Loss of customer goodwill in the uncapacitated lot-sizing problem

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Abstract

Loss of customer goodwill in uncapacitated single level lot-sizing is studied with a mixed integer programming model extending the well-known Wagner-Whitin (WW) model. The objective is to maximize profit from production and sales of a single good over a finite planning horizon. Demand, costs, and prices vary with time. Unsatisfied demand cannot be backordered. It leads to the immediate loss of profit from sales. Previous models augment the total cost objective by this lost profit. The difference of the proposed model is that unsatisfied demand in a given period causes the demand in the next period to shrink due to the loss of customer goodwill. A neighborhood search and restoration heuristic is developed that tries to adjust the optimal lot sizes of the original no-goodwill-loss model to the situation with goodwill loss. Its performance is compared with the Wagner-Whitin solution, and with the commercial solver CPLEX 8.1 on 360 test problems of various period lengths.

Keywords: lot-sizing; neighborhood search; customer goodwill; profit

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1. Introduction and background

The deterministic dynamic lot-sizing problem has been first introduced and solved by Wagner and Whitin [1]. It is one of the most famous discrete decision problems of production and inventory planning. The way this problem is modeled differs from the classical economic order quantity (EOQ) model in that demand is neither stationary nor continuous. Instead, both inventory replenishments and demand are instantaneous and discretized over a finite planning horizon of several periods. Demand may change from one period to another, implying that it is time-varying. Without capacity restrictions, the Wagner-Whitin (WW) problem can be viewed as a shortest path problem. An early work by Evans [2] describes an efficient computer implementation of the original forward recursive dynamic programming (*DP*) algorithm of Wagner and Whitin. The implementation of Evans requires low core storage and solves the WW problem to optimality in $O(T^2)$ time where T denotes the number of periods in the planning horizon. A later report by Saydam and Evans [3] provides a comparative performance analysis of the WW and other heuristics for the lot-sizing problem when demand cannot be backordered. Similar *DP* based algorithms have been developed which guarantee optimality in polynomial time. As the variation in the problem data lessens, the computational complexity of such exact methods decreases too. There are now algorithms running in $O(T \log T)$ or even in linear time when costs are constant for all periods. The

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algorithms by Federgruen and Tzur [4], Wagelmans *et al.* [5], and by Aggarwal and Park [6] are prominent examples.

This paper focuses exclusively on the lost demand dimension of the basic WW model with revenue considerations. Lost demand in inventory planning due to the erosion of customer goodwill has not received as much attention in the operations research literature as the other challenging characteristics of lot-sizing. Yet, even a Google search for "*loss of customer goodwill*" finds above 900 hits on the Web as of August 2005. This concept has been touched upon in management science textbooks of various business school curricula. Today it is widely accepted that product unavailability no matter for what reason incurs a goodwill cost that may be difficult to estimate [7–11]. Loss of sales revenue from cancelled orders is a natural consequence. The cost of such loss is also referred to as shortage cost. However, estimating the loss in future sales because of customer dissatisfaction is typically harder. In Lawrence and Pasternack [7], for example, it is argued that many businesses do not have any idea what the long-term goodwill cost for an unsatisfied customer might be. Empirical evidence found in Blazenko and Vandezande [11] ties the likelihood of lost revenue arising from the loss of customer goodwill to the alternative sources of supply that are available. If there are good alternative sources or substitutable products, the prospect of long-term revenue loss for companies is greater.

Motivated by the inherent intricacy of modeling the cost of stockouts, we take on the profit-maximizing lost sales model of Aksen *et al.* [12] and add the impact of customer goodwill loss. The goodwill loss concept is relatively new in the literature of lot-sizing as will be elaborated in Section 2. The rest of this paper is organized as follows: We first review the background of lot-sizing models in the literature. Then we present the basic relationships between customer goodwill loss and demand. The fading impact of goodwill loss is illustrated in a short example. A mixed integer programming (*MIP*) formulation is described for which we propose a heuristic search method. This method is demonstrated on a numerical example, and then benchmarked on 22 test problems found in Aksen *et al*. Computational experiments with 360 randomly generated test problems are also reported. Finally, we conclude with a summary of our study, and suggest further research directions in the modeling of customer goodwill loss.

2. Background of lot-sizing models

Though the WW model approaches its 50th anniversary, it is still the starting point of myriad models that deal with a wide spectrum of medium to long term planning problems of the manufacturing and process industry. In developing variants of the WW model, researchers aim to capture more real world phenomena. Capacity or resource restrictions, set-up times besides set-up costs, sequence-dependent set-up costs, multi-level and multi-item versions, loading and scheduling decisions concurrent with the lot-sizing decisions, backlogging of demand, all-units or incremental quantity discounts, rolling planning horizons, deteriorating inventory due to limited shelf life of items, stockouts, and lot-sizing with supplier selection are among such phenomena addressed by present-day models. The majority of these extensions inflate the model's complexity, thereby worsen its solution time and quality. The abundance of deterministic dynamic lot-sizing models has given birth to several distinguished surveys and reviews. Two highlights in this area are the comprehensive review of Bahl *et al.* [13] and the excellent study by Kuik *et al.* [14]. Lately; Karimi *et al.* [15] review models and algorithms for the single-level capacitated lot-sizing problem (CLSP). The reader is also referred to Drexl and Kimms [16] for an in-depth summary and analysis of CLSP that integrates scheduling decisions.

Several articles have combined backordering and stockouts in lot-sizing. Backordering means that unsatisfied customer demand (shortage) can be met in a later period as opposed to being immediately lost. There is sometimes a maximum number of periods until the end of which demand can be backordered. Any unsatisfied demand beyond that grace period is lost. Backordering in a period is assumed to trigger a partial loss of demand in that period in addition to the shortage cost. An example of this approach is the deteriorating inventory model of Wee [17] with partial backordering and profit maximization. When computing the total cost of inventory shortage in his model, Wee assumes that a fraction of unsatisfied demand will be backordered whereas the rest will be lost right away. He uses a shortage cost per unit backordered per period and a penalty cost per unit lost that covers the loss of marginal profit. A recent paper by Wang [18] discusses an inventory model with shortages for time-varying demand of deteriorating items. Wang mentions the opportunity cost due to lost sales during each shortage period. Also Wang's model pursues a profit maximization objective. He argues that from an economic point of view shortages may be desirable in practice when the unit value of the inventory is very high and hence the holding cost is high. While some customers can be convinced to wait for backlogging during the shortage periods, others would withdraw their orders, thus cause an opportunity cost due to lost sales.

Pricing with the aim of profit maximization in a CLSP setting is modeled and solved with an efficient Lagrangian relaxation algorithm in Haugen *et al.* [19]. Their model can be considered a monopolistic generalization of the CLSP. The manufacturer facing the lot-sizing problem is a price setter, and his customers' demand is sensitive to price. This price sensitivity is leveraged in order to adapt the demand so as to conform to the capacity constraints throughout the planning horizon. When demand in a period is too much to meet, then price is increased, which leads to a decline in demand, thus prevents shortage. On the other hand, too high a price in that period would cause demand to shrink so much that maximum possible profit would not be attained. Thus, prices and lot sizes should be decided concurrently. Haugen *et al.*'s algorithm competes favorably with the known methods of solving the cost minimization version of the same problem without pricing. The authors argue that profit maximization and pricing in the CLSP may prove both practically more interesting as well as numerically more feasible.

Sandbothe and Thompson [20,21] present a total cost minimization model that replaces backordering with production capacity constraints and inventory bounds. In their model, unsatisfied demand in a stockout period is permanently lost, incurring a cost at a fixed rate per unit lost. Aksen *et al.* [12] omit capacity constraints in the model of Sandbothe and Thompson, and allow all costs and prices to vary dynamically over the planning horizon. In addition to stockout periods, they introduce the so-called conservation period, in which it is more profitable for the producer to lose the demand in spite of available inventory at hand. Structural properties of an optimal solution to their lost sales model are proven using a concave cost network representation with a single source node. This conservation model is reiterated in Chu and Chu [22] with time-varying storage capacities. Chu and Chu assert that as companies put an increasing emphasis on customer satisfaction, deliberate lost sales occur seldom. In their view, to have the demand of a stockout period out-sourced is more realistic than to consider it lost. This way, the producer suffers an outsourcing cost instead of shortage or backordering cost.

The goodwill loss concept in lot-sizing is first mentioned in a paper by Hsu and Lowe [23]. The authors comment that production loss or customer goodwill loss may lead to a unit penalty cost for unsatisfied demand that grows in a nonlinear fashion, and is dependent on how long that demand has been backordered. Also inventory holding costs may be dependent on how long the items have been in stock. In Graves [24] we come across goodwill loss again. Graves formulates two models, one for the lost sales case, and another for the backorders. He assumes that demand that cannot be met in a period is lost, and a loss of customer goodwill would manifest itself in terms of reduced future sales. According to Graves, this lost sales cost is very difficult to quantify as it represents the future unknown impact from poor service today. In his lost sales model too, the cumulative cost of goodwill loss, which is linearly proportional to the sum of unmet demands by a unit penalty, is subtracted from the total profit function. In our model, on the other hand, a certain ratio of realized demand that goes unsatisfied in the current period t is deducted from the original demand of the next period (t+1), which in turn yields the realized demand for (t+1). We use the term *effective demand* for the realized amount of original demand. It could be less than the original value if a goodwill loss comes from the preceding period. For the producer, this goodwill loss in demand is exogenous and irreversible. A fraction of those customers whose demand is not satisfied in the current period reacts to this by not returning the next period. On the other hand, the producer might not meet an effective demand either due to stockout or for the sake of conserving inventory. Such a decision of the producer leads to lost sales. This type of loss, which we call *shortage*, is endogenous since it is under the control of the producer. We assume that any shortage in period t causes a goodwill loss in (t+1) only. The goodwill impact vanishes or becomes negligible after that period. As far as we know, our model is the first dynamic lot-sizing model that quantifies customer goodwill loss as a reduction in future demand. Lawrence and Pasternack [7] point out that marketing surveys and focus groups can yield reasonable estimates of future reduction in a firm's profitability due to goodwill loss. The rationale behind our model is the effectiveness of such instruments in forecasting the impact of goodwill loss as shrinking future demand.

3. Representing goodwill loss

In this section we give two different specific versions of representing customer goodwill loss as a reduction in original demand. The second version will be adopted and incorporated into the mathematical lot-sizing model with lost sales. In Appendix A we merge these two versions into a unified goodwill loss representation, and show that it converges to either specific version at the limits. We make a distinction between original, realized, and satisfied demand. Realized demand is the effective demand, which yields after the deduction of the goodwill loss from the original demand. If all of the effective demand in period *t* is satisfied, then this implies there is no shortage in *t*. Otherwise, satisfied demand in *t* is the rest of the effective demand after the deduction of the shortage. The following relationship is common to both versions of goodwill loss representation.

$$\textit{Satisfied Demand in}\,(t) \;=\; \underbrace{\textit{Original Demand in}\,(t) \;-\; \textit{Goodwill Loss in}\,(t)}_{\textit{Effective Demand in}\,(t)} \;-\; \textit{Shortage in}\,(t)$$

3.1. Version-A

In Version-A, the amount of goodwill loss in period *t* depends on the size of lost sales in the previous period relative to that period's original demand. The larger percentage of original demand in (*t*–1) is unsatisfied, the larger percentage of original demand in (*t*) will be lost due to goodwill. This may happen when a company's customers move with a collective memory, and make their decisions of buying or not buying by looking at the past performance of the company. If company statistics reveal that it adequately met demand in the previous period, then customers choose buying from it. Otherwise, they get concerned about the company's failure, and do not feel like buying from it in the current period. The relationship between demand and goodwill loss in Version-A is given below where β ($0 < \beta \leq 1$) is a known coefficient indicating the rate of goodwill loss.

$$\textit{Goodwill Loss in}\,(t) \;=\; \beta \times \frac{\textit{Shortage in}\,(t-1)}{\textit{Orig.Demand in}\,(t\text{ - }1)} \times \textit{Orig.Demand in}\,(t)$$

3.2. Version-B

In Version-B, there exists an absolute dependence of the amount of goodwill loss in period *t* on the size of lost sales in the previous period. Two concurrent relationships between demand and loss in Version-B are given below where the latter directly follows from the former. Note that these relationships can be rewritten such that any nonlinearity is avoided when they are incorporated into a mathematical model.

$$\textit{Goodwill Loss in}\,(t) \;=\; \textit{min}\left\{\; \textit{Original Demand in}\,(t),\;\; \beta \times \textit{Shortage in}\,(t-1) \;\right\}$$

$$\textit{Effective Demand in}\,(t) \;=\; \textit{max}\left\{\; 0,\;\; \textit{Original Demand in}\,(t) \;-\; \beta \times \textit{Shortage in}\,(t-1) \;\right\}$$

To explain the goodwill loss in Version-B with a naive example, assume that the demand of customers in period (*t*–1) is both effectively and originally 10 units. Let β (the rate of goodwill loss) equal one. Assume now this effective demand entirely goes unmet by the producer. Then, the goodwill loss in *t* will be the minimum of those 10 units and the current period's original demand, no matter how big it is. In Version-A, however, since the last period's demand is fully lost and β is one, goodwill loss will be as high as the current period's original demand. This means, even if the current period's original demand is 10000, all of it will be lost due to goodwill. Version-B of the goodwill loss phenomenon applies particularly when the producer serves during the planning horizon a multitude number of individual repeat buyers each having more or less a unit demand. If this is the case then 10 buyers can hardly have the power of affecting the opinion of 10000 buyers at once. In contrast, if the producer serves the very same customer whose demand randomly fluctuates over time, then the representation of goodwill loss in Version-A could be a better fit to the producer's situation. We adopt Version-B and solve our goodwill loss model according to that representation. The same solution techniques could be used for Version-A or for a unified goodwill loss representation that combines both versions as proposed in Appendix A.

3.3. Diminishing impact of goodwill loss

In both representations of the customer goodwill loss, the impact of a single loss period on the demand of the first succeeding no-loss period is likely higher than the impact of two or more consecutive loss periods. Such a block of loss periods eventually makes a diminishing impact. We ascribe this counterintuitive situation to two reasons. First one is the short time span of goodwill impact. No matter how big a demand is lost in the current period; its effect is felt only in the next period. Secondly, as shortages pile up in consecutive periods, effective demand in the last loss period shrinks. Shortage in the last loss period is made out of this effective demand, and goodwill loss in the next period amounts to a certain ratio of this shortage in both Version-A and -B. The lesser effective demand in the last loss period is, the lesser possible shortage can happen, thus the smaller goodwill loss arises in the first no-loss period. We give a short example with four periods to illustrate this situation. Two scenarios and the corresponding impact on the last period are shown in Table 1. Arrows in the table indicate the direction of impact. In scenario 1 we do not meet demand until the last month whereas in scenario 2 only the third month's demand is lost. A smaller effective demand goes unsatisfied in the last loss period July (2500 in scenario 1 vs. 3000 in scenario 2), which translates into a less impact on the first no-loss period August (1250 in scenario 1 vs. 1500 in scenario 2). For the same rate of goodwill loss (50%), one can verify that if Version-A was adopted, a diminishing goodwill impact would be observed again. That is, in scenario 1 the goodwill loss of August would be 1500 units as opposed to 2000 units in scenario 2.

Table 1 Goodwill impacts of different number of loss periods with $\beta = 50\%$

	Period	Original Demand	Effective Demand	Satisfied Demand	Shortage	Period Loss
Scenario 1	May	2000	2000	0	2000	2000
	June	2000	1000	0	1000	2000
	July	3000	2500	0	2500	3000
	August	4000	2750	2750	0	**1250**
Scenario 2	May	2000	2000	2000	0	0
	June	2000	2000	2000	0	0
	July	3000	3000	0	3000	3000
	August	4000	2500	2500	0	**1500**

4. The mathematical model with goodwill loss

We add several new decision variables and logical constraints to the lost sales model **P** of Aksen *et al.* [12] while we modify their balance constraint for inventory flows. The following parameters appear in our goodwill loss model.

T : Number of periods in the planning horizon.

s_t : Set-up cost *(fixed cost)* of production in period t.

p_t : Unit revenue (unit selling price) in period t.

c_t : Unit production cost *(variable cost)* in period t.

h_t : Unit inventory holding cost in period t. This is charged for inventory at the end of period t.

d_t : Demand in period t.

β : Rate of customers who do not return the next time if their current effective demand is not satisfied.

M : A large number acting as a Big-M value in logical constraints. ($M = \sum_{t=1}^{T} d_t$).

ε : A very small number used in the definition of the effective demand to convert *less-than-or-equal-to* constraints to strictly inequality constraints. We calculate it as $\varepsilon = max\ \{M^{-1},\ 0.10\beta\ , 0.05\}$ to mitigate the quite possible scaling problem between this constant and the value M.

The decision variables of our goodwill loss model are given below.

X_t : Production quantity in period t.

I_t : End-of-period inventory for period t.

LU_t : Shortage (or unsatisfied effective demand) in period t. This variable is the same as L_t in **P**.

LG_t : Goodwill loss (or demand of customers lost to the goodwill impact) in period t.

E_t : Effective (or realized) demand in period t.

y_t : Indicator variable of the production activity in period t. $y_t = \begin{cases} 1 & if\ X_t > 0 \\ 0 & otherwise \end{cases}$

δ_t : Indicator variable signaling whether or not the expression $\left(d_t - \lceil \beta \cdot LU_{t-1} \rceil\right)$ in the definition of the effective demand E_t is nonnegative. $\delta_t = \begin{cases} 1 & if\ \ d_t \geq \lceil \beta \cdot LU_{t-1} \rceil \\ 0 & otherwise \end{cases}$

The operator $\lceil \cdot \rceil$ in the definition of δ_t returns the smallest integer number greater than or equal to its argument. We use this operator to measure loss values as integers. Demand in our new model occurs in discrete units. Hence, loss values have to be integer too. There are two additional assumptions in Aksen *et al*'s lost sales model **P** besides those of the standard Wagner-Whitin model: Any demand not satisfied in its period is considered lost, and the gross marginal profit ($p_t - c_t$) is nonnegative in each period t. We adopt these assumptions as is. Since realized sales for period t are given by ($d_t - LG_t - LU_t$) in the new model, profit function Π of the problem **P** needs a modification. An *MIP* formulation of the new goodwill loss model is presented below.

Problem PG

$$\textbf{Max.}\ \text{PROFIT}\ \Pi_G = \sum_{t=1}^{T} p_t\,(d_t - LG_t - LU_t) - \sum_{t=1}^{T}(s_t y_t + c_t X_t + h_t I_t) \tag{1}$$

s.t.

$$X_t \le M y_t \qquad t = 1, \ldots, T \tag{2}$$

$$X_1 - I_1 + LU_1 = d_1 \tag{3}$$

$$I_{t-1} + X_t - I_t + LU_t = E_t \qquad t = 2, \ldots, T \tag{4}$$

$$E_1 = d_1 \tag{5}$$

$$LU_t \le E_t \qquad t = 1, \ldots, T \tag{6}$$

$$LG_t + E_t = d_t \qquad t = 1, \ldots, T \tag{7}$$

$$E_t \le d_t - \beta LU_{t-1} + M(1-\delta_t) \qquad t = 2, \ldots, T \tag{8}$$

$$E_t \ge d_t - \beta LU_{t-1} - M(1-\delta_t) - (1-\varepsilon) \qquad t = 2, \ldots, T \tag{9}$$

$$E_t \le d_t \delta_t \qquad t = 2, \ldots, T \tag{10}$$

$$d_t \le \beta LU_{t-1} + M\delta_t \qquad t = 2, \ldots, T \tag{11}$$

$$d_t \ge \beta LU_{t-1} - M(1-\delta_t) \qquad t = 2, \ldots, T \tag{12}$$

$$X_t \ge 0,\ \ I_t \ge 0,\ \ E_t \ge 0 \qquad t = 1, \ldots, T \tag{13}$$

$$LG_t, LU_t \in \mathbb{Z}^+ \qquad t = 1, \ldots, T \tag{14}$$

$$y_t, \delta_t \in \{0, 1\} \qquad t = 1, \ldots, T \tag{15}$$

The profit function in (1) shows the difference between total realized revenue and total inventory cost. The first set of constraints (2) and binary property of y_t in (15) enforce a set-up cost with positive production in each period. Constraints (3)-(5) provide balance for inventory flow from the previous period (t–1) to the current period. Constraint (5) also implies there can be no goodwill loss in the first period. Constraints (6) ensure that lost sales in a period cannot exceed that period's effective demand. Constraints (7) establish the simple relationship between effective demand, original demand and goodwill loss in a period. Constraints (8)-(12) supported by the binary property of δ_t in (15) linearize the definitions of indicator variables δ_t, goodwill loss, and its impact on demand. In other words, they ensure that $E_t = max\left\{0,\, d_t - \left\lceil \beta \times LU_{t-1} \right\rceil\right\}\ \forall t \ge 2$. Loss variables LU_t and LG_t are restricted to nonnegative integer values in (14). This way, since demand occurs also in discrete units, we are allowed in (13) to declare X_t, I_t and E_t as nonnegative linear variables only. This *MIP* model has 3×T linear, 2×T integer, and (2×T− 1) binary decision variables used in (9×T−4) constraint equations. The T equality constraints in (7) and T linear variables representing effective demand values E_t are used for the purpose of clarity. They can actually be dropped from the model. Subsequently all E_t's in constraints from (4) through (10) need to be substituted by (d_t− LG_t). This would reduce the size of the proposed model in (1)-(15); however, since the number of binary and integer variables would remain the same, the model would not be substantially easier to solve with a general purpose *MIP* solver.

Ignoring the impact of goodwill loss in problem **PG** directly requires the goodwill rate β to be zero. In this case, **PG** transforms to the lost sales model **P**, because then the extra variables LG_t, E_t and δ_t all become redundant, and the variables LU_t in **PG** substitute for L_t in **P** while the integrality requirements on them drops out. We can bracket the optimal objective value of **PG** between a lower and upper bound. Let Π_G^*, Π_{WW}^* and Π^* denote optimal objectives of the problems **PG**, its associated WW problem and lost sales problem **P** without goodwill loss, respectively. Likewise, let σ_G^*, σ_{WW}^* and σ^* respectively denote the optimal production schedules (X_t^* values) of these three associated problems. Then, the following lemma and corollary hold true for Π_G^*.

Lemma 1. $\Pi_{WW}^* \le \Pi_G^* \le \Pi^*$.

***Proof**:* The production schedule σ_{WW}^* minimizes the total cost by satisfying all periods' demand, leaving no space for a shortage, thus preventing also goodwill losses. It is a feasible schedule for the associated lost goodwill problem **PG**. However, demand in the WW model is satisfied regardless of profitability, even when potential gross profit from sales is too low to meet the set-up cost. The profit obtained from a total cost minimizing objective cannot exceed the optimal value of a profit maximizing objective. On the other hand, in the no-goodwill-loss problem **P** the goodwill rate β equals 0%, which means that losing the effective demand to achieve a higher profitability does not cut off potential profits from any future demand. Hence, the profit obtained in the no-goodwill-loss problem **P** cannot be lower than that in the lost goodwill problem **PG**. □

Corollary 1. If production schedule σ^* is feasible for the associated problem **PG**, then $\Pi_G^* = \Pi^*$.

***Proof**:* The proof of Corollary 1 immediately follows from Lemma 1. □

5. A Search-and-Restoration heuristic for problem PG

We implement basically a neighborhood search method to find a high-quality heuristic solution to the lost goodwill problem **PG**. This method called search-and-restoration heuristic [SRH] exploits the optimal solution of the equivalent no-goodwill problem **P**. We first use the *DP* algorithm [AAC] described in [12] to obtain in $O(T^2)$ time an initial production schedule σ^* which is optimal for the corresponding problem **P**. If σ^* is feasible for **PG**, it will be also optimal for it due to Corollary 1. If not, then we scan σ^* for (blocks of) loss periods starting from period 1 through period *T*. During this inspection of σ^*, we try to restore the feasibility of balance of inventory flow constraints (4), and in conjunction with them the feasibility of constraints (6)-(12) which define the effective demand and goodwill loss values in **PG**. A number of restoration alternatives are checked for the minimum extra cost incurred. These restoration alternatives make the backbone of the [SRH] heuristic. Each of them can be viewed as a scenario of either satisfying or losing the effective demand in the last loss period analyzed. It is not impossible for the least cost scenario to modify the production schedule such that new loss periods arise ahead of the one currently analyzed. We focus on only one loss period at a time, which is either a singled-out period or the last in a block of loss periods. As we progress from the first towards the last loss period, we do not look back on the previous ones for which a least cost alternative has been already chosen. Therefore, our heuristic is a local search method as a whole.

During the local search with [SRH], we benefit from three lemmas proven for the no-goodwill problem **P** in Aksen *et al*. To fit the lemmas into the goodwill problem, terms d_t and L_t are updated with E_t and LU_t respectively. The first updated lemma ($LU_t^* X_t^* = 0 \ \ \forall t$) suggests that effective demand in a given period be fully satisfied if production is made in that period. The second updated lemma ($I_{t-1}^* X_t^* = 0 \ \ \forall t$) refers to the so-called *zero-inventory ordering policy* of the WW solution. It suggests that production be made in a given period only if the inventory at the beginning of that period equals zero. The third updated lemma ($LU_t^* (E_t^* - LU_t^*) = 0 \ \ \forall t$) dictates that partial shortage of effective demand be avoided. This implies that each period we lose either none or all of effective demand. Our algorithm [SRH] is developed upon these updated lemmas. Fine programming details and pseudo code of [SRH] are left out for the sake of brevity. A short verbal explanation of the cost analysis performed in each restoration is provided in sections 5.1 and 5.2. This is followed by a small example illustrating the effectiveness of [SRH] for the lost goodwill problem **PG**. Thorough description of the restoration alternatives with formal notations are left to Appendix B.

Some terms (*production*, *loss*, *stockout* and *conservation period*) that help elucidate the algorithm [SRH] are defined next. These are supplemented with three more definitions below. The major steps of [SRH] are outlined in Figure 1.

Definition 1a. A period *t* is a *production period* if $X_t > 0$.

Definition 1b. A period *t* is a *loss period* if $LU_t > 0$.

Definition 1c. A loss period *t* is also a *stockout period* if $I_t = 0$.

Definition 1d. A loss period *t* is also a *conservation period* if $I_t > 0$.

Definition 1e. A period *t* in the optimal production schedule σ^* of problem **P** is a *regeneration period* if the last production that occurred in or before *t* is exhausted in *t*. To be exact, period *t* is a regeneration period if $I_t = 0$ and $LU_t = 0$ while $E_t > 0$.

Definition 2a. A block of loss periods $\langle t_1,t_2\rangle$ is said to be *interior* if $t_1 > 1$ and $t_2 < T$.

Definition 2b. A block of loss periods $\langle t_1,t_2\rangle$ is said to *span k periods* if $(t_2 - t_1) = k$.

Search-and-Restoration Heuristic [SRH] for problem **PG**

<u>*Initialization*</u>*:*

Solve the equivalent no-goodwill problem **P** with the *DP* algorithm **[**AAC**]**. An optimum of **P** is the production schedule σ^* with production amounts X_t^*.

<u>*Step 1*</u>*:* Set the counter $t = 1$.

<u>*Step 2*</u>*:* Starting from period t onward, look for the first block of loss period(s) $\langle t_1,t_2\rangle$ where $t \le t_1 \le t_2$. Compute the current effective demand values $E_{t_1+1}, E_{t_1+1}, \ldots, E_{t_2}$ within this block. If the last loss period t_2 coincides with the final period T $(t_2 = T)$ then stop.

<u>*Step 3*</u>*:* If E_{t_2} is calculated as zero, then advance the counter t to (t_2+2) and go to step 2.

<u>*Step 4*</u>*:* Depending on the location (interior or at the very beginning of the planning horizon?), size (spans at least 2 periods or only 1 period?), and type of loss periods (stockout or conservation periods?), examine appropriate alternatives of satisfying E_{t_2} (such that $LG_{t_2+1} = 0$) or losing E_{t_2} (such that $LG_{t_2+1} > 0$).

<u>*Step 5*</u>*:* Choose the one that charges the least additional cost on the maximum profit Π^* of problem **P**. Decrease Π^* by that least cost, accommodate the production amounts X_t^*, effective demand E_t and total loss values ($LG_t + LU_t$) as determined in this least cost alternative.

<u>*Step 6*</u>*:* Advance the counter t beyond the last analyzed loss period t_2 for as many periods as required by the least cost alternative chosen in step 5. Repeat steps 2-6 until all loss periods have been analyzed.

Figure 1 Major steps of [SRH]

In the following restoration alternatives, $\langle t_1,t_2\rangle$ denotes a block of one or more loss periods. The last production period preceding $\langle t_1,t_2\rangle$ is u. If $\langle t_1,t_2\rangle$ consists of conservation periods, then q is the first regeneration period coming after this block. If $\langle t_1,t_2\rangle$ consists of stockout periods, then (t_2+1) is a production period by definition, and j denotes the regeneration period of this production, while $\langle j+1, q\rangle$ represents a possible second block of stockout periods followed by a third production in period (q+1). Having $j = q$ implies that there exists no stockout period between the consecutive production periods (t_2+1) and (q+1). Figure 2 and Figure 3 below help better visualize conservation and stockout events together with the relevant regeneration, production and loss periods as well as effective demands and shortages. On the time axis in the figures are also ending inventories indicated. Solid, dashed, and dotted arrows show demands, production activities, and shortages, respectively.

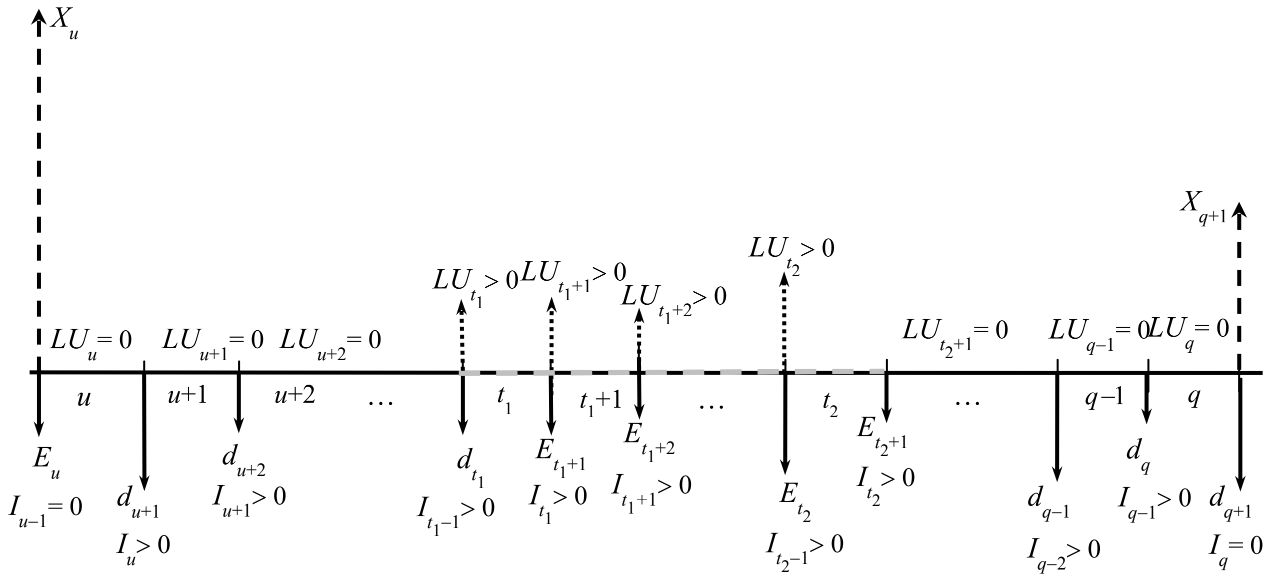

Figure 2 A block of conservation periods $\langle t_1,t_2\rangle$

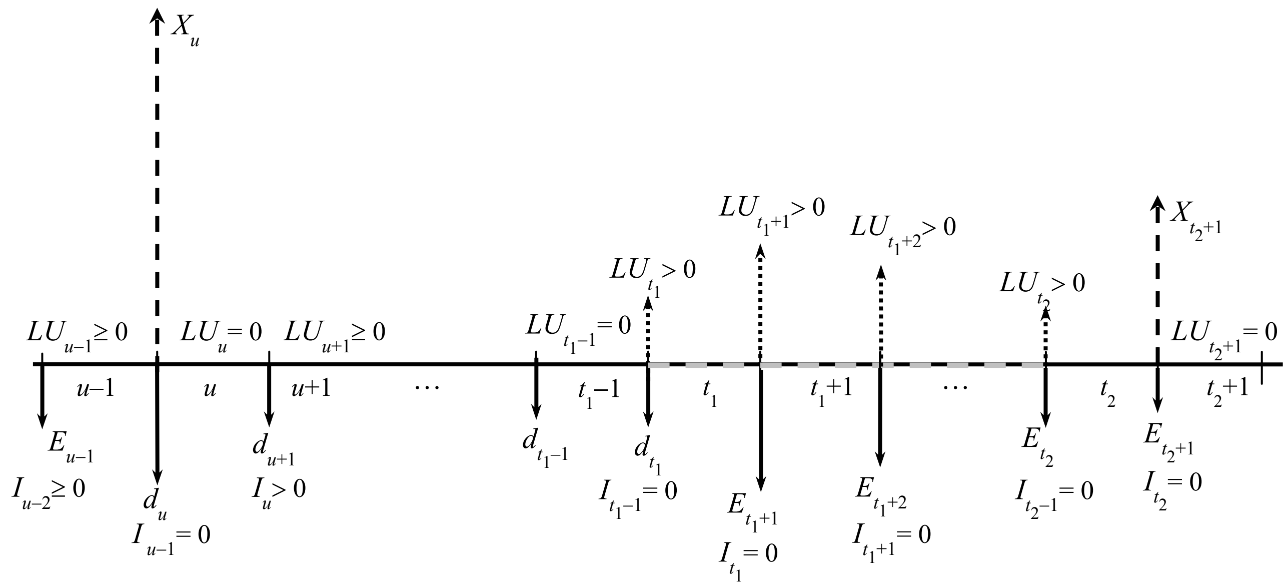

Figure 3 An interior block of stockout periods $\langle t_1,t_2\rangle$

5.1. Alternatives of restoration for conservation periods $\langle t_1,t_2\rangle$

Alt.1: Endure the goodwill loss in period (t_2+1). Decrease the production in period u to accommodate this loss in the demand of (t_2+1).

Alt.2: Avoid the shortage in period t_2, this way the goodwill loss in (t_2+1), by increasing the production in period u accordingly.

Alt.3: Make a new production in t_2 to avoid the shortage in this period. This will cover the effective demand of t_2 and demands of the succeeding periods $\langle t_2+1,q\rangle$ unless there is one or more demands to be conserved among them. Adjust the production level in period u accordingly.

Alt.4: Repeat Alt.3 for some period before t_2 and after u. Attention should be paid again to the potential conservation periods of the new production.

Alt.5: Shift the production in $(q+1)$ to an earlier period strictly between u and (t_2+1) such that the shortage in t_2, thus the goodwill loss in (t_2+1) is prevented. Adjust then the original production in u.

5.2. Alternatives of restoration for stockout periods $\langle t_1,t_2\rangle$

Alt.6: Endure the goodwill loss in period (t_2+1). Decrease the original production in period (t_2+1) by as much as this goodwill loss.

Alt.7: Endure the goodwill loss in period (t_2+1). Then, either cancel the entire production in that period, or shift and adjust it to a later period that comes before the next production period $(q+1)$.

Alt.8: In case of interior stockout period(s), cancel the production in (t_2+1) and endure the goodwill loss in this period. Its remaining effective demand is to be covered by the previous production in u. Provided that there is no production activity in period (t_2+2), recheck Alt.7 to see if the cancelled production of (t_2+1) should be shifted to a later period.

Alt.9: Avoid the shortage in period t_2, this way the goodwill loss in (t_2+1), by increasing the production in period u accordingly.

Alt.10: Make an extra intermediate production in period t_2 in the amount of this period's effective demand in order to restore the goodwill loss in (t_2+1).

Alt.11: If the block $\langle t_1,t_2\rangle$ is comprised of three or more stockout periods, then consider either only one or two new production activities inside this block. These new production activities have to cover the effective demand of period t_2 so as to avoid the goodwill loss in (t_2+1).

Alt. 12-14: In order to meet the effective demand in period t_2 shift the original production in period (t_2+1) backward either to the previous period t_2 (**Alt.12**), or to some least cost period in $\langle t_1,t_2-1\rangle$ (**Alt.13**), or further back to some least cost period in $\langle u+1,t_1-1\rangle$ (**Alt.14**). For **Alt.13** there must be at least three stockout periods, while **Alt.14** is applicable to interior stockout periods only. In the latter alternative, the production quantity in period u must be recalculated by observing the conservation periods ahead of u.

Alt.15: First, check if stockout periods appear right at the start of the planning horizon, and if the original production in period (t_2+1) of the no-goodwill schedule covers the demands of at least two periods. If both yes, then replace that production with two production activities. The first of these will be made in some stockout period in $\langle 1,t_2\rangle$ while the second will be made (t_2+1) periods after the first one.

Alt.16: First, check if $\langle t_1,t_2\rangle$ is an interior block of stockout periods, and if the preceding production activity in u covers at least two periods. If both yes, then discard the demand of period (t_1-1) from that production in u and lose it. Calculate new effective demands for $\langle t_1,t_2\rangle$, and make an exclusive production in t_2 so as to meet the effective demand in that period and avoid the goodwill loss in the next period.

Alt.17: Same as **Alt.11** except for the period of the first new production activity fixed as t_1.

Alt.18: Begin with the same initial checks of **Alt. 16**. If $\langle t_1,t_2\rangle$ pass these checks, then shift it backward on the time axis by one period. As a result of this shift the production in u will contract by the demand of period (t_1-1) and new stockout periods will be $\langle t_1-1,t_2-1\rangle$. Depending on whether production is made in period (t_2+2), there are two subcases to be analyzed both of which are carefully explained in Appendix B.

Searching and restoring a block of loss periods in each of these 18 alternatives takes at most $O(T^2)$ time. Thus, once algorithm [AAC] finds the optimal no-goodwill production schedule σ^* in $O(T^2)$ time, the entire cost analysis and restoration of loss blocks can be made in another $O(T^2)$ time. As a result, the local search method [SRH] has a quadratic order of time complexity overall. As explained in section 4 before, any instance of the no-goodwill problem **P** reduces to a special instance of the goodwill problem **PG** with the goodwill rate β simply set to zero, i.e. we can write **P** $\propto_P$ **PG** where $\propto_P$ indicates reduction in polynomial time. For time-varying data **P** is solvable in no less than $O(T^2)$ time. Then, even if there exists a polynomial time algorithm that solves **PG** optimally, it cannot have a time complexity less than $O(T^2)$. Therefore, our proposed method [SRH] with its quadratic running time has merit.

5.3. An illustrative example

We demonstrate the algorithm [SRH] on an 8-period test problem called AKSEN2. Problem parameters and an optimal production schedule σ^* ignoring the impact of goodwill loss are given in Table 2 and Table 3, respectively. There is an interior block of four stockout periods $\langle 4,\ 7\rangle$ in σ^*. The impact of goodwill loss with the rate β = 50% supplements σ^* with two new production activities within the loss block. The new production plan results in one less stockout period and approx. 7.5% decrease in profit. Among 12 alternatives that are eligible for inspection in this sample problem, it is *Alternative 17*, which achieves the least fall in the profit of the no-goodwill problem. We verify the optimality of the new plan by solving the model of the corresponding problem **PG** with the *MIP* solver CPLEX 8.1. We give the profit values also for two suboptimal cases. One of these is the WW solution of the problem. Since all demand is met in this solution, there is no need to consider the lost goodwill cost. However, the profit obtained from the WW solution is \$44 less than the profit we find with [SRH]. The second case, which is the [AAC] solution exposed to 50% lost goodwill impact, does not promise a better profit, either. As can be seen in Table 3, goodwill impact deteriorates the original profit of [AAC] solution by 14.75% (from \$1017 to \$867).

Table 2 Cost, unit revenue and demand data for AKSEN2

Period t	s_t	c_t	h_t	p_t	Demand d_t
1	\$ 100	\$ 10	\$ 5	\$ 30	9
2	100	10	5	22	12
3	100	10	5	42	9
4	100	10	5	14	25
5	100	10	5	15	9
6	100	10	5	12	20
7	100	10	5	12	20
8	100	10	5	40	25

Table 3 [AAC] without goodwill loss, [AAC] with goodwill loss, WW, and [SRH] solutions for AKSEN2

σ^*	t	I^*_{t-1}	X^*_t	d_t	LG^*_t	E^*_t	LU^*_t	Total Loss in t	I^*_t	Revenue in t	Costs in t	Profit in t
[AAC] without goodwill loss	1	0	9	9	–	9	0	0	0	\$ 270	\$ 190	\$ 80
	2	0	21	12	–	12	0	0	9	264	355	(91)
	3	9	0	9	–	9	0	0	0	378	0	378
	4	0	0	25	–	25	25	25	0	0	0	0
	5	0	0	9	–	9	9	9	0	0	0	0
	6	0	0	20	–	20	20	20	0	0	0	0
	7	0	0	20	–	20	20	20	0	0	0	0
	8	0	25	25	–	25	0	0	0	1000	350	650
											Total Profit:	**\$ 1017**
[AAC] with goodwill loss	1	0	9	9	0	9	0	0	0	\$ 270	\$ 190	\$ 80
	2	0	21	12	0	12	0	0	9	264	355	(91)
	3	9	0	9	0	9	0	0	0	378	0	378
	4	0	0	25	0	25	25	25	0	0	0	0
	5	0	0	9	9	0	0	9	0	0	0	0
	6	0	0	20	0	20	20	20	0	0	0	0
	7	0	0	20	10	10	10	20	0	0	0	0
	8	0	20	25	5	20	0	5	0	800	300	500
											Total Profit:	**\$ 867**
WW	1	0	9	9	0	9	0	0	0	\$ 270	\$ 190	\$ 80
	2	0	21	12	0	12	0	0	9	264	355	(91)
	3	9	0	9	0	9	0	0	0	378	0	378
	4	0	34	25	0	25	0	0	9	350	485	(135)
	5	9	0	9	0	9	0	0	0	135	0	135
	6	0	40	20	0	20	0	0	20	240	600	(360)
	7	20	0	20	0	20	0	0	0	240	0	240
	8	0	25	25	0	25	0	0	0	1000	350	650
											Total Profit:	**\$ 897**
[SRH]	1	0	9	9	0	9	0	0	0	\$ 270	\$ 190	\$ 80
	2	0	21	12	0	12	0	0	9	264	355	(91)
	3	9	0	9	0	9	0	0	0	378	0	378
	4	0	25	25	0	25	0	0	0	350	350	0
	5	0	0	9	0	9	9	9	0	0	0	0
	6	0	0	20	5	15	15	20	0	0	0	0
	7	0	12	20	8	12	0	8	0	144	220	(76)
	8	0	25	25	0	25	0	0	0	1000	350	650
											Total Profit:	**\$ 941**

6. Computational experiments and benchmarking

Algorithm [SRH] has been coded in C, compiled in Microsoft Visual C++ 6.0®, and run on a desktop computer with 1.7 GHz Pentium 4 processor and 512 MB RAM. The solution quality and time of [SRH] has been examined first on Aksen *et al.*'s 22 test problems. In all problems, a uniform impact rate β = 50% is assumed for goodwill loss. In addition, a GAMS [25] model has been created for each test problem, and solved on the same platform to optimality where possible using the *MIP* solver CPLEX 8.1. Together with the CPU time, each test problem's optimal or best feasible profit value found by CPLEX serves as a benchmark for [SRH]. In *MIP* models of GAMS, solution accuracy obtained from the employed solver is mainly controlled by upper limits on the number of iterations (`ITERLIM`) and on the resource usage (`RESLIM`) in terms of CPU seconds. A relative optimality criterion (`OPTCR`) can be set for the *MIP* master problem determining when the solver should terminate its branch and bound or cut procedure. `OPTCR` is defined as the ratio (|BP–BF|) / (1.0e-10 + |BF|) where BF is the objective function value of the current best integer solution while BP is the best known (current) upper bound in case of maximization. The solver stops trying to improve upon the integer solution BF when this ratio drops below the specified value [26].

During the first phase of experimentation the triplets of accuracy parameters `ITERLIM`, `RESLIM` and `OPTCR` in GAMS models are (750000 / 18000 / 0.01%) and (7750000 / 21600 / 0.1%) for test problems with $T \leq 200$ and for those with $T \geq 240$, respectively. Table 4 shows the results of experiments with the 22 test problems. On this test bed [SRH] performs apparently well. While CPLEX 8.1 runs out of memory in three large-size problems, [SRH] can solve all of them under 0.1 sec. Only in problem AKSEN1, the optimal profit of the solver is slightly better than the outcome of [SRH].

In the second phase of experimentation, we test [SRH] on a larger set of randomly generated problems, and again benchmark with the *MIP* solver CPLEX 8.1 of GAMS suit 21.1. We use the accuracy parameters (750000 / 18000 / 0.01%) in most of the GAMS models in this phase. However, in 19 problems of T = 150 periods, the solver runs out of memory before the specified limit on resource usage is reached. Therefore, in the GAMS models of those problems, we raise the relative optimality criterion to 1% and limit the resource usage to 3¼ to 4.0 CPU hours.

The randomized scheme of test problem generation is given in Table 5. Variation in five problem parameters for five demand patterns (using five different initial seeds) as shown in the table leads to 360 random problems. In generating demand values, we sample from three probability distributions. In two of these, namely normal and exponential distributions, we round random variables to the nearest integer to make sure that there is no fractional demand. In the case of discrete uniform distribution, there is obviously no rounding. Different values of unit costs and prices due to supposed irregular seasonality in the problem data are presented in Table 6 and Table 7. Gross marginal profits ($p_t - c_t$) are relatively small, making loss of demand more appealing. The ratio of unit holding cost h_t to unit production cost c_t varies between 3.85% and 15%. This is an acceptable range in most real-life lot-sizing problems, because holding cost is usually treated as the capital cost of production.

Table 4 Performance of [SRH] in the first phase of experimentation

Prob. Name	*T*	Profit of [SRH]	CPU of [SRH] (s)	CPU of CPLEX 8.1 (s)	Opt. Profit of CPLEX 8.1
AKSEN1	8	82.00	0.000	0.078	84.00
AKSEN2	8	941.00	0.000	0.046	941.00
BRENNAN	10	355.85	0.000	0.078	356.66
MAES6	12	88,604.30	0.000	0.093	88,604.30
RD7-ALL	12	58,550.00	0.000	0.046	58,550.00
AKSEN3	16	1,184.00	0.000	0.234	1,184.00
RD10-ALL	16	725.00	0.000	0.125	725.00
RD2-ALL	18	570.00	0.000	0.109	570.00
RD4-ALL	24	467.00	0.000	1.250	467.00
RD6-ALL	24	1,617.00	0.000	0.156	1,617.00
RD8-ALL	30	5,369.00	0.000	0.109	5,369.00
RD3-ALL	36	1,665.00	0.000	0.156	1,665.00
RD5-ALL	48	6,748.00	0.000	2.609	6,748.00
RD01-ALL	48	310,749.00	0.000	0.109	310,749.00
RD9-ALL	72	25,090.00	0.000	3.062	25,090.00
RD100	100	782,149.00	0.000	0.234	782,150.97
RD144	144	46,243.00	0.050	698.453	46,243.00
RD200	200	1,564,298.00	0.000	1.171	1,564,298.00
RD240	240	93,488.00	0.000	13,749.812	OUT OF MEMORY
RD312	312	118,578.00	0.050	12,876.453	OUT OF MEMORY
RD350	350	109,105.46	0.060	16,472.921	OUT OF MEMORY
RD400	400	3,128,596.00	0.050	1.734	3,128,596.00

Table 5 Randomized generation scheme for 360 test problems

Parameter	Explanation	No. values	Values
T	Length of the planning horizon	6	24, 36, 60, 96, 120, 150
$pdf\ of\ d_t$	Probability distribution function of demand	3	• *Normal* with $\mu = 150$ and $\sigma^2 = 1600$ • *Discrete Uniform*[30, 270, 10] • *Exponential* with $\mu = 150$
c_t and h_t	Unit production and holding costs	2	• Both constant: $c_t = 13.0$ and $h_t = 1.0$ • Both seasonally varying.
p_t	Unit selling prices	2	• Discrete uniformly distributed between 15.0 and 25.0 in increments of 5.0. • Seasonally varying.
s_t	Set-up costs	1	Constant: $s_t = 1000.0$
SEED	Initial random number seed	5	622, 1371, 2003, 99, 1971

Table 6 Values for unit cost and unit price parameters with seasonality

c_t with seasonality:	c_{I}	c_{II}	c_{III}	c_{IV}	c_{V}	c_{VI}	c_{VII}	c_{VIII}	c_{IX}	c_{X}
Values:	12	13	13	11	13	13	10	11	12	13
h_t with seasonality:	h_{I}	h_{II}	h_{III}	h_{IV}	h_{V}	h_{VI}				
Values:	0.5	1.0	1.5	1.5	1.0	0.5				
p_t with seasonality:	p_{I}	p_{II}	p_{III}	p_{IV}	p_{V}	p_{VI}	p_{VII}	p_{VIII}	p_{IX}	p_{X}
Values:	18	25	16	19	22	20	15	15	18	20

6.1. Results

Comparative results obtained from 360 test problems are summarized in Table 8. The first half of the table shows the average best profit values of WW, CPLEX and [SRH] solutions for each subset of 60 problems. Average CPU times for CPLEX and for [SRH] are displayed in the last two columns of the first half. These results validate the solution time superiority of [SRH] over commercial solvers. They also confirm that [SRH] can find close-to-optimal production schedules for the lost goodwill problem **PG**, assuring a higher profit value than the corresponding WW solution. The second half of Table 8 shows relative discrepancies between the best profit values of WW, CPLEX and [SRH] solutions. Cumulative results of 360 test problems indicate that [SRH] achieves an average improvement of 2.41% and maximum improvement of 14.32% over the WW approach. On the other hand, CPLEX can generate a higher profit than [SRH] in approx. 60% of test problems (220 out of 360). Yet, profits found by [SRH] fall short of the objective values of the optimal (or best feasible integer) CPLEX solutions by only 0.18% on the average. The biggest discrepancy between a [SRH] and associated CPLEX profit is measured as 3.27%. Our experiments verify that [SRH] is not an exhaustive but a local search algorithm; hence, it may not always yield the true optimal solution of a given instance of the lost goodwill problem **PG**.

An interesting but not too much surprising empirical finding of our extensive GAMS runs is that solution times of CPLEX 8.1 decrease dramatically as gross marginal profits ($p_t - c_t$) grow. Preserving p_t's, when we drop unit production costs by 3 units, i.e. when we increase the average gross marginal profit by 3 units, then the maximum CPU time even for 150-period problems reduces to 72 seconds. A reasonable justification for this high sensitivity of commercial *MIP* solvers to gross marginal profits in problem **PG** is as follows. Endogenous loss of demand (shortage) becomes less profitable as gross marginal profits tend upwards. Thus, stockouts and conservation of demand arise less; consequently, the lost goodwill problem to be solved does not differ much from a classical WW model without lost goodwill cost.

Table 7 Range of periods in which values of seasonality are applicable

c_t with seasonality:	c_{I}	c_{II}	c_{III}	c_{IV}	c_{V}	c_{VI}	c_{VII}	c_{VIII}	c_{IX}	c_{X}
$T = 24$	1–6	7–12	13–18	19–24						
$T = 36$	1–6	7–12	13–18	19–24	25–30	31-36				
$T = 60$	1–10	11–20	21–30	31–40	41–50	51–60				
$T = 96$	1–12	13–24	25–36	37–48	49–60	61–72	73–84	85–96		
$T = 120$	1–12	13–24	25–36	37–48	49–60	61–72	73–84	85–96	97–108	109-120
$T = 150$	1–15	16–30	31–45	46–60	61–75	76–90	91–105	106–120	121–135	136–150
h_t with seasonality:	h_{I}	h_{II}	h_{III}	h_{IV}	h_{V}	h_{VI}				
$T = 24$	1–4	5–8	9–12	13–16	17–20	21–24				
$T = 36$	1–6	7–12	13–18	19–24	25–30	31–36				
$T = 60$	1–6, 37–42	7–12, 43–48	13–18, 49–54	19–24, 55–60	25–30	31–36				
$T = 96$	1–6, 37–42, 73–78	7–12, 43–48, 79–84	13–18, 49–54, 85–90	19–24, 55–60, 91–96	25–30, 61–66	31–36, 67–72				
$T = 120$	1–12, 73–84	13–24, 85–96	25–36, 97–108	37–48, 109–120	49–60	61–72				
$T = 150$	1–15, 91–105	16–30, 106–120	31–45, 121–135	46–60, 136–150	61–75	76–90				
p_t with seasonality:	p_{I}	p_{II}	p_{III}	p_{IV}	p_{V}	p_{VI}	p_{VII}	p_{VIII}	p_{IX}	p_{X}
$T = 24$	1–6	7–12	13–18	19–24						
$T = 36$	1–6, 25–30	7–12, 31–36	13–18	19–24						
$T = 60$	1–10	11–20	21–30	31–40	41–50	51–60				
$T = 96$	1–12	13–24	25–36	37–48	49–60	61–72	73–84	85–96		
$T = 120$	1–12	13–24	25–36	37–48	49–60	61–72	73–84	85–96	97–108	109-120
$T = 150$	1–15	16–30	31–45	46–60	61–75	76–90	91–105	106–120	121–135	136–150

Table 8 Performance comparisons between WW, GAMS/CPLEX and [SRH] in 360 test problems

T	N_T	Avg.$\left(\Pi^*_{WW}\right)$	Avg.$\left(\Pi^*_{G}\right)^{CPLEX}$	Avg.$\left(\Pi_{G}\right)^{[SRH]}$	Avg. CPLEX CPU time (s)	Avg. [SRH] CPU time (s)
24	60	13752.53	14195.69	14158.83	0.345	0.000
36	60	22423.97	22885.25	22842.96	0.353	0.000
60	60	40294.72	40879.51	40803.53	3.488	0.000
96	60	57747.07	59094.03	58981.18	208.843	0.002
120	60	71817.62	73323.76	73211.39	2686.485	0.003
150	60	89969.00	91758.99	91715.87	8393.388	0.004
T	N_T	Avg. %(WW-SRH)	Max. %(WW-SRH)	Avg. %(CPLEX-SRH)	Max. %(CPLEX -SRH)	$N_{CPLEX > \text{SRH}}$
24	60	3.55	14.32	0.30	3.27	24
36	60	2.10	9.36	0.19	3.13	28
60	60	1.41	6.36	0.20	1.63	38
96	60	2.71	11.01	0.19	1.27	43
120	60	2.34	7.95	0.15	0.87	48
150	60	2.36	8.04	0.05	0.90	39
Cum. Res.	**360**	**2.41**	**14.32**	**0.18**	**3.27**	**220**

7. Summary and conclusions

This paper dwells upon the customer goodwill loss due to unsatisfied demand in lot-sizing problems. Previous research treats the goodwill loss by including it directly in the objective function as an additional cost component. The traditional approach is to intensify the gross profit loss due to stockouts with an extra unit cost which is multiplied by the volume of unsatisfied demand. The resulting amount is usually assumed to reflect both the actual loss of profit and the eventual cost of losing customer goodwill. Our paper brings about a first-time change to the treatment of goodwill loss in dynamic lot-sizing. We propose an uncapacitated single-item model that allows lost sales to maximize profit in a production environment with fluctuating costs and prices. The impact of goodwill loss due to unsatisfied demand in a given period is felt as a decline in the realized (effective) demand of the succeeding period. This decline occurs only in one period ahead, and is proportional to the quantity of unsatisfied demand. The deterministic nature of goodwill impact in our model is supported by observations of Wee [11] who says "*in real world applications, it is not unrealistic to assume that the retailers have some knowledge on buyers' behaviors such as their responses to shortages and price increase.*"

The impact of customer goodwill loss has been imbedded in the lost sales version of the well-known WW model. This yields a *MIP* formulation of our proposed model. We differentiate between endogenous and exogenous loss of demand, the latter of which represents the goodwill loss as an uncontrollable consequence of the former. The solution time of the new model with the benchmark *MIP* solver CPLEX 8.1 is noticeably sensitive to the average gross marginal profit in the problem. Lower gross marginal profits delay the convergence to a proven optimal solution. As an alternative to the *MIP* optimization, we present a quadratic-time two-stage neighborhood search and restoration heuristic called [SRH]. It first uses the *DP* algorithm [AAC] to generate an optimal production schedule to the problem without goodwill impact. An extensive local search is then applied to the (blocks of) loss periods in this schedule, and feasibility is restored at the minimum possible cost. On 22 problems reported in [12] as well as on another test bed of 360 randomly generated problems, the solution quality of [SRH] is competitive. Moreover, [SRH] requires significantly less CPU time than the *MIP* solver. In comparison to WW solutions which do not allow lost sales, average [SRH] profits on the same test bed are by 2.41% higher.

The immediate future extension of our study will be an in-depth computational analysis of the first extreme version of customer goodwill representation and its comparison to the second specific version which we have thoroughly studied in this paper. The continuation on customer goodwill loss has to look into such generic perspectives as capacity constraints, backordering, quantity discounts and set-up times. They could be incrementally incorporated into the goodwill loss model, which would certainly increase the problem complexity. Alternatively, the challenge for a polynomial time algorithm to solve the lost goodwill problem optimally might also be undertaken by future researchers.

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Appendix A. A unified representation of customer goodwill loss

Version-A and Version-B of customer goodwill loss representation proposed in section 3 can be merged into a unified representation that would converge to either of them at the limits. For this, in addition to the notations and symbols introduced in section 4, let coefficients β_{A} and β_{B} denote the given ratios of goodwill loss in Version-A and -B, respectively. Also let α be the ratio of original demand that comes from customers of Version-A. This refers to those customers of the company who make up their mind for a purchase in period t by looking at the company's success of satisfying overall demand in (t–1). The rest of the demand is then due to repeat customers whose goodwill loss is defined as in Version-B. This decomposition of overall demand d_t is equally projected onto the unsatisfied demand (shortage) LU_t and goodwill loss LG_t. The relationships defining the goodwill loss can be rewritten according to the demand decomposition as follows.

$$LG_t \text{ due to customers of Version-A} = \beta_A \frac{d_t}{d_{t-1}} \alpha\, LU_{t-1} \qquad t = 2, \ldots, T$$

$$LG_t \text{ due to customers of Version-B} = \min\{d_t, \beta_B (1-\alpha)\, LU_{t-1}\} \qquad t = 2, \ldots, T$$

Since $d_{t-1} \geq LU_{t-1}$ always holds true, the expression $min\{\cdot\}$ in the second relationship can be dropped out if a monotonically increasing demand stream ($d_t \geq d_{t-1}$) is assumed for simplicity. Once the relationships are added side by side, the total goodwill loss in period t is obtained.

$$Total\ LG_t \ = \ \left[\beta_A \frac{d_t}{d_{t-1}}\ \alpha \ + \ \beta_B(1-\alpha)\right] LU_{t-1} \qquad t = 2, \ldots, T$$

The coefficient preceding LU_{t-1} in the above expression can be treated as a new, period dependent rate of customer goodwill loss. This leads to the following unified goodwill loss expression.

$$LG_t = \hat{\beta}_t \ LU_{t-1} \qquad t = 2, \ldots, T \tag{16}$$

The only difference between this unified goodwill loss expression and the one in Version-B is the time dependency of goodwill loss rate. Time dependency of $\hat{\beta}$ would not affect the general algorithmic procedure in [SRH], the search-and-restoration heuristic. [SRH] can be tailored to solve the lot-sizing problem with profit maximization and the unified customer goodwill loss representation given in (16). In the limits, this is, for the extreme values 1 and 0 of the ratio α, (16) converges to Version-A and Version-B, respectively. One last remark should be made about the necessary modifications to the mathematical model of problem **PG** in the limit α = 1, i.e. when Version-A is preferred instead of Version-B as goodwill loss representation. In order to adapt the model of **PG** to Version-A, binary variables δ_t have to be discarded, and equations (8)-(12) have to be replaced by three new constraints provided below. The third constraint in (19) only ensures that the goodwill loss in period t be zero if there was no demand, hence there was also no shortage in the previous period. The first two constraints enforce LG_t to attain an integer value, which equals the ceiling of $\left(\beta \frac{d_t}{d_{t-1}}\ LU_{t-1}\right)$.

$$LG_t \ \le \ \beta \frac{d_t}{d_{t-1}}\ LU_{t-1} \ + \ (1-\varepsilon) \qquad \forall t = 2, \ldots, T \quad \ni \ d_{t-1} > 0 \tag{17}$$

$$LG_t \ \ge \ \beta \frac{d_t}{d_{t-1}}\ LU_{t-1} \qquad \forall t = 2, \ldots, T \quad \ni \ d_{t-1} > 0 \tag{18}$$

$$LG_t \ \le \ d_{t-1} d_t \qquad \forall t = 2, \ldots, T \tag{19}$$

Appendix B. Formal description of 18 restoration alternatives in [SRH]

The following restoration alternatives (five in B1 and thirteen in B2) are applicable for all kinds of conservation and stockout periods, respectively, unless specific conditions of applicability are explicitly stated next to the numbers of alternatives. Notations and symbols used therein are the same as those given in section 5 and depicted in Figure 2 and Figure 3.

B1. Restoring conservation periods $\langle t_1, t_2 \rangle$

<u>Alternative 1</u>

Lose E_{t_2} such that $LG_{t_2+1} = min\left\{d_{t_2+1}\ ,\ \lceil \beta \times E_{t_2} \rceil\right\}$ is deducted from the original demand d_{t_2+1} to yield E_{t_2+1}. Also X_u^* drops then by LG_{t_2+1}.

Alternative 2

Produce E_{t_2} in period u, and carry it from u through t_2 to avoid a shortage in t_2, this way to avoid the goodwill loss in period (t_2+1). X_u^* increases then by E_{t_2}.

Alternative 3

Make a new production in t_2. Include in $X_{t_2}^{new}$ effective demand E_{t_2} and demand of periods (t_2+1) through q which were originally covered by the production X_u^*. If there is a period r between t_2 and q whose demand was not covered in X_u^*, then include d_r in $X_{t_2}^{new}$ only if r is not a conservation period with respect to t_2. Deduct those demand values from X_u^* which will be included now in $X_{t_2}^{new}$.

Alternative 4

Split the original production X_u^* into two parts at some least expensive period $k^* \in \langle u+1,\ t_2-1\rangle$. In this period k^*, produce the dynamically determined value of ($E_k + E_{k+1} + \ldots + E_{t_2-1}$) in addition to E_{t_2}. Reduced X_u^* and new effective demand and total loss values need to be computed by taking into account the conservation periods of production in u as well as in k^*.

Alternative 5: *Valid only if* $q < T$

If it is nonzero, then shift the original production X_{q+1}^* that comes right after the regeneration period q to an earlier period $k \in \langle u+1,\ t_2\rangle$ such that effective demand in t_2 is also met. Compute new effective demand and total loss values by looking out for the conservation periods of production in u as well as in k^*.

B2. Restoring stockout periods $\langle t_1,t_2\rangle$

Alternative 6

Lose E_{t_2} such that $LG_{t_2+1} = min\left\{d_{t_2+1}\ ,\ \left\lceil \beta \times E_{t_2} \right\rceil\right\}$ and $E_{t_2+1} = d_{t_2+1} - LG_{t_2+1}$. Drop also $X_{t_2+1}^*$ by LG_{t_2+1}.

Alternative 7

Lose E_{t_2} and let $E_{t_2+1} = d_{t_2+1} - LG_{t_2+1}$. Either cancel the entire production $X_{t_2+1}^*$, which means to lose all demand met from (t_2+1) through q, or shift it to a later period $k^* \in \langle t_2+2,\ q\rangle$ such that it covers less demand than it does at its current period (t_2+1). Adjust effective demand and total loss values accordingly by looking out for the conservation periods of production in (t_2+1) as well as in k^* if $X_{t_2+1}^*$ is shifted to k^*.

Alternative 8: *Applies to interior stockout periods only.*

The entire production $X_{t_2+1}^*$ is cancelled, and E_{t_2} is lost. Hence we will have $E_{t_2+1} = d_{t_2+1} - LG_{t_2+1}$. The cost analysis is done in three subcases as shown below. In each subcase, effective demand and total loss values need to be adjusted in accordance with the respective changes on the current production schedule.

Subcase 1. $X^*_{t_2+2} > 0$

Just increase X^*_u by E_{t_2+1}.

Subcase 2. $X^*_{t_2+2} = 0$ and $j = q$

Increase X^*_u by E_{t_2+1}. Shift the cancelled production $X^*_{t_2+1}$ to period (t_2+2).

Subcase 3. $X^*_{t_2+2} = 0$ and $j < q$

Again, increase X^*_u by E_{t_2+1}. Make a new production in period (t_2+2). This new production $X^*_{t_2+2}$ will cover all or some of the demand values of the interval $\langle t_2+2, q\rangle$ depending on the conservation periods of (t_2+2) in that interval.

Alternative 9

Produce E_{t_2} in period u and carry it from u through t_2 to avoid a shortage in t_2, this way to avoid the goodwill loss in period (t_2+1). X^*_u increases then by E_{t_2}.

Alternative 10

Produce E_{t_2} in period t_2 and do not carry it in inventory. We will have $X^{new}_{t_2} = E_{t_2}$.

Alternative 11: *Valid only if the block $\langle t_1,t_2\rangle$ spans 2 or more periods.*

Choose first a least expensive extra production period $k^*_1 \in \langle t_1, t_2-1\rangle$ to meet the effective demand in t_2 and to avoid goodwill loss in (t_2+1). This new extra production $X^{new}_{k^*_1}$ will meet $E^{new}_{k^*_1}$, $E^{new}_{t_2}$ and new values of effective demand in those periods between (k^*_1+1) and (t_2–1) which are not a conservation period for $X^{new}_{k^*_1}$. Next, investigate the profitability of dividing $X^{new}_{k^*_1}$ into two parts, one part still produced in k^*_1 and the rest in a later period k^*_2 with $t_1 \le k^*_1 < k^*_2 \le t_2$. If it turns out less expensive than making only one extra production, then modify $X^{new}_{k^*_1}$ such that its coverage is limited to periods $\langle k^*_1, k^*_2-1\rangle$. Produce $X^{new}_{k^*_2}$ to meet $E^{new}_{k^*_2}$ and $E^{new}_{t_2}$ alongside with several or all new effective demand values of periods between (k^*_2+1) and (t_2–1). After we determine the least expensive extra production period(s) k^*_1 (and k^*_2) we need to adjust also total loss values in $\langle t_1,t_2\rangle$. This alternative might incur the least cost among others if in particular set-up costs s_t in **PG** are either negligible or of a magnitude comparable with unit production and holding costs.

Alternative 12:

Produce ($E_{t_2} + X^*_{t_2+1}$) in period t_2. Carry $X^*_{t_2+1}$ in inventory during t_2 and do not produce in (t_2+1).

Alternative 13: *Valid only if the block $\langle t_1,t_2\rangle$ spans 2 or more periods.*

Shift $X^*_{t_2+1}$ backward from (t_2+1) to some least expensive period $k^* \in \langle t_1, t_2-1\rangle$. The new production $X^{new}_{k^*}$ will include previous E_{k^*}, new $E^{new}_{t_2}$ and several or all new effective demand values of periods $\langle k^*+1, t_2-1\rangle$ in addition to $X^*_{t_2+1}$. Also recalculate new total loss values for periods (k^*+1) through t_2.

Alternative 14: *Applies to interior stockout periods only.*

Shift $X^*_{t_2+1}$ backward from (t_2+1) to some least expensive period $k^* \in \langle u+1, t_1-1\rangle$. $X^{new}_{k^*}$ will include previous E_{k^*}, new $E^{new}_{t_2}$ and several or all new effective demand values of periods $\langle k^*+1, t_2-1\rangle$ in addition to $X^*_{t_2+1}$. Total loss and effective demand values for periods $\langle k^*+1, t_2\rangle$ as well as $\left(X^*_u\right)^{new}$ must be recalculated according to the conservation periods of production in u and in k^*.

Alternative 15: *Invalid for interior stockout periods. Also invalid if $X^*_{t_2+2} > 0$.*

In this alternative, stockout periods begin immediately at $t_1 = 1$. Meet E_{t_2} to avoid LG_{t_2+1}. For this, replace the original production $X^*_{t_2+1}$ with two productions activities in periods r_1^* and r_2^* where $r_1^* \in \langle 1, t_2\rangle$ and $r_2^* = (t_2+1+r_1^*)$. Remember that original $X^*_{t_2+1}$ is used up by the end of the first regeneration period j following the block $\langle 1, t_2\rangle$. Since $X^*_{t_2+1}$ cannot be shifted beyond j, at most $min\{t_2,\ j-t_2-1\}$ candidate pairs are inspected starting from $r_1= 1$ and $r_2= t_2+2$ to find the least expensive pair of production periods (r_1^*, r_2^*). Once found, new values of effective demand and total losses for periods $\langle 1, q\rangle$ must be recalculated where (q+1) is the second original production period after $\langle 1, t_2\rangle$.

Alternative 16: *Applies to interior stockout periods only. Invalid if $u = t_1 - 1$.*

Lose d_{t_1-1}. Thus, decrease original X^*_u by d_{t_1-1}. Compute new effective demand values for periods $\langle t_1,t_2\rangle$. Meet $E^{new}_{t_2}$ with an exclusive production in t_2 such that $X^{new}_{t_2} = E^{new}_{t_2}$.

Alternative 17: *Valid only if the block $\langle t_1,t_2\rangle$ spans 2 or more periods.*

This alternative is an easier version of *Alternative 12* with a smaller search space. To meet the effective demand in t_2 and to avoid goodwill loss in (t_2+1), produce twice during the stockout periods; once in t_1 and once in a least expensive period $k^* \in \langle t_1+1, t_2\rangle$. So, different than in *Alternative 12*, the time of the first extra production activity is fixed to t_1. The rest of this alternative is the same as *Alternative 12*.

Alternative 18: *Applies to interior stockout periods only where $u < (t_1 - 1)$.*

Provided that the original X_u^* covers at least two periods before t_1, shift in this last alternative the block of stockout periods from $\langle t_1,t_2\rangle$ to $\langle t_1-1, t_2-1\rangle$ such that X_u^* contracts by d_{t_1-1}. There are two subcases to be considered depending on period (t_2+2).

<u>Subcase 1</u>. $X_{t_2+2}^* > 0$

Either make new extra production in t_2 to meet $E_{t_2}^{new}$, or shift original $X_{t_2+1}^*$ backward to t_2. In both cases, a goodwill loss in period (t_2+1) will be avoided.

<u>Subcase 2</u>. $X_{t_2+2}^* = 0$

Cancel the production $X_{t_2+1}^*$ and shift it to the next period (t_2+2). Produce $X_{t_2}^{new} = E_{t_2}^{new} + d_{t_2+1}$ after the new effective demands in periods $\langle t_1, t_2+1\rangle$ have been computed. In this subcase, too, we avoid a goodwill loss in period (t_2+1).

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