Those Ubiquitous Cut Polyhedra

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Abstract

I first met Paul Erdös on August 3, 1975 in the Stanford apartment of my Ph.D. supervisor, Vasek Chvátal. Erdös asked me what he asked everyone: "What are you working on?" I was working on two topics involving finite metrics, which I knew was one of Erdös' favourite subjects.

I started by defining what I called the Hamming cone, and is now known as the cut cone, CUT_n. This is the conic hull of the 0/1 edge incidence vectors of the 2^{n-1} cuts in the complete graph K_n. The edge incidence vectors corresponding to a cut have the form $x = (x_{ij} : 1 \leq i < j \leq n)$, where $x_{ij} = 1$ if and only if vertices i and j are on different sides of the cut. I proceeded to explain what seemed like a rather esoteric conjecture of Michel Deza on the form of the facets of the cut cone.

The second topic was about determining the extreme rays of the metric cone, MET_n. This cone is the set of solutions $x = (x_{ij} : 1 \leq i < j \leq n)$ to the triangle inequalities:

$$x_{ik} \leq x_{ij} + x_{jk}, \quad 1 \leq i, j, k \leq n,$$

where the i, j, k are distinct, and we identify x_{ij} with x_{ji}. These solutions are often called finite semi-metrics. The cut vectors, which generate CUT_n, are also extreme rays of MET_n. An encyclopedic treatment of these polyhedra is contained in the book by Michel Deza and Monique Laurent [2].

Erdös listened politely but I suspected that he was wondering why I was interested in these two questions. Later, in 1977, at my Ph.D. defence in the now defunct Department of Operations Research, I was asked what what was the connection between the two topics of my thesis. I did not have the answer until later.

In 1979, during my first trip to Japan, I met Masao Iri. He explained to me what is often called the "Japanese theorem"[3]: a generalization of Ford and Fulkerson's max flow/min cut theorem, which is a condition based on the cut cone, to fractional multicommodity flows, where the condition is based on the metric cone. So here was a first connection and application.

Around the same time as Iri's work, physicists had been asking similar questions to Deza's in terms of what they called the Slater hull, which is isomorphic to the cut cone (eg., see [4]). This was a second application.

After relating this to Ron Graham, he remarked that it was impossible to escape from one's Ph.D. thesis. Indeed, with remarkable regularity I come across yet another application of these and related polyhedra. In this talk I will outline a few of my favourites, some old and some new, as time allows:

- Given a set of pairwise distances between n points, can you locate the points in space so that the L_1 distance between each pair of points matches its given distance?
- Consider five properties that a man may have: tall, handsome, rich, strong, intelligent. It is quite possible to have a population of males so that two thirds of them are either tall or handsome, but not both. The same is true for any other pair of properties. But it is not possible that this can *simultaneously* hold for every pair of the five properties. (It can hold for any pair of four properties.)
- Two well separated physics labs perform measurements on some quantum system and later compute correlations between their results. Could the same set of correlations have been obtained by simply sampling coloured balls from two urns?
- An open pit mining company has core samples of blocks in the ground. Can they achieve a profit of $\$K$ by mining at most M tons of material?

I am not sure if any of this would have interested Erdös, but I am sure he enjoyed the irony of it helping my computer *mutt* earn an Erdös number of two [1].

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References

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