textit{and} \quad a_1 - \alpha_1 + b_1 > 0, \tag{3.127}$$

$$-1 < \alpha_2 < q-1 \quad \textit{and} \quad a_2 - \alpha_2 + b_2 > 0. \tag{3.128}$$

Then, $(\mathbf{P})$ *admits positive solutions* $(u,v) \in \mathrm{W}_0^{1,p}(\Omega) \times \mathrm{W}_0^{1,q}(\Omega)$ *satisfying:*

$$u(x) \sim d(x)|\ln(d(x))|^{\frac{1}{p}} \quad \textit{and} \quad v(x) \sim d(x) \quad \textit{in } \Omega. \tag{3.129}$$

In addition, we have $(u,v) \in \mathscr{C}^{0,\alpha}\left(\overline{\Omega}\right) \times \mathscr{C}^{1,\alpha}\left(\overline{\Omega}\right)$, *for some* $0 < \alpha < 1$.

A A useful Hölder regularity result

We consider the following quasilinear elliptic boundary value problem,

$$-\Delta_r w = f \quad \text{in } \Omega; \quad w|_{\partial\Omega} = 0, \quad w > 0 \quad \text{in } \Omega. \tag{A.1}$$

In this equation, f is a $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ function such that there exist two constants $C > 0$ and $\delta > 0$ satisfying

$$|f(x)| \leq C d(x)^{-\delta}, \quad \text{a.e. in } \Omega. \tag{A.2}$$

Then, we have the following Hölder regularity result on the solutions to (A.1).

Theorem A.1. *Assume that* f *satisfies the growth hypothesis* (A.2). *Let* $u \in \mathrm{W}_0^{1,r}(\Omega)$ *be a positive weak solution to* (A.1). *Let* $\overline{u} \in \mathrm{W}_0^{1,r}(\Omega)$ *be a supersolutions to* (A.1) *such that*

$$-\Delta_r \overline{u} \geq |f| \quad \textit{in } \Omega, \tag{A.3}$$

in the sense of distributions in $\mathrm{W}^{-1,r'}(\Omega)$. *In addition, assume that there exists* $C' > 0$ *such that*

$$0 \leq u \leq \overline{u} \leq C' d(x)^{\delta'} \quad \textit{a.e in } \Omega, \tag{A.4}$$

with $0 < \delta' < \delta$. *Finally, let* α *be an arbitrary number such that*

$$0 < \alpha < \frac{r}{r - 1 + \delta/\delta'} < 1.$$

Then, there exists a constant $M > 0$, *depending solely on* Ω, r *and* N, *on the constants* C *and* δ *in* (A.2), *on the constants* C' *and* δ' *in* (A.4), *and on the constant* α, *such that* $u \in \mathscr{C}^{0,\alpha}(\overline{\Omega})$ *and*

$$\|u\|_{\mathscr{C}^{0,\alpha}(\overline{\Omega})} \leq M.$$

Proof. The proof is quite similar to the Theorem 1.1's in [11] with $\mathbf{a} : (x, \eta) \mapsto |\eta|^{p-2}\eta$ in $\Omega \times \mathbb{R}^N$. Indeed, to overcome the non-positivity of f, we add conditions (A.3) and (A.4). Then, introducing the same boundary value problem (2.12),instead of inequality (2.14), we get here

$$|u(x) - v(x)| \leq \overline{u}(x) \leq Cx_N^{\delta'} \quad \text{for all } x = (x', x_N) \in B_R^+(0). \tag{A.5}$$

Then, estimate (A.18) still holds and the end of the proof is exactly the same. □

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