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brief notes

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Formulation of the Eigenvalue Problem for an Unconstrained Circular Bar

M. Nikkhah-Bahrami[1]

Many eminent mathematicians and applied mechanicists have been interested in the vibration problems of naturally curved and twisted bars [1–6].[2] In all these cases the equations of motion are based on assumptions equivalent to those of the Euler-Bernoulli theory of vibrations of a straight beams.

These equations are solved for some cases, such as an elastic solid full toroid [1, 2, 5], whereas for an incomplete circular bar the solution, because of tedious calculations, has been left in a closed form [3]. Even today, with the existence of computers, much unwanted algebra needs to be done before using computers, to obtain numerical results from these existing classical equations. In order to avoid this tedious and unwanted algebra, a new approach for the formulation of the equations of motion, and thus the frequency equations, will be presented here which is more apt for numerical calculations. This paper will show how an eigenvalue problem can be formulated for unconstrained incomplete circular bars by superposition of rigid-body and deformation modes.

Derivations of the Eigenvalue Problem

The homogenous elastic circular bar of constant cross section is referred to a fixed system of orthogonal Cartesian coordinates (Q, X^1, X^2, X^3) and to a curvilinear right-handed coordinate system (0, θ^1, θ^2, θ^3), where the origin 0 coincides with the generic point on the central axis of the circular bar. Coordinate θ^1 coincides with the central axis and has positive direction the direction of the tangent vector of the bar center line. Coordinate θ^2 coincides with the principal normal and is directed positively toward the center of curvature. Coordinate θ^3 coincides with the binormal and has a positive direction the direction of the binormal.

The kinetic energy of the elastic bar is defined as

$$T = \frac{1}{2}\int_v (\bar r; t)\cdot(\bar r; t)\rho dv \tag{1}$$

where

$$\bar r = \bar\zeta(t) + \bar\xi(\theta^1, \theta^2, \theta^3) + \bar u(\theta^1, t)$$

For the in-plane motion (that is, the displacement of the center line of the circular bar is predominantly in the plane containing the undeformed center line), the position vector $\bar r$ in terms of the unit vectors may be expressed as

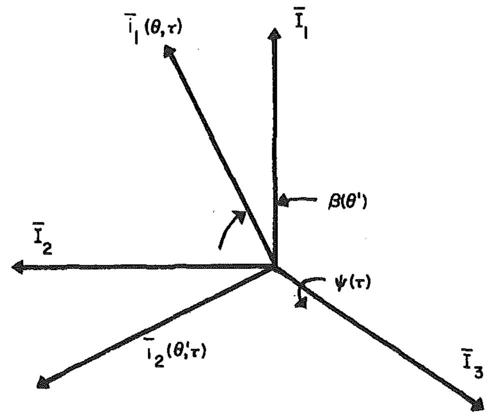

Fig. 1

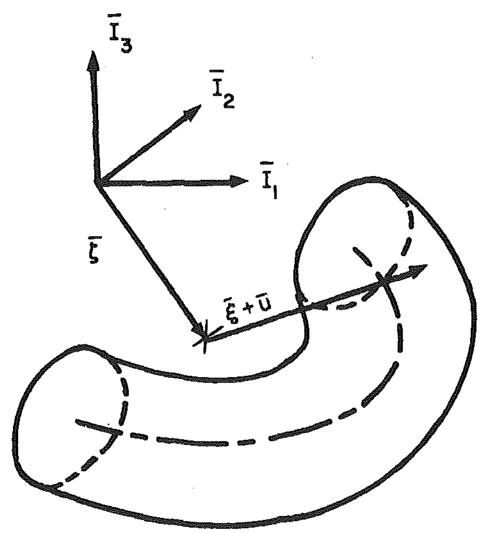

Fig. 2

$$\bar r = \bar I_1\zeta^1(t) + \bar I_2\zeta^2(t) + \bar I_3\zeta^3(t) + \bar I_3\xi^3(\theta^1, \theta^2, \theta^3) - \bar i_2\xi^2(\theta^1, \theta^2, \theta^3) + \bar i_1 u^1(\theta^1, t) + \bar i_2 u^2(\theta^1, t), \tag{2}$$

differentiating and noting $\bar i_2;\, t = \bar{\dot\psi}(t) x \bar i_2 = -\dot\psi(t)\bar i_1$ one obtains that

$$\bar r;\, t = \bar I_1\dot\zeta^1 + \bar I_2\dot\zeta^2 + \bar i_1 u^1;\, t + \bar i_2 u^2;\, t + \bar i_1\dot\psi\xi^2 \tag{3}$$

Therefore

$$|\bar r;\, t|^2 = (u^1;\, t)^2 + (u^2;\, t)^2 + 2u';\, t\dot\psi\xi^2 + (\xi^2)^2(\dot\psi)^2 + 2u^1;\, t\dot\zeta^1\cos\beta + 2u^1;\, t\dot\zeta^2\sin\beta + (\dot\zeta^1)^2 + (\dot\zeta^2)^2 - 2u^2;\, t\dot\zeta^1\sin\beta + 2u^2;\, t\dot\zeta^2\cos\beta + 2\dot\psi\xi^2\dot\zeta^1\cos\beta + 2\dot\psi\xi^2\dot\zeta^2\sin\beta \tag{4}$$

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[2] Numbers in brackets designate References at end of Note.

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By letting $u^a(\theta^1, t) = p_c{}^a(\theta^1) f_E{}^c(t)$ [9, p. 152], it follows that

$$(\dot{r})^2 = \dot{f}_E{}^c(p_c{}^1p_a{}^1 + p_c{}^2p_a{}^2)\dot{f}_E{}^a + (\xi^2)^2(\dot{\psi})^2 + (\dot{\zeta}^1)^2 + (\dot{\zeta}^2)^2 + 2\dot{f}_E{}^c p_c{}^1\dot{\psi}\xi^2 + 2\dot{f}_E{}^c p_c{}^1\dot{\zeta}^1\cos\beta + 2\dot{f}_E{}^c p_c{}^1\dot{\zeta}^2\sin\beta - 2\dot{f}_E{}^c p_c{}^2\dot{\zeta}^1\sin\beta + 2\dot{f}_E{}^c p_c{}^2\dot{\zeta}^2\cos\beta + 2\dot{\psi}\xi^2\dot{\zeta}^1\cos\beta + 2\dot{\psi}\xi^2\dot{\zeta}^2\sin\beta. \quad (5)$$

Therefore in matrix notation the kinetic energy is

$$2T = \begin{bmatrix}\dot{\zeta}^1\\ \dot{\zeta}^2\\ \dot{\psi}\\ \{\dot{f}_E\}'\end{bmatrix} \begin{bmatrix} M_{\zeta^1\zeta^1} & 0 & M_{\zeta^1\psi} & \{M_{\zeta^1 f_E}\}' \\ 0 & M_{\zeta^2\zeta^2} & M_{\zeta^2\psi} & \{M_{\zeta^2 f_E}\}' \\ M_{\zeta^1\psi} & M_{\zeta^2\psi} & M_{\psi\psi} & \{M_{\psi f_E}\}' \\ \{M_{\zeta^1 f_E}\} & \{M_{\zeta^2 f_E}\} & \{M_{\psi f_E}\} & \{M_{f_E f_E}\}\end{bmatrix} \begin{bmatrix}\dot{\zeta}^1\\ \dot{\zeta}^2\\ \dot{\psi}\\ \{\dot{f}_E\}\end{bmatrix} \quad (6)$$

where

$$M_{\zeta^1\zeta^1} = M_{\zeta^2\zeta^2} = \int_v \rho dv = \int_0^{\pi R}\int_A \rho\sqrt{G}\, d\theta^1 d\theta^2 d\theta^3 = \rho\int_0^{\pi R}\int_A (1-\theta^2 c)\, d\theta^1 d\theta^2 d\theta^3 = \rho\pi R(\pi r^2) = M,$$

$$M_{\zeta^1\psi} = \int_v \rho\xi^2\cos\beta dv = \int_0^{\pi R}\int_A \rho(R-\theta^2)\cos\frac{\theta^1}{R}\sqrt{G}\, d\theta^1 d\theta^2 d\theta^3 = 0,$$

$$M_{\zeta^2\psi} = \int_{\rho\xi^2}\sin\beta dv = \int_0^{\pi R}\int_A \rho(R-\theta^2)\sin\frac{\theta^1}{R}\sqrt{G}\, d\theta^1 d\theta^2 d\theta^3 = \frac{2MR}{\pi} + \frac{Mr^2}{2\pi R},$$

$$M_{\psi\psi} = \int_v \rho(\xi^2)^2 dv = \int_0^{\pi R}\rho\int_A (R-\theta^2)^2\sqrt{G}\, d\theta^1 d\theta^2 d\theta^3 = MR^2 + \frac{3}{4}Mr^2,$$

If u^1 is defined as

$$u^1(\theta^1, t) = \cos\frac{N_c\theta^1}{R} f_E{}^c(t)$$

then since $u_1; 1 = (1/R)u^2$ (the condition for unstretching the center line) it follows that

$$u^2 = -N_c\sin\frac{N_c\theta^1}{R} f_E{}^c(t)$$

$$[M_{f_E f_E}] = \int_v (\{p^1\}\{p^1\}' + \{p^2\}\{p^2\}')\rho dv = \int_v\left[\cos\frac{N_c\theta^1}{R}\cos\frac{N_m\theta^1}{R} + N_cN_m\sin\frac{N_c\theta^1}{R}\sin\frac{N_m\theta^1}{R}\right]\rho dv$$

$$= \begin{cases}\rho\pi r^2\left(\dfrac{\pi R}{2}\right)[1+N_cN_m] & \text{for } N_c = N_m,\ c = m\\ 0 & \text{for } N_c \neq N_m,\end{cases}$$

$$\{M_{\zeta^1 f_E}\} = \int_v\left(\{p^1\}\cos\frac{\theta^1}{R} - \{p^2\}\sin\frac{\theta^1}{R}\right)\rho dv = 0 \text{ for all } N_c,$$

$$\{M_{\zeta^2 f_E}\} = \int_v\left(\{p^2\}\cos\frac{\theta^1}{R} + \{p^1\}\sin\frac{\theta^1}{R}\right)\rho dv$$

$$= \begin{cases} 0 & \text{for } N_c \text{ odd}\\ R\rho\pi r^2\left[\dfrac{-2N^2(c)}{N_c{}^2-1} + \dfrac{2}{1-N_c{}^2}\right] = 2\rho\pi r^2 R\left(\dfrac{1+N_c{}^2}{1-N_c{}^2}\right),\end{cases}$$

Table 1

$E = 20.6 \times 10^7$ kPa $R = 12.7$ cm $\rho = 7.8$ gr/cm^3

$r = 2.54$ cm		$r = 0.63$ cm	
Mode no.	Frequency (Hz)	Mode no.	Frequency (Hz)
1	3395	1	849
2	7154	2	1789
3	12239	3	3059
4	18666	4	4666

$$\{M_{\psi f_E}\} = \int_v \{p^1\}\xi^2\rho dv = 0 \text{ for all } N_c$$

The potential energy is

$$U = \frac{1}{2}\int_0^{\pi R} EI_{22}(Ru^1;111 + \frac{1}{R}u^1;1)d\theta^1 \quad (7)$$

or

$$2U = \{f_E\}'[\omega^2 M_{f_E f_E}]\{f_E\}. \quad (8)$$

where

$$\omega^2 = \frac{EI_{22}N^2(N^2-1)^2}{R^4\rho A(N^2+1)}, \qquad N = 2,3,\ldots.$$

If the rigid-body displacements are defined as

$$\{f_R\}' = [\zeta^1, \zeta^2, \psi] \quad (9)$$

then upon using Hamilton's principle the equation of motion may be written as

$$\begin{bmatrix}[M_{RR}] & [M_{RE}]\\ [M_{ER}] & [M_{EE}]\end{bmatrix}\begin{bmatrix}\{\ddot{f}_R\}\\ \{\ddot{f}_E\}\end{bmatrix} + \begin{bmatrix}[0] & [0]\\ [0] & [K_{EE}]\end{bmatrix}\begin{bmatrix}\{f_R\}\\ \{f_E\}\end{bmatrix} = \begin{bmatrix}\{0\}\\ \{0\}\end{bmatrix} \quad (10)$$

where

$$[M_{RR}] = \begin{bmatrix}M_{\zeta^1\zeta^1} & 0 & M_{\zeta^1\psi}\\ 0 & M_{\zeta^2\zeta^2} & M_{\zeta^2\psi}\\ M_{\zeta^1\psi} & M_{\zeta^2\psi} & M_{\psi\psi}\end{bmatrix}, \qquad [M_{EE}] = [M_{f_E f_E}]$$

$$[M_{RE}] = \begin{bmatrix}\{M_{\zeta^1 f_E}\}'\\ \{M_{\zeta^2 f_E}\}'\\ \{M_{\psi f_E}\}'\end{bmatrix}, \qquad [K_{EE}] = \omega^2[M_{f_E f_E}]$$

Thus with $\{f_E\} = \{\Psi\}f_f$ the free-free eigenvalues are obtained from

$$([\text{—}1\text{—}] - [\chi])\{\Psi\} = \{0\} \quad (11)$$

where

$$[\chi] = [K_{EE}]^{-1}([M_{EE}] - [M_{ER}][M_{RR}]^{-1}[M_{RE}]) \quad (12)$$

The resulting eigenvalue problems were numerically solved for two circular bars for which the results are tabulated in Table 1. These results were equivalent to those obtained by solving the classical equations.

Conclusion

This paper presents a new approach for finding frequencies of the incomplete circular bar which is different from the classical approach. The method presented is more apt for numerical calculation than the classical method. In addition, it provides a method for extending the elementary method to more exact methods [7, 8]. A similar approach could be used for the out-of-plane motion, that is, for the motion in the direction normal to the in-plane motion.

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An Experimental Verification of Boundary-Layer Solutions of Karman's Equations of Plates[1]

Chin-Hao Chang[2]

The so-called edge zone or boundary-layer solutions are widely used in large deflection problems of thin elastic plates [1–3][3] and shells [4–6]; but a little experimental work has been seen in the literature of verification of these approximate solutions. The boundary-layer solutions for the bending of a buckled rectangular plate were obtained in [3]. Of the plate, two opposite edges are free and the other two are simply supported; along the supported edges uniformly distributed compressive forces are acted on. Because of simplicity of this plate, an experimental study was made. Good agreement was observed between the analytical and experimental results.

For comprehension, the boundary-layer solutions are briefly given.

Boundary-Layer Solutions

The von Karman equations for a dimensionless lateral deflection W and Airy's stress function F of an elastic plate may be presented in the form

$$\gamma^2\Delta\Delta W - Q(F, W) = 0, \qquad \alpha^2\Delta\Delta F + Q(W, W)/2 = 0 \qquad (1a, b)$$

in which

$$\gamma^2 = 1/12(1 - \nu^2), \qquad \alpha = h/A \qquad (2a, b)$$

$$Q(r, s) = r_{,xx}s_{,yy} + r_{,yy}s_{,xx} - 2r_{,xy}s_{,xy} \qquad (2c)$$

ν = Poisson'e ratio, h = thickness, A = amplitude, and Δ = Laplacian operator in Cartesian xy − coordinates; a subscript preceded by a comma represents the appropriate derivative.

For a plate of $a \times 2b$ with $x = 0$, a simply supported, $y = \pm b$ free of Fig. 1 functions

$$W^0 = \sin(\pi x/a), \qquad F^0 = -\gamma^2\pi^2y^2/(2a^2) \qquad (3a, b)$$

can satisfy equations (1) and all the required boundary conditions except along the free edge $y = +b$, the normal moment

$$M^0_{yy}(x, \pm b) = -EAh^3\gamma^2[W^0_{,yy} + \nu W^0_{,xx}] \neq 0 \qquad (4a)$$

since

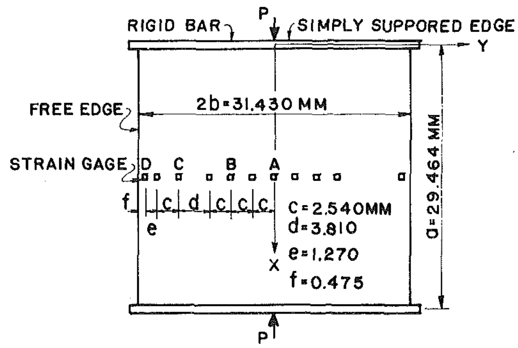

Fig. 1 Plate

$$W^0_{,xx}(x, \pm b) \neq 0 \qquad (4b)$$

Now let

$$W = W^0(x) + \alpha W^1(x, \eta) + \ldots \qquad (5a)$$

$$F = F^0(y) + f^0(x, \eta) + \alpha f^1(x, \eta) + \ldots \qquad (5b)$$

in which

$$\eta = (b - y)\alpha^{-1/2} \qquad (6)$$

being a boundary-layer coordinate measured from $y = +b$. Substituting series (5) into equations (1) results into a set of equations associated with different orders of α. The set with the lowest order of α has the form

$$\gamma^2W^1_{,\eta\eta\eta\eta} - W^0_{,xx}f^0_{,\eta\eta} = 0 \qquad (7a)$$

$$W^0_{,xx}W^1_{,\eta\eta} + f^0_{,\eta\eta\eta\eta} = 0 \qquad (7b)$$

With the following boundary conditions at the free edge ($\eta = 0$)

$$W^1_{,\eta\eta} + \nu W^0_{,xx} = 0; \qquad W^1_{,\eta\eta\eta} = 0 \qquad (8a, b)$$

$$f^0_{,xx} = 0; \qquad f^0_{,xy} = 0 \qquad (9a, b)$$

The solutions for the boundary-layer equations (7)–(9) are

$$W^1(x, \eta) = \nu\gamma e^{-\lambda\eta}(\cos\lambda\eta - \sin\lambda\eta) \qquad (10a)$$

$$f^0(x, \eta) = -\nu\gamma^2e^{-\lambda\eta}(\cos\lambda\eta + \sin\lambda\eta) \qquad (10b)$$

in which

$$\lambda^2 = |W^0_{,xx}|/2\gamma \qquad (11)$$

At the central section ($x = a/2$), the normal stresses are

$$N_{xx} = h^3E(F^0 + f^0)_{,yy}$$
$$= -h^3E(\gamma\pi/a)^2 + AE(\pi h/a)^2\nu\gamma e^{-\lambda\eta}(\cos\gamma\eta - \sin\gamma\eta) \qquad (12a)$$

$$N_{yy} = h^3Ef^0_{,xx} = -E(\pi h/a)^3\nu\gamma(\gamma/2)^{1/2}\eta e^{-\lambda\eta}\sin\lambda\eta \qquad (12b)$$

Thus N_{yy} vanishes at the free edge ($\eta = 0$) and also at the center of the plate ($\eta = \infty$); furthermore, N_{yy} is of higher order of α in comparison with N_{xx}, therefore its effect is negligible in computation of normal strain e_{xx}, i.e.,

$$e_{xx} = (N_{xx} - \nu N_{yy})/Eh \simeq N_{xx}/Eh \qquad (13)$$

which is to be compared with the experimental results.

Experimental Results

A plate of bare (unclad) aluminum sheet of 2024-T3 of h = 1.59 mm (1/16 in.) was used. The average modulus of elasticity, E = 76.60 × 10^9 Pa (11.11 × 10^6 psi), and ν = 0.331. The simply supported edges were so constructed that the ends could freely rotate. The Euler buckling load of the plate shown in Fig. 1 is

$$P_e = \frac{\pi^2Eh^3b}{12(1 - \nu^2)a^2} = 1023\,N \qquad (14)$$

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[3] Numbers in brackets designate References at end of Note.

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