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## Research Article

# Lightlike Hypersurfaces of a Semi-Riemannian Product Manifold and Quarter-Symmetric Nonmetric Connections

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We study lightlike hypersurfaces of a semi-Riemannian product manifold. We introduce a class of lightlike hypersurfaces called screen semi-invariant lightlike hypersurfaces and radical anti-invariant lightlike hypersurfaces. We consider lightlike hypersurfaces with respect to a quarter-symmetric nonmetric connection which is determined by the product structure. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifolds and the quarter-symmetric nonmetric connection, and we obtain some results.

## 1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of important topics of differential geometry. The geometry of lightlike submanifolds of a semi-Riemannian manifold, was presented in [1] (see also [2, 3]) by Duggal and Bejancu. In [4], Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [5], Kılıç and Şahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold and gave some examples and results for lightlike submanifolds. The lightlike hypersurfaces have been studied by many authors in various spaces (for example [6, 7]).

In [8], Hayden introduced a metric connection with nonzero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semisymmetric (symmetric) and nonmetric connection have been studied by many authors [9–14]. In [15], Yaşar et al. have studied lightlike hypersurfaces in semi-Riemannian manifolds with semisymmetric nonmetric connection. The idea of quarter-symmetric linear connections in a differential

manifold was introduced by Golab [11]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor  $\bar{T}$  is of the form:

$$\bar{T}(X, Y) = u(Y)\varphi X - u(X)\varphi Y, \quad (1.1)$$

for any vector fields  $X, Y$  on a manifold, where  $u$  is a 1-form and  $\varphi$  is a tensor of type (1,1).

In this paper, we study lightlike hypersurfaces of a semi-Riemannian product manifold. As a first step, in Section 3, we introduce screen semi-invariant lightlike hypersurfaces and radical anti-invariant lightlike hypersurfaces of a semi-Riemannian product manifold. We give some examples and study their geometric properties. In Section 4, we consider lightlike hypersurfaces of a semi-Riemannian product manifold with quarter-symmetric nonmetric connection determined by the product structure. We compute the Riemannian curvature tensor with respect to the quarter-symmetric nonmetric connection and give some results.

## 2. Lightlike Hypersurfaces

Let  $(\bar{M}, \bar{g})$  be an  $(m+2)$ -dimensional semi-Riemannian manifold with index  $(\bar{g}) = q \geq 1$  and let  $(M, g)$  be a hypersurface of  $\bar{M}$ , with  $g = \bar{g}|_M$ . If the induced metric  $g$  on  $M$  is degenerate, then  $M$  is called a lightlike (null or degenerate) hypersurface [1] (see also [2, 3]). Then there exists a null vector field  $\xi \neq 0$  on  $M$  such that

$$g(\xi, X) = 0, \quad \forall X \in \Gamma(TM). \quad (2.1)$$

The radical or the null space of  $T_x M$ , at each point  $x \in M$ , is a subspace  $\text{Rad } T_x M$  defined by

$$\text{Rad } T_x M = \{ \xi \in T_x M | g_x(\xi, X) = 0, \forall X \in \Gamma(TM) \}, \quad (2.2)$$

whose dimension is called the nullity degree of  $g$ . We recall that the nullity degree of  $g$  for a lightlike hypersurface of  $\bar{M}$  is 1. Since  $g$  is degenerate and any null vector being perpendicular to itself,  $T_x M^\perp$  is also null and

$$\text{Rad } T_x M = T_x M \cap T_x M^\perp. \quad (2.3)$$

Since  $\dim T_x M^\perp = 1$  and  $\dim \text{Rad } T_x M = 1$ , we have  $\text{Rad } T_x M = T_x M^\perp$ . We call  $\text{Rad } TM$  a radical distribution and it is spanned by the null vector field  $\xi$ . The complementary vector bundle  $S(TM)$  of  $\text{Rad } TM$  in  $TM$  is called the screen bundle of  $M$ . We note that any screen bundle is nondegenerate. This means that

$$TM = \text{Rad } TM \perp S(TM). \quad (2.4)$$

Here  $\perp$  denotes the orthogonal-direct sum. The complementary vector bundle  $S(TM)^\perp$  of  $S(TM)$  in  $T\bar{M}$  is called screen transversal bundle and it has rank 2. Since  $\text{Rad } TM$  is a lightlike subbundle of  $S(TM)^\perp$  there exists a unique local section  $N$  of  $S(TM)^\perp$  such that

$$\bar{g}(N, N) = 0, \quad \bar{g}(\xi, N) = 1. \quad (2.5)$$

Note that  $N$  is transversal to  $M$  and  $\{\xi, N\}$  is a local frame field of  $S(TM)^\perp$  and there exists a line subbundle  $\text{ltr}(TM)$  of  $T\bar{M}$ , and it is called the lightlike transversal bundle, locally spanned by  $N$ . Hence we have the following decomposition:

$$T\bar{M} = TM \oplus \text{ltr}(TM) = S(TM) \perp \text{Rad } TM \oplus \text{ltr}(TM), \quad (2.6)$$

where  $\oplus$  is the direct sum but not orthogonal [1, 3]. From the above decomposition of a semi-Riemannian manifold  $\bar{M}$  along a lightlike hypersurface  $M$ , we can consider the following local quasiorthonormal field of frames of  $\bar{M}$  along  $M$ :

$$\{X_1, \dots, X_m, \xi, N\}, \quad (2.7)$$

where  $\{X_1, \dots, X_m\}$  is an orthonormal basis of  $\Gamma(S(TM))$ . According to the splitting (2.6), we have the following Gauss and Weingarten formulas, respectively:

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \bar{\nabla}_X N &= -A_N X + \nabla_X^t N, \end{aligned} \quad (2.8)$$

for any  $X, Y \in \Gamma(TM)$ , where  $\nabla_X Y, A_N X \in \Gamma(TM)$  and  $h(X, Y), \nabla_X^t N \in \Gamma(\text{ltr}(TM))$ . If we set  $B(X, Y) = \bar{g}(h(X, Y), \xi)$  and  $\tau(X) = \bar{g}(\nabla_X^t N, \xi)$ , then (2.8) become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (2.9)$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N. \quad (2.10)$$

$B$  and  $A$  are called the second fundamental form and the shape operator of the lightlike hypersurface  $M$ , respectively [1]. Let  $P$  be the projection of  $S(TM)$  on  $M$ . Then, for any  $X \in \Gamma(TM)$ , we can write

$$X = PX + \eta(X)\xi, \quad (2.11)$$

where  $\eta$  is a 1-form given by

$$\eta(X) = \bar{g}(X, N). \quad (2.12)$$

From (2.9), we get

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \quad \forall X, Y, Z \in \Gamma(TM), \quad (2.13)$$

and the induced connection  $\nabla$  is a nonmetric connection on  $M$ . From (2.4), we have

$$\begin{aligned}\nabla_X W &= \nabla_X^* W + h^*(X, W) \\ &= \nabla_X^* W + C(X, W)\xi, \quad X \in \Gamma(TM), W \in \Gamma(S(TM)), \\ \nabla_X \xi &= -A_\xi^* X - \tau(X)\xi,\end{aligned}\tag{2.14}$$

where  $\nabla_X^* W$  and  $A_\xi^* X$  belong to  $\Gamma(S(TM))$ .  $C$ ,  $A_\xi^*$  and  $\nabla^*$  are called the local second fundamental form, the local shape operator and the induced connection on  $S(TM)$ , respectively. Also, we have the following identities:

$$\begin{aligned}g(A_\xi^* X, W) &= B(X, W), \quad g(A_\xi^* X, N) = 0, \\ B(X, \xi) &= 0, \quad g(A_N X, N) = 0.\end{aligned}\tag{2.15}$$

Moreover, from the first and third equations of (2.15) we have

$$A_\xi^* \xi = 0.\tag{2.16}$$

Now, we will denote  $\bar{R}$  and  $R$  the curvature tensors of the Levi-Civita connection  $\bar{\nabla}$  on  $\bar{M}$  and the induced connection  $\nabla$  on  $M$ . Then the Gauss equation of  $M$  is given by

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X \\ &\quad + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \quad \forall X, Y, Z \in \Gamma(TM),\end{aligned}\tag{2.17}$$

where  $(\nabla_X h)(Y, Z) = \nabla_X^t(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$ . Then the Gauss-Codazzi equations of a lightlike hypersurface are given by

$$\begin{aligned}\bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) \\ &\quad + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW), \\ \bar{g}(\bar{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &\quad + B(Y, Z)\tau(X) - B(X, Z)\tau(Y), \\ \bar{g}(\bar{R}(X, Y)Z, N) &= g(R(X, Y)Z, N), \\ \bar{g}(\bar{R}(X, Y)\xi, N) &= g(R(X, Y)\xi, N) \\ &= C(Y, A_\xi^* X) - C(X, A_\xi^* Y) - 2d\tau(X, Y),\end{aligned}\tag{2.18}$$

for any  $X, Y, Z, W \in \Gamma(TM)$ ,  $\xi \in \Gamma(\text{Rad } TM)$ .

For geometries of lightlike submanifolds, hypersurfaces and curves, we refer to [1–3].

### 2.1. Product Manifolds

Let  $\bar{M}$  be an  $n$ -dimensional differentiable manifold with a tensor field  $F$  of type  $(1,1)$  on  $\bar{M}$  such that

$$F^2 = I. \quad (2.19)$$

Then  $\bar{M}$  is called an almost product manifold with almost product structure  $F$ . If we put

$$\pi = \frac{1}{2}(I + F), \quad \sigma = \frac{1}{2}(I - F), \quad (2.20)$$

then we have

$$\begin{aligned} \pi + \sigma &= I, & \pi^2 &= \pi, & \sigma^2 &= \sigma, \\ \sigma\pi &= \pi\sigma = 0, & F &= \pi - \sigma. \end{aligned} \quad (2.21)$$

Thus  $\pi$  and  $\sigma$  define two complementary distributions and  $F$  has the eigenvalue of  $+1$  or  $-1$ . If an almost product manifold  $\bar{M}$  admits a semi-Riemannian metric  $\bar{g}$  such that

$$\bar{g}(FX, FY) = \bar{g}(X, Y), \quad (2.22)$$

for any vector fields  $X, Y$  on  $\bar{M}$ , then  $\bar{M}$  is called a semi-Riemannian almost product manifold. From (2.19) and (2.22), we have

$$\bar{g}(FX, Y) = \bar{g}(X, FY). \quad (2.23)$$

If, for any vector fields  $X, Y$  on  $\bar{M}$ ,

$$\bar{\nabla}F = 0, \quad \text{that is } \bar{\nabla}_X FY = F\bar{\nabla}_X Y, \quad (2.24)$$

then  $\bar{M}$  is called a semi-Riemannian product manifold, where  $\bar{\nabla}$  is the Levi-Civita connection on  $\bar{M}$ .

### 3. Lightlike Hypersurfaces of Semi-Riemannian Product Manifolds

Let  $M$  be a lightlike hypersurface of a semi-Riemannian product manifold  $(\bar{M}, \bar{g})$ . For any  $X \in \Gamma(TM)$  we can write

$$FX = fX + w(X)N, \quad (3.1)$$

where  $f$  is a  $(1,1)$  tensor field and  $w$  is a 1-form on  $M$  given by  $w(X) = \bar{g}(FX, \xi) = \bar{g}(X, F\xi)$ .

*Definition 3.1.* Let  $M$  be a lightlike hypersurface of a semi-Riemannian product manifold  $(\overline{M}, \overline{g})$ :

- (i) if  $F \text{ Rad } TM \subset S(TM)$  and  $F \text{ ltr}(TM) \subset S(TM)$  then we say that  $M$  is a screen semi-invariant lightlike hypersurface;
- (ii) if  $FS(TM) = S(TM)$  then we say that  $M$  is a screen invariant lightlike hypersurface;
- (iii) if  $F \text{ Rad } TM = \text{ltr}(TM)$  then we say that  $M$  is a radical anti-invariant lightlike hypersurface.

We note that a radical anti-invariant lightlike hypersurface is a screen invariant lightlike hypersurface.

*Remark 3.2.* We recall that there are some lightlike hypersurfaces of a semi-Riemannian product manifold which differ from the above definition, that is, this definition does not cover all lightlike hypersurfaces of a semi-Riemannian product manifold  $(\overline{M}, \overline{g})$ . In this paper we will study the hypersurfaces determined above.

Now, let  $M$  be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold. If we set  $\mathbb{D}_1 = F \text{ Rad } TM, \mathbb{D}_2 = F \text{ ltr}(TM)$  then we can write

$$S(TM) = \mathbb{D} \perp \{\mathbb{D}_1 \oplus \mathbb{D}_2\}, \quad (3.2)$$

where  $\mathbb{D}$  is a  $(m-2)$ -dimensional distribution. Hence we have the following decomposition:

$$\begin{aligned} TM &= \mathbb{D} \perp \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \perp \text{Rad } TM, \\ \overline{TM} &= \mathbb{D} \perp \{\mathbb{D}_1 \oplus \mathbb{D}_2\} \perp \{\text{Rad } TM \oplus \text{ltr}(TM)\}. \end{aligned} \quad (3.3)$$

**Proposition 3.3.** *The distribution  $\mathbb{D}$  is an invariant distribution with respect to  $F$ .*

*Proof.* For any  $X \in \Gamma(\mathbb{D})$  and  $U \in \Gamma(\mathbb{D}_1), V \in \Gamma(\mathbb{D}_2)$  we obtain

$$\begin{aligned} g(FX, U) &= g(X, FU) = 0, \\ g(FX, V) &= g(X, FV) = 0. \end{aligned} \quad (3.4)$$

Thus there are no components of  $FX$  in  $\mathbb{D}_1$  and  $\mathbb{D}_2$ . Furthermore, we have

$$\begin{aligned} g(FX, \xi) &= g(X, F\xi) = 0, \\ g(FX, N) &= g(X, FN) = 0. \end{aligned} \quad (3.5)$$

Proof is completed. □

If we set  $\overline{\mathbb{D}} = \mathbb{D} \perp \text{Rad } TM \perp F \text{ Rad } TM$ , we can write

$$TM = \overline{\mathbb{D}} \oplus \mathbb{D}_2. \quad (3.6)$$

From the above proposition we have the following corollary.

**Corollary 3.4.** *The distribution  $\overline{\mathbb{D}}$  is invariant with respect to  $F$ .*

*Example 3.5.* Let  $(\overline{M} = R_2^5, \overline{g})$  be a 5-dimensional semi-Euclidean space with signature  $(-, +, -, +, +)$  and  $(x, y, z, s, t)$  be the standard coordinate system of  $R_2^5$ . If we set  $F(x, y, z, s, t) = (x, y, -z, -s, -t)$ , then  $F^2 = I$  and  $F$  is a product structure on  $R_2^5$ . Consider a hypersurface  $M$  in  $\overline{M}$  by the equation:

$$t = x + y + z. \quad (3.7)$$

Then  $TM = \text{Span}\{U_1, U_2, U_3, U_4\}$ , where

$$U_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad U_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \quad U_3 = \frac{\partial}{\partial z} + \frac{\partial}{\partial t}, \quad U_4 = \frac{\partial}{\partial s}. \quad (3.8)$$

It is easy to check that  $M$  is a lightlike hypersurface and

$$TM^\perp = \text{Span}\{\xi = U_1 - U_2 + U_3\}. \quad (3.9)$$

Then take a lightlike transversal vector bundle as follow:

$$\text{ltr}(TM) = \text{Span}\left\{N = -\frac{1}{4}\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right\}\right\}. \quad (3.10)$$

It follows that the corresponding screen distribution  $S(TM)$  is spanned by

$$\{W_1 = U_4, W_2 = U_1 - U_2 - U_3, W_3 = U_1 + U_2 - U_3\}. \quad (3.11)$$

If we set  $\mathbb{D} = \text{Span}\{W_1\}$ ,  $\mathbb{D}_1 = \text{Span}\{W_2\}$  and  $\mathbb{D}_2 = \text{Span}\{W_3\}$ , then it can be easily checked that  $M$  is a screen semi-invariant lightlike hypersurface of  $\overline{M}$ .

*Example 3.6.* Let  $(x, y, z, t)$  be the standard coordinate system of  $R^4$  and  $ds^2 = -dx^2 - dy^2 + dz^2 + dt^2$  be a semi-Riemannian metric on  $R^4$  with 2-index. Let  $F$  be a product structure on  $R^4$  given

by  $F(x, y, z, t) = (z, t, x, y)$ . We consider the hypersurface  $M$  given by  $t = x + (1/2)(y + z)^2$  [1]. One can easily see that  $M$  is a lightlike hypersurface and

$$\begin{aligned} \text{Rad } TM &= \text{Span} \left\{ \xi = \frac{\partial}{\partial x} + (y+z) \frac{\partial}{\partial y} - (y+z) \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right\}, \\ \text{ltr}(TM) &= \text{Span} \left\{ N = -\frac{1}{2(1+(y+z)^2)} \left( \frac{\partial}{\partial x} + (y+z) \frac{\partial}{\partial y} + (y+z) \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right) \right\}, \\ S(TM) &= \text{Span} \left\{ W_1 = -(y+z) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, W_2 = \frac{\partial}{\partial z} + (y+z) \frac{\partial}{\partial t} \right\}. \end{aligned} \quad (3.12)$$

We can easily check that

$$F\xi = W_1 + W_2, \quad FN = \frac{1}{2(1+(y+z)^2)} \{W_1 - W_2\}. \quad (3.13)$$

Thus  $M$  is a screen semi-invariant lightlike hypersurface with  $\mathbb{D} = \{0\}$ ,  $\mathbb{D}_1 = \text{Span}\{F\xi\}$  and  $\mathbb{D}_2 = \text{Span}\{FN\}$ .

*Example 3.7.* Let  $(R_2^4, \bar{g})$  be a 4-dimensional semi-Euclidean space with signature  $(-, -, +, +)$  and  $(x_1, x_2, x_3, x_4)$  be the standard coordinate system of  $R_2^4$ . Consider a Monge hypersurface  $M$  of  $R_2^4$  given by

$$x_4 = Ax_1 + Bx_2 + Cx_3, \quad A^2 + B^2 - C^2 = 1, \quad A, B, C \in R. \quad (3.14)$$

Then the tangent bundle  $TM$  of the hypersurface  $M$  is spanned by

$$\left\{ U_1 = \frac{\partial}{\partial x_1} + A \frac{\partial}{\partial x_4}, U_2 = \frac{\partial}{\partial x_2} + B \frac{\partial}{\partial x_4}, U_3 = \frac{\partial}{\partial x_3} + C \frac{\partial}{\partial x_4} \right\}. \quad (3.15)$$

It is easy to check that  $M$  is a lightlike hypersurface (p.196, Ex.1, [3]) whose radical distribution  $\text{Rad } TM$  is spanned by

$$\xi = AU_1 + BU_2 - CU_3 = A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial x_2} - C \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}. \quad (3.16)$$

Furthermore, the lightlike transversal vector bundle is given by

$$\text{ltr}(TM) = \text{Span} \left\{ N = -\frac{1}{2(C^2 + 1)} \left( A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial x_2} + C \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) \right\}. \quad (3.17)$$

It follows that the corresponding screen distribution  $S(TM)$  is spanned by

$$\left\{ W_1 = \frac{1}{A^2 + B^2} \left( B \frac{\partial}{\partial x_1} - A \frac{\partial}{\partial x_2} \right), W_2 = \frac{1}{A^2 + B^2} \left( \frac{\partial}{\partial x_3} + C \frac{\partial}{\partial x_4} \right) \right\}. \quad (3.18)$$



If we define a mapping  $F$  by  $F(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4)$  then  $F^2 = I$  and  $F$  is a product structure on  $R_2^4$ . One can easily check that  $FS(TM) = S(TM)$  and  $F \text{Rad } TM = \text{ltr}(TM)$ . Thus  $M$  is a radical anti-invariant lightlike hypersurface of  $R_2^4$ . Furthermore, this lightlike hypersurface is a screen invariant lightlike hypersurface.

**Theorem 3.8.** *Let  $(\overline{M}, \overline{g})$  be a semi-Riemannian product manifold and  $M$  be a screen semi-invariant lightlike hypersurface of  $\overline{M}$ . Then the following assertions are equivalent.*

- (i) *The distribution  $\overline{\mathbb{D}}$  is integrable with respect to the induced connection  $\nabla$  of  $M$ .*
- (ii)  *$B(X, fY) = B(Y, fX)$ , for any  $X, Y \in \Gamma(\overline{\mathbb{D}})$ .*
- (iii)  *$g(A_\xi^*X, PfY) = g(A_\xi^*Y, PfX)$ , for any  $X, Y \in \Gamma(\overline{\mathbb{D}})$ .*

*Proof.* For any  $X, Y \in \Gamma(\overline{\mathbb{D}})$ , from (2.9), (2.24), and (3.1), we obtain

$$f\nabla_X Y + w(\nabla_X Y)N + B(X, Y)FN = \nabla_X fY + B(X, fY)N. \tag{3.19}$$

Interchanging role of  $X$  and  $Y$  we have

$$f\nabla_Y X + w(\nabla_Y X)N + B(Y, X)FN = \nabla_Y fX + B(Y, fX)N. \tag{3.20}$$

From (3.19), (3.20) we get

$$w([X, Y]) = B(X, fY) - B(Y, fX) \tag{3.21}$$

and this is (i)  $\Leftrightarrow$  (ii). From the first equation of (2.15), we conclude (ii)  $\Leftrightarrow$  (iii). Thus we have our assertion. □

From the decomposition (3.6), we can give the following definition.

*Definition 3.9.* Let  $M$  be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold  $\overline{M}$ . If  $B(X, Y) = 0$ , for any  $X \in \Gamma(\overline{\mathbb{D}}), Y \in \Gamma(\mathbb{D}_2)$ , then we say that  $M$  is a mixed geodesic lightlike hypersurface.

**Theorem 3.10.** *Let  $(\overline{M}, \overline{g})$  be a semi-Riemannian product manifold and  $M$  be a screen semi-invariant lightlike hypersurface of  $\overline{M}$ . Then the following assertions are equivalent.*

- (i)  *$M$  is mixed geodesic.*
- (ii) *There is no  $\mathbb{D}_2$ -component of  $A_N$ .*
- (iii) *There is no  $\mathbb{D}_1$ -component of  $A_\xi^*$ .*

*Proof.* Suppose that  $M$  is mixed geodesic screen semi-invariant lightlike hypersurface of  $\overline{M}$  with respect to the Levi-Civita connection  $\overline{\nabla}$ . From (2.24), (2.9), (2.10), and (3.1), we obtain

$$\nabla_X FN + B(X, FN)N = -fA_N X + \tau(X)FN - w(A_N X)N, \tag{3.22}$$

for any  $X \in \Gamma(\overline{\mathbb{D}})$ . If we take tangential and transversal parts of this last equation we have

$$\begin{aligned}\nabla_X FN &= -fA_N X + \tau(X)FN, \\ B(X, FN) &= -w(A_N X).\end{aligned}\tag{3.23}$$

Furthermore, since  $w(A_N X) = g(A_N X, F\xi)$ , we get (i)  $\Leftrightarrow$  (ii). Since  $\overline{g}(FN, \xi) = \overline{g}(N, F\xi) = 0$ , we obtain

$$g(A_N X, F\xi) = -g(A_\xi^* X, FN).\tag{3.24}$$

This is (ii)  $\Leftrightarrow$  (iii). □

From the decomposition (3.6), we have the following theorem.

**Theorem 3.11.** *Let  $M$  be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold  $\overline{M}$ . Then  $M$  is a locally product manifold according to the decomposition (3.6) if and only if  $f$  is parallel with respect to induced connection  $\nabla$ , that is  $\nabla f = 0$ .*

*Proof.* Let  $M$  be a locally product manifold. Then the leaves of distributions  $\overline{\mathbb{D}}$  and  $\mathbb{D}_2$  are both totally geodesic in  $M$ . Since the distribution  $\overline{\mathbb{D}}$  is invariant with respect to  $F$  then, for any  $Y \in \Gamma(\overline{\mathbb{D}})$ ,  $FY \in \Gamma(\overline{\mathbb{D}})$ . Thus  $\nabla_X Y$  and  $\nabla_X fY$  belong to  $\Gamma(\overline{\mathbb{D}})$ , for any  $X \in \Gamma(TM)$ . From the Gauss formula, we obtain

$$\nabla_X fY + B(X, fY)N = f\nabla_X Y + w(\nabla_X Y)N + B(X, Y)FN.\tag{3.25}$$

Comparing the tangential and normal parts with respect to  $\overline{\mathbb{D}}$  of (3.25), we have

$$\nabla_X fY = f\nabla_X Y, \quad \text{that is } (\nabla_X f)Y = 0,\tag{3.26}$$

$$B(X, Y) = 0.\tag{3.27}$$

Since  $fZ = 0$ , for any  $Z \in \Gamma(\mathbb{D}_2)$ , we get  $\nabla_X fZ = 0$  and  $f\nabla_X Z = 0$ , that is  $(\nabla_X f)Z = 0$ . Thus we have  $\nabla f = 0$  on  $M$ .

Conversely, we assume that  $\nabla f = 0$  on  $M$ . Then we have  $\nabla_X fY = f\nabla_X Y$ , for any  $X, Y \in \Gamma(\overline{\mathbb{D}})$  and  $\nabla_U fW = f\nabla_U W = 0$ , for any  $U, W \in \Gamma(\mathbb{D}_2)$ . Thus it follows that  $\nabla_X fY \in \Gamma(\overline{\mathbb{D}})$  and  $\nabla_U W \in \Gamma(\mathbb{D}_2)$ . Hence, the leaves of the distributions  $\overline{\mathbb{D}}$  and  $\mathbb{D}_2$  are totally geodesic in  $M$ . □

From Theorem 3.11 and (3.27) we have the following corollary.

**Corollary 3.12.** *Let  $M$  be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold  $\overline{M}$ . If  $M$  has a local product structure, then it is a mixed geodesic lightlike hypersurface.*

Let  $M$  be a radical anti-invariant lightlike hypersurface of a semi-Riemannian product manifold  $\overline{M}$ . Then we have the following decomposition:

$$T\overline{M} = S(TM) \perp \{\text{Rad } TM \oplus F \text{Rad } TM\}. \quad (3.28)$$

**Theorem 3.13.** *Let  $M$  be a radical anti-invariant lightlike hypersurface of a semi-Riemannian product manifold  $\overline{M}$ . Then the screen distribution  $S(TM)$  of  $M$  is an integrable distribution if and only if  $B(X, FY) = B(Y, FX)$ .*

*Proof.* If a vector field  $X$  on  $M$  belongs to  $S(TM)$  if and only if  $\eta(X) = 0$ . Since  $M$  is a radical anti-invariant lightlike hypersurface, for any  $X \in \Gamma(S(TM))$ ,  $FX \in \Gamma(S(TM))$ . For any  $X, Y \in \Gamma(S(TM))$ , we can write

$$\overline{\nabla}_X FY = \nabla_X FY + B(X, FY)N. \quad (3.29)$$

In this last equation interchanging role of  $X$  and  $Y$ , we obtain

$$F[X, Y] = \nabla_X FY - \nabla_Y FX + (B(X, FY) - B(Y, FX))N. \quad (3.30)$$

Since  $\eta([X, Y]) = \overline{g}([X, Y], N) = \overline{g}(F[X, Y], FN)$ , we get

$$\eta([X, Y]) = (B(X, FY) - B(Y, FX))\overline{g}(N, FN). \quad (3.31)$$

Since  $\overline{g}(N, FN) \neq 0$ ,  $\eta([X, Y]) = 0$  if and only if  $B(X, FY) = B(Y, FX)$ . This is our assertion.  $\square$

#### 4. Quarter-Symmetric Nonmetric Connections

Let  $(\overline{M}, \overline{g}, F)$  be a semi-Riemannian product manifold and  $\overline{\nabla}$  be the Levi-Civita connection on  $\overline{M}$ . If we set

$$\overline{D}_X Y = \overline{\nabla}_X Y + u(Y)FX, \quad (4.1)$$

for any  $X, Y \in \Gamma(T\overline{M})$ , then  $\overline{D}$  is a linear connection on  $\overline{M}$ , where  $u$  is a 1-form on  $\overline{M}$  with  $U$  as associated vector field, that is

$$u(X) = \overline{g}(X, U). \quad (4.2)$$

The torsion tensor of  $\overline{D}$  on  $\overline{M}$  denoted by  $\overline{T}$ . Then we obtain

$$\overline{T}(X, Y) = u(Y)FX - u(X)FY, \quad (4.3)$$

$$(\overline{D}_X \overline{g})(Y, Z) = -u(Y)\overline{g}(FX, Z) - u(Z)\overline{g}(FX, Y), \quad (4.4)$$

for any  $X, Y \in \Gamma(T\bar{M})$ . Thus  $\bar{D}$  is a quarter-symmetric nonmetric connection on  $\bar{M}$ . From (2.24) and (4.1) we have

$$(\bar{D}_X F)Y = u(FY)FX - u(Y)X. \quad (4.5)$$

Replacing  $X$  by  $FX$  and  $Y$  by  $FY$  in (4.5) and using (2.19) we obtain

$$(\bar{D}_{FX} F)FY = u(Y)X - u(FY)FX. \quad (4.6)$$

Thus we have

$$(\bar{D}_X F)Y + (\bar{D}_{FX} F)FY = 0. \quad (4.7)$$

If we set

$$'F(X, Y) = \bar{g}(FX, Y), \quad (4.8)$$

for any  $X, Y \in \Gamma(T\bar{M})$ , from (4.1) we get

$$(\bar{D}_X 'F)(Y, Z) = (\bar{\nabla}_X 'F)(Y, Z) - u(Y)\bar{g}(X, Z) - u(Z)\bar{g}(X, Y). \quad (4.9)$$

From (4.1) the curvature tensor  $\bar{R}^D$  of the quarter-symmetric nonmetric connection  $\bar{D}$  is given by

$$\bar{R}^D(X, Y)Z = \bar{R}(X, Y)Z + \bar{\lambda}(X, Z)FY - \bar{\lambda}(Y, Z)FX, \quad (4.10)$$

for any  $X, Y, Z \in \Gamma(T\bar{M})$ , where  $\bar{\lambda}$  is a  $(0, 2)$ -tensor given by  $\bar{\lambda}(X, Z) = (\bar{\nabla}_X u)(Z) - u(Z)u(FX)$ . If we set  $\bar{R}^D(X, Y, Z, W) = \bar{g}(\bar{R}^D(X, Y)Z, W)$ , then, from (4.10), we obtain

$$\bar{R}^D(X, Y, Z, W) = -\bar{R}^D(Y, X, Z, W). \quad (4.11)$$

We note that the Riemannian curvature tensor  $\bar{R}^D$  of  $\bar{D}$  does not satisfy the other curvature-like properties. But, from (4.10), we have

$$\begin{aligned} \bar{R}^D(X, Y)Z + \bar{R}^D(Y, Z)X + \bar{R}^D(Z, X)Y &= (\bar{\lambda}(Z, Y) - \bar{\lambda}(Y, Z))FX \\ &+ (\bar{\lambda}(X, Z) - \bar{\lambda}(Z, X))FY \\ &+ (\bar{\lambda}(Y, X) - \bar{\lambda}(X, Y))FZ. \end{aligned} \quad (4.12)$$

Thus we have the following proposition.

**Proposition 4.1.** *Let  $M$  be a lightlike hypersurface of a semi-Riemannian product manifold  $\overline{M}$ . Then the first Bianchi identity of the quarter-symmetric nonmetric connection  $\overline{D}$  on  $M$  is provided if and only if  $\overline{\lambda}$  is symmetric.*

Let  $M$  be a lightlike hypersurface of a semi-Riemannian product manifold  $(\overline{M}, \overline{g})$  with quarter-symmetric nonmetric connection  $\overline{D}$ . Then the Gauss and Weingarten formulas with respect to  $\overline{D}$  are given by, respectively,

$$\overline{D}_X Y = D_X Y + \overline{B}(X, Y)N \quad (4.13)$$

$$\overline{D}_X N = -\overline{A}_N X + \overline{\tau}(X)N \quad (4.14)$$

for any  $X, Y \in \Gamma(TM)$ , where  $D_X Y, \overline{A}_N X \in \Gamma(TM)$ ,  $\overline{B}(X, Y) = \overline{g}(\overline{D}_X Y, \xi)$ ,  $\overline{\tau}(X) = \overline{g}(\overline{D}_X N, \xi)$ . Here,  $D, \overline{B}$  and  $\overline{A}_N$  are called the induced connection on  $M$ , the second fundamental form, and the Weingarten mapping with respect to  $\overline{D}$ . From (2.9), (2.10), (3.1), (4.1), (4.13), and (4.14) we obtain

$$D_X Y = \nabla_X Y + u(Y)fX, \quad (4.15)$$

$$\overline{B}(X, Y) = B(X, Y) + u(Y)\omega(X), \quad (4.16)$$

$$\overline{A}_N X = A_N X - u(N)fX, \quad (4.17)$$

$$\overline{\tau}(X) = \tau(X) + u(N)\omega(X),$$

for any  $X, Y \in \Gamma(TM)$ . From (4.1), (4.4), (4.13), and (4.16) we get

$$\begin{aligned} (D_X g)(Y, Z) &= B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ &\quad - u(Y)g(fX, Z) - u(Z)g(fX, Y). \end{aligned} \quad (4.18)$$

On the other hand, the torsion tensor of the induced connection  $D$  is

$$T^D(X, Y) = u(Y)fX - u(X)fY. \quad (4.19)$$

From last two equations we have the following proposition.

**Proposition 4.2.** *Let  $M$  be a lightlike hypersurface of a semi-Riemannian product manifold  $(\overline{M}, \overline{g})$  with quarter-symmetric nonmetric connection  $\overline{D}$ . Then the induced connection  $D$  is a quarter-symmetric nonmetric connection on the lightlike hypersurface  $M$ .*

For any  $X, Y \in \Gamma(TM)$ , we can write

$$D_X PY = D_X^* PY + \overline{C}(X, PY)\xi, \quad (4.20)$$

$$D_X \xi = -\overline{A}_\xi^* X + \varepsilon(X)\xi,$$

where  $D_X^*PY \bar{A}_\xi^*X \in \Gamma(S(TM))$ ,  $\bar{C}(X, PY) = \bar{g}(D_X PY, N)$ , and  $\varepsilon(X) = \bar{g}(D_X \xi, N)$ . From (2.14), (16), and (4.15), we obtain

$$\bar{C}(X, PY) = C(X, PY) + u(PY)\eta(fX), \quad (4.21)$$

$$\bar{A}_\xi^*X = A_\xi^*X - u(\xi)PfX, \quad \varepsilon(X) = -\tau(X) + u(\xi)\eta(fX). \quad (4.22)$$

Using (2.15), (4.16) and (4.22) we obtain

$$\begin{aligned} \bar{B}(X, PY) &= g(\bar{A}_\xi^*X, PY) + u(PY)w(X) \\ &\quad + u(\xi)\bar{g}(FX, PY), \end{aligned} \quad (4.23)$$

for any  $X, Y \in \Gamma(TM)$ .

Now, we consider a screen semi-invariant lightlike hypersurface  $M$  of a semi-Riemannian product manifold  $\bar{M}$  with respect to the quarter symmetric connection  $\bar{D}$  given by (4.1). Since  $w(X) = g(FX, \xi)$ , for any  $X \in \Gamma(\mathbb{D})$ ,  $w(X) = 0$ . Thus we have the following propositions.

**Proposition 4.3.** *Let  $M$  be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold  $(\bar{M}, \bar{g})$  with quarter-symmetric nonmetric connection. The second fundamental form  $\bar{B}$  of quarter-symmetric nonmetric connection  $\bar{D}$  is degenerate.*

**Proposition 4.4.** *Let  $(\bar{M}, \bar{g})$  be a semi-Riemannian product manifold and  $M$  be a screen semi-invariant lightlike hypersurfaces of  $\bar{M}$ . If  $M$  is  $\mathbb{D}$  totally geodesic with respect to  $\bar{\nabla}$ , then  $M$  is  $\mathbb{D}$  totally geodesic with respect to quarter-symmetric nonmetric connection.*

**Theorem 4.5.** *Let  $(\bar{M}, \bar{g})$  be a semi-Riemannian product manifold and  $M$  be a screen semi-invariant lightlike hypersurfaces of  $\bar{M}$ . Then the following assertions are equivalent.*

- (i) *The distribution  $\mathbb{D}$  is integrable with respect to the quarter symmetric nonmetric connection  $D$ .*
- (ii)  *$\bar{B}(X, fY) = \bar{B}(Y, fX)$ , for any  $X, Y \in \Gamma(\mathbb{D})$ .*
- (iii)  *$g(\bar{A}_\xi^*X, PfY) = g(\bar{A}_\xi^*Y, PfX)$ , for any  $X, Y \in \Gamma(\mathbb{D})$ .*

The proof of this theorem is similar to the proof of the Theorem 3.8.

From (4.23), for any  $X \in \Gamma(\mathbb{D})$  and  $Y \in \Gamma(\mathbb{D}_2)$ , we have  $\bar{B}(X, PY) = g(\bar{A}_\xi^*X, PY)$ . If we set  $\mathbb{D}' = \mathbb{D} \perp \mathbb{D}_2$ , then, from Theorem 3.10, we have the following corollary.

**Corollary 4.6.** *Let  $(\bar{M}, \bar{g})$  be a semi-Riemannian product manifold and  $M$  be a screen semi-invariant lightlike hypersurface of  $\bar{M}$ . Then the distribution  $\mathbb{D}'$  is a mixed geodesic foliation defined with respect to quarter symmetric nonmetric connection if and only if there is no  $\mathbb{D}_1$  component of  $\bar{A}_\xi^*$ .*

From (4.15), we obtain

$$\begin{aligned}
 R^D(X, Y)Z &= R(X, Y)Z + u(Z)\{(\nabla_X f)Y - (\nabla_Y f)X\} \\
 &\quad + \lambda(X, Z)fY - \lambda(Y, Z)fX,
 \end{aligned}
 \tag{4.24}$$

where  $\lambda$  is a  $(0, 2)$  tensor on  $M$  given by  $\lambda(X, Z) = (\nabla_X u)(Z) - u(Z)u(fX)$ .

From (4.24), we have the following proposition which is similar to the Proposition 4.1.

**Proposition 4.7.** *Let  $M$  be a lightlike hypersurface of a semi-Riemannian product manifold  $\bar{M}$ . One assumes that  $f$  is parallel on  $M$ . Then the first Bianchi identity of the quarter-symmetric nonmetric connection  $D$  on  $M$  is provided if and only if  $\lambda$  is symmetric.*

Now we will compute Gauss-Codazzi equations of lightlike hypersurfaces with respect to the quarter-symmetric nonmetric connection:

$$\begin{aligned}
 \bar{g}(\bar{R}^D(X, Y)Z, PW) &= g(R(X, Y)Z, PW) \\
 &\quad + B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW) \\
 &\quad + \bar{\lambda}(X, Z)g(fY, PW) - \bar{\lambda}(Y, Z)g(fX, PW), \\
 \bar{g}(\bar{R}^D(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\
 &\quad + \bar{\lambda}(X, Z)w(Y) - \bar{\lambda}(Y, Z)w(X), \\
 \bar{g}(\bar{R}^D(X, Y)Z, N) &= g(R(X, Y)Z, N) \\
 &\quad + \bar{\lambda}(X, Z)\eta(fY) - \bar{\lambda}(Y, Z)\eta(fX),
 \end{aligned}
 \tag{4.25}$$

for any  $X, Y, Z, W \in \Gamma(TM)$ .

Now, let  $M$  be a screen semi-invariant lightlike hypersurface of a  $(m + 2)$ -dimensional semi-Riemannian product manifold with the quarter-symmetric nonmetric connection  $\bar{D}$  such that the tensor field  $f$  is parallel on  $M$ . We consider the local quasiorthonormal basis  $\{E_i, F\xi, FN, \xi, N\}$ ,  $i = 1, \dots, m - 2$ , of  $\bar{M}$  along  $M$ , where  $\{E_1, \dots, E_{m-2}\}$  is an orthonormal basis of  $\Gamma(\mathbb{D})$ . Then, the Ricci tensor of  $M$  with respect to  $D$  is given by

$$\begin{aligned}
 R^{D(0,2)}(X, Y) &= \sum_{i=1}^{m-2} \varepsilon_i g(R^D(X, E_i)Y, E_i) + g(R^D(X, F\xi)Y, FN) \\
 &\quad + g(R^D(X, FN)Y, F\xi) + g(R^D(X, \xi)Y, N).
 \end{aligned}
 \tag{4.26}$$

From (4.24) we have

$$\begin{aligned}
 R^{D(0,2)}(X, Y) &= R^{(0,2)}(X, Y) \\
 &+ \sum_{i=1}^{m-2} \varepsilon_i \{ \lambda(X, Y)g(fE_i, E_i) - \lambda(E_i, Y)g(fX, E_i) \} \\
 &- \lambda(F\xi, Y)\eta(X) - \lambda(\xi, Y)\eta(fX),
 \end{aligned} \tag{4.27}$$

where  $R^{(0,2)}(X, Y)$  is the Ricci tensor of  $M$ . Thus we have the following corollary.

**Corollary 4.8.** *Let  $M$  a screen semi-invariant lightlike hypersurface of a  $(m + 2)$ -dimensional semi-Riemannian product manifold with the quarter-symmetric nonmetric connection  $\bar{D}$  such that the tensor field  $f$  is parallel on  $M$  and  $R^{(0,2)}(X, Y)$  is symmetric. Then  $R^{D(0,2)}$  is symmetric on the distribution  $\mathbb{D}$  if and only if  $\lambda$  is symmetric and  $\lambda(fX, Y) = \lambda(fY, X)$ .*

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