

## Research Article

# A Lower Bound on the Sinc Function and Its Application

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Received 12 March 2014; Revised 24 June 2014; Accepted 26 June 2014; Published 8 July 2014

Academic Editor: Shanhe Wu

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A lower bound on the sinc function is given. Application for the sequence  $\{b_n\}_{n=1}^{\infty}$  which related to Carleman inequality is given as well.

## 1. Introduction

The sinc function is defined to be

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0, \\ 1 & x = 0. \end{cases} \quad (1)$$

This function plays a key role in many areas of mathematics and its applications [1–6].

The following result that provides a lower bound for the sinc is well known as Jordan inequality [7]:

$$\operatorname{sinc}(x) \geq \frac{2}{\pi}, \quad x \in \left[0, \frac{\pi}{2}\right], \quad (2)$$

where equality holds if and only if  $x = \pi/2$ .

This inequality has been further refined by many authors in the past few years [8–35].

In [36], it was proposed that

$$\operatorname{sinc}(x) \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \neq 0. \quad (3)$$

We noticed that the lower bound in (3) is the fractional function. Similar result has been reported as follows [1]:

$$\operatorname{sinc}(x) \geq \frac{53}{53 + 9x^2}, \quad 0 \leq x \leq \frac{1}{3}. \quad (4)$$

To the best of the authors' knowledge, few results have been obtained on fractional lower bound for the sinc function. It is the first aim of the present paper to establish the following fractional lower bound for the sinc function.

**Theorem 1.** For any  $x \in [0, \pi]$ , one has

$$\operatorname{sinc}(x) \geq \frac{16\pi^4}{(3\pi^2 + x^2)^2} - 1. \quad (5)$$

In [37], Yang proved that for any positive integer  $m$ , the following Carleman type inequality holds:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^m \frac{b_k}{(n+1)^k}\right) a_n, \quad (6)$$

whenever  $a_n \geq 0$ ,  $n = 1, 2, 3, \dots$ , with  $0 < \sum_{n=1}^{\infty} a_n < \infty$ , where

$$b_0 = 1,$$

$$b_n = \frac{1}{n} \left( \frac{1}{n+1} - \sum_{k=0}^{n-2} \frac{b_{n-1-k}}{k+1} \right), \quad (n = 1, 2, \dots). \quad (7)$$

From a mathematical point of view, the sequence  $\{b_n\}_{n=1}^{\infty}$  has very interesting properties. Yang [38] and Gyllenberg and Ping [39] have proved that, for any positive integer  $n$ ,

$$b_n > 0, \quad (8)$$

$$b_n < \frac{1}{n(n+1)}.$$

In [40], the authors proved that

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = 1, \tag{9}$$

$$eb_n = \int_0^1 x^{n-2} h(x) dx, \quad n \geq 2, \tag{10}$$

where

$$h(x) = x^{x+1} (1-x)^{1-x} \text{sinc}(\pi x). \tag{11}$$

As an application of Theorem 1, it is the second aim of the present paper to give a better upper bound on the sequence  $\{b_n\}_{n=1}^{\infty}$ .

**Theorem 2.** For any positive integer  $n \geq 2$ , one has

$$eb_n < \frac{1}{n(n+1)} - \frac{2-4/\pi}{n(n+1)(n+2)}. \tag{12}$$

### 2. The Proof of Theorem 1

The proof is not based on (3). We first prove the following result.

**Lemma 3.** For any  $x \in (\pi - 1/3, \pi]$ , one has

$$\text{sinc}(x) \geq \frac{16\pi^4}{(3\pi^2 + x^2)^2} - 1. \tag{13}$$

*Proof.* Set  $x = \pi - t$ ,  $0 \leq t < 1/3$ . Then inequality (13) is equivalent to

$$\pi - t + \sin t \geq \frac{16\pi^4 (\pi - t)}{(3\pi^2 + (\pi - t)^2)^2}. \tag{14}$$

To prove (14) by (4), it is enough to prove that

$$(\pi - t) + \frac{53t}{53 + 9t^2} \geq \frac{16\pi^4 (\pi - t)}{(3\pi^2 + (\pi - t)^2)^2}; \tag{15}$$

namely,

$$\frac{(\pi - t)(53 + 9t^2) + 53t}{(53 + 9t^2)} \geq \frac{16\pi^4 (\pi - t)}{(3\pi^2 + (\pi - t)^2)^2}. \tag{16}$$

Next we prove (16). Let

$$\begin{aligned} g(t) &= (\pi - t)(53 + 9t^2)(3\pi^2 + (\pi - t)^2)^2 \\ &\quad + 53t(3\pi^2 + (\pi - t)^2)^2 \\ &\quad - 16\pi^4 (\pi - t)(53 + 9t^2). \end{aligned} \tag{17}$$

We need only to prove that  $g(t) \geq 0$ . Elementary calculations reveal that

$$\begin{aligned} g(t) &= t^2 \left( -9t^5 + 45\pi t^4 - 144\pi^2 t^3 + (53\pi + 252\pi^3) t^2 \right. \\ &\quad \left. - 4\pi^2 (36\pi^2 + 53)t + 636\pi^3 \right). \end{aligned} \tag{18}$$

Noting that, for  $0 \leq t < 1/3$ , we have

$$\begin{aligned} -9t^5 &> -\frac{1}{27}, \\ -144\pi^2 t^3 &> -\frac{16\pi^2}{3}, \\ -4\pi^2 (36\pi^2 + 53)t &> -\frac{4}{3}\pi^2 (36\pi^2 + 53). \end{aligned} \tag{19}$$

Thus, from (19) and (18), we get

$$g(t) \geq t^2 \left( 636\pi^3 - \frac{1}{27} - \frac{16\pi^2}{3} - \frac{4\pi^2 (36\pi^2 + 53)}{3} \right) \geq 0. \tag{20}$$

This completes the proof. Now we prove Theorem 1.  $\square$

*Proof.* By using the power series expansions of  $\text{sinc}(x)$  and  $16\pi^4/(3\pi^2 + x^2)^2$ , we find that

$$\begin{aligned} 1 + \text{sinc}(x) - \frac{16\pi^4}{(3\pi^2 + x^2)^2} \\ = \frac{2}{9} + \sum_{n=1}^{\infty} (-1)^{n-1} u_n \left( \frac{x^2}{\pi^2} \right)^n, \end{aligned} \tag{21}$$

where

$$u_n = \frac{16(n+1)}{3^{n+2}} - \frac{\pi^{2n}}{(2n+1)!}. \tag{22}$$

Set  $x^2/\pi^2 = t$ ,  $0 \leq t \leq 1$ . Consider the function  $f(t)$  defined by

$$f(t) = \frac{2}{9} + \sum_{n=1}^{\infty} (-1)^{n-1} u_n t^n. \tag{23}$$

From (21), we get  $f(0) = 2/9$  and  $f(1) = 0$ . Lemma 3 implies

$$f(t) \geq 0, \quad t_0 < t \leq 1, \tag{24}$$

where

$$t_0 = \left( 1 - \frac{1}{3\pi} \right)^2 \approx 0.79 \dots. \tag{25}$$

Elementary calculations reveal that for  $n \geq 4$ ,

$$\frac{16(2n+1)}{3^3} > \frac{(3\pi^2)^n}{(2n+1)!}. \tag{26}$$

Hence, for  $n \geq 4$ , we have

$$\begin{aligned} u_n &> 0, \\ u_n - u_{n+1} &= \frac{16(2n+1)}{3^{n+3}} - \frac{\pi^{2n}}{(2n+1)!} + \frac{\pi^{2n+2}}{(2n+3)!} > 0. \end{aligned} \tag{27}$$

Therefore,

$$f(t) \geq \frac{2}{9} + \sum_{n=1}^6 (-1)^{n-1} u_n t^n. \tag{28}$$

If we set

$$g(t) = \frac{2}{9} + \sum_{n=1}^6 (-1)^{n-1} u_n t^n, \tag{29}$$

then we have

$$g(0) = \frac{2}{9} > 0, \quad g(1) < f(1) = 0. \tag{30}$$

The intermediate value theorem implies that there must be at least one root  $c$  with  $(0, 1)$  such that  $g(c) = 0$ . Using Maple, we find that on the open interval  $(0, 1)$  the equation  $g(t) = 0$  has a unique real root  $t_1 \approx 0.89 \dots$ .

Hence, from (28) we get

$$f(t) \geq 0, \quad t \in [0, t_1]. \tag{31}$$

By (21), (24), and (31), Theorem 1 follows. □

### 3. The Proof of Theorem 2

First, we need an auxiliary result.

**Lemma 4.** For any  $x \in [0, 1/2]$ , one has

$$\text{sinc}^2(\pi x) \geq \frac{1 - 2x + x^2}{1 - x + x^2}. \tag{32}$$

*Proof.* By letting  $x = 1/2 - t/2\pi$ ,  $0 \leq t \leq \pi$ , the requested inequality can be equivalently written as

$$\cos t \geq \frac{t^2}{2} + \frac{8\pi^4}{t^2 + 3\pi^2} - \frac{5\pi^2 + 2}{2}, \tag{33}$$

so it suffices to show that the function

$$G(t) = \cos t - \frac{t^2}{2} - \frac{8\pi^4}{t^2 + 3\pi^2} + \frac{5\pi^2 + 2}{2} \tag{34}$$

is negative on  $0 \leq t \leq \pi$ . Theorem 1 implies

$$G'(t) < 0. \tag{35}$$

Hence,

$$G(t) \geq G(\pi) = 0. \tag{36}$$

The required inequality follows. Now we prove Theorem 2. □

*Proof.* Let

$$H(x) = \begin{cases} x^x(1-x)^{-x} \text{sinc}(\pi x), & 0 < x < 1 \\ 1 & x = 0, 1. \end{cases} \tag{37}$$

We first consider the case  $0 \leq x \leq 1/2$ .

Taking the natural log gives

$$\ln H(x) = (x-1) \ln x - x \ln(1-x) + \ln \text{sinc}(\pi x) - \ln \pi. \tag{38}$$

Taking the second derivative of both sides of (38), we have

$$\frac{HH'' - H'^2}{H^2} = \frac{x^2 - x + 1}{x^2(1-x)^2} - \pi^2 \text{csc}^2(\pi x). \tag{39}$$

By Lemma 4, it follows that

$$\frac{HH'' - H'^2}{H^2} > 0. \tag{40}$$

Thus,

$$H'' > 0, \tag{41}$$

and therefore for  $0 \leq x \leq 1/2$ , we have

$$\begin{aligned} H(x) &\leq (1-2x)H(0) + 2xH\left(\frac{1}{2}\right) \\ &= \left(2 - \frac{4}{\pi}\right)(-x) + 1. \end{aligned} \tag{42}$$

For the case  $1/2 < x \leq 1$ , since  $H(1/2) = 2/\pi$ ,  $H(1) = 1$ , and  $H$  is concave up, it follows that

$$\begin{aligned} H(x) &\leq 2(1-x)H\left(\frac{1}{2}\right) + (2x-1)H(1) \\ &= \left(2 - \frac{4}{\pi}\right)(x-1) + 1. \end{aligned} \tag{43}$$

Using (10) from (42) and (43), we have

$$\begin{aligned} eb_n &= \int_0^1 x^{n-2} h(x) dx = \int_0^1 H(x) (x^{n-1} - x^n) dx \\ &\leq \int_0^{1/2} \left[1 - \left(2 - \frac{4}{\pi}\right)x\right] (x^{n-1} - x^n) dx \\ &\quad + \int_{1/2}^1 \left[1 + \left(2 - \frac{4}{\pi}\right)(x-1)\right] (x^{n-1} - x^n) dx \\ &= \frac{1}{n(n+1)} - \left(2 - \frac{4}{\pi}\right) \frac{2^{n+2} - n - 4}{n(n+1)(n+2)2^{n+1}} \\ &< \frac{1}{n(n+1)} - \frac{2 - 4/\pi}{n(n+1)(n+2)}. \end{aligned} \tag{44}$$

This proves Theorem 2. □

### Conflict of Interests

There is no conflict of interests regarding the publication of this paper.

### Acknowledgments

The authors are very grateful to the anonymous referees and the editor for their insightful comments and suggestions. The work of the second author was supported by a Grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI project no. PN-II-ID-PCE-2011-3-0087.

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